EXTENDED SUPERKINEMATICS

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The supersymmetric extensions of the kinematical groups with N bispinor generators are considered. The case of anti-de Sitter group is treated in detail. It is proved that the maximal internal symmetry group is then SO(N).

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1. *Introduction*

In the previous paper [1] the possible superkinematics with one Majorana bispinor generator have been considered. It has been shown that if the extensions with $\{Q, Q\} = 0$ are neglected then all superalgebras are obtained by extension of the anti-de Sitter SO(3, 2) family. The supergroups obtained can be viewed as the quantum-mechanical ones. Then the problem of incorporating of internal symmetries arises. This question was answered for some groups (see, for example, the famous Haag-Lopuszanski-Sohnius theorem for supersymmetric extensions of Poincare group [2]). The most general formulation of the problem is the following: find all possible extensions of the given quantum-mechanical group by the finite number of bispinor generators. In this paper such general analysis is presented. The case of anti-de Sitter group SO(3, 2) is treated in some detail. It was suggested [3], [4] that the maximal internal symmetry group of the super-anti-de Sitter group is an SO(N) and not SU(N) group. However, the arguments given in Ref. [3] were based on the assumption that the anti-de Sitter algebra is embedded in SU(2, 2/N) superconformal algebra. Here we give the straightforward proof that in fact the maximal internal symmetry group is SO(N).

2. *General formulation*

The classification of possible kinematics given by Bacry and Levy-Leblond (see also [1]) [5] was based on the following assumptions:
(i) space is isotropic,

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(*ii*) parity and time reversal are the automorphisms of kinematical group,
(*iii*) inertial transformations in any given direction form a non-compact one-parameter subgroup.
The generators H, $\vec{P}$ (space-time translations), $\vec{J}$ (rotations) and $\vec{K}$ (boosts) satisfy the basic commutation rules

$$[J_k, J_l] = i\varepsilon_{klm}J_m, \quad [J_k, P_l] = i\varepsilon_{klm}P_m,$$

$$[J_k, H] = 0, \quad [J_k, K_l] = i\varepsilon_{klm}K_m,$$

$$[H, P_k] = i\alpha K_k, \quad [H, K_k] = i\lambda P_k, \quad [P_k, K_l] = i\varphi\delta_{kl}H,$$

$$[P_k, P_l] = i\beta\varepsilon_{klm}J_m, \quad [K_k, K_l] = \mu\varepsilon_{klm}J_m.$$

The values of parameters α, β, λ, μ, φ are given in Table I of Ref. [1].
Let us introduce N Majorana bispinor generators Q_α^A (α — space-time index, $A = 1, \ldots, N$ — internal index):

$$[J_k, Q] = \tfrac{1}{2}\gamma_0\gamma_5\gamma_k Q.$$

We adopt here the conventions of Ref. [1]. In particular γ's are purely imaginary. Under the space inversion operation **P** the generators Q transform as follows

$$\mathbf{P}Q\mathbf{P}^{-1} = i\gamma_0 Q.$$

To obtain the possible extensions we follow the straightforward method which consists in:
(a) writing the unknown commutators and anticommutators as linear forms of H, $\vec{P}$, $\vec{J}$, $\vec{K}$, Q,
(b) taking into account the assumption ot space-parity invariance,
(c) assuming that the bispinor generators span the representation space of some compact internal symmetry group commuting with space-time transformations (with generators X_λ and X_λ^5),
(d) imposing the generalized Jacobi identities [6]

$$[X[Y, Q]]+[Y, [Q, X]]+[Q, [X, Y]] = 0,$$

$$[X, \{Q, Q'\}]+\{[Q, X], Q'\}+\{[Q', X], Q\} = 0,$$

$$[Q, \{Q', Q''\}]+[Q', \{Q'', Q\}]+[Q'', \{Q, Q'\}] = 0,$$

where X, Y are bosonic generators.

After the steps (a), (b) and (c) we get

$$[H, Q] = (ia+b\gamma_0)Q, \quad [P_k, Q] = (c+id\gamma_0)\gamma_k Q, \quad [K_k, Q] = (e+if\gamma_0)\gamma_k Q,$$

$$[X_\lambda, Q] = t_\lambda Q, \quad [X_\lambda^5, Q] = \gamma_5 t_\lambda^5 Q,$$

$$\{Q, Q\} = (A+iB\gamma_0)H+(iC+D\gamma_0)\gamma_k P_k+(iE+F\gamma_0)\gamma_k K_k$$

$$+(R+iS\gamma_0)\gamma_5\gamma_k J_k+(N_\lambda+i\gamma_0 M_\lambda)X_\lambda+(iN_\tau^5+\gamma_0 M_\tau^5)\gamma_5 X_\tau^5 \tag{1}$$

with

$$[X_\lambda, X_\tau] = ic^\varepsilon_{\lambda\tau}X_\varepsilon, \quad [X_\lambda, X^5_\tau] = id^\varepsilon_{\lambda\tau}X^5_\varepsilon, \quad [X^5_\lambda, X^5_\tau] = ie^\varepsilon_{\lambda\tau}X_\varepsilon, \tag{2}$$

where X's and X^5's are scalars and pseudoscalars respectively. The $N\times N$ matrices in formulae (1) have the following properties: a, b, c, d, e, f are real, A, C, D, E, F, R, N — real and symmetric, B, S, M, N^5, M^5 — real and antisymmetric, t — imaginary and antisymmetric and finally t^5 — real and symmetric.

The most general form ot the antiunitary time-reversion operation is

$$TQT^{-1} = \tau\gamma_0\gamma_5 Q,$$

where the $N\times N$ real matrix τ commutes with b, c, f and anticommutes with a, d, e. According to the point (d) the bilinear conditions on the matrices $a, \ldots, t^5$ are imposed which guarantee that the Jacobi identities are fulfilled. The general analysis of such conditions is very tedious task. For the reasons explained in the Introduction we concentrate ourselves on the case of SO(3, 2).

3. *The anti-de Sitter superalgebra*

For the anti-de Sitter group ($\alpha = -1, \beta = -1, \lambda = 1, \varphi = 1, \mu = -1$, see Table I in Ref. [1]) b, c, d, e, f are symmetric and a — antisymmetric. This follows from the fact that the generators $\vec{J}$ and H are compact while $\vec{K}$ and $\vec{P}$ are noncompact. Let us point out that $[e, f] = 0$ (from the Jacobi identities) so by a point transformation $\exp(i\omega\gamma_0)$ of Q, where $[\omega, e] = 0$, $[\omega, f] = 0$, we can put $e = 0$. From the analysis of the Jacobi identities we get

$$\begin{gathered}
a = d = e = 0, \quad b = \mp c, \quad f = \pm\tfrac{1}{2}I, \quad c^2 = \tfrac{1}{4}I, \\
B = C = F = S = 0, \quad A = \kappa I, \quad D = \mp\kappa I, \\
E = -2\kappa c, \quad R = \mp 2\kappa c, \quad N_\lambda = N^5_\varepsilon = 0, \\
[c, M_\lambda] = \{c, M^5_\varepsilon\} = [c, t_\lambda] = \{c, t^5_\varepsilon\} = 0, \\
[t_\lambda M_\theta] = ic^\theta_{\lambda\tau}M_\tau, \quad [t_\lambda, M^5_\varepsilon] = id^\varepsilon_{\lambda\tau}M^5_\tau, \\
\{t^5_\theta, M_\lambda\} = -d^\lambda_{\theta\varepsilon}M^5_\varepsilon, \quad \{t^5_\theta, M^5_\varepsilon\} = e^\varepsilon_{\theta\lambda}M_\lambda, \\
iM^{AB}_\lambda t^{CD}_\lambda = \pm\kappa(c^D + c^C)(\delta^{BC}\delta^{AD} - \delta^{CA}\delta^{BD}), \\
M^{5AB}_\varepsilon t^{5CD}_\varepsilon = \kappa(c^D - c^C)(\delta^{BC}\delta^{AD} - \delta^{CA}\delta^{BD})
\end{gathered} \tag{3}$$

in the basis where c is diagonal ($c^{AB} = \delta^{AB}c^A$). The parameter κ is positive because of the positive definiteness of the underlying space of states.

Using the fact that for any (in general reducible) nontrivial representation of semisimple noncompact group one can choose the generators T in such a way that

$$T_\gamma(T_\alpha T_\beta) = \theta(T)\delta_{\alpha\beta}, \quad \theta(T) - \text{constant},$$

we obtain

$$M_\lambda = \mp \frac{4i\kappa}{\theta} ct_\lambda, \quad M^5_\varepsilon = \pm \frac{4\kappa}{\theta} ct^5_\varepsilon. \tag{4}$$

Rescaling Q's we can put $\kappa = 1$. From (3) and (4) we get

$$\sum_\lambda t^{AB}_\lambda t^{CD}_\lambda = (\delta^{BC}\delta^{AD} - \delta^{CA}\delta^{BD})c^A(c^D + c^C)\theta(t),$$

$$\sum_\varepsilon t^{5AB}_\varepsilon t^{5CD}_\varepsilon = (\delta^{BC}\delta^{AD} - \delta^{CA}\delta^{BD})c^A(c^D - c^C)\theta(t).$$

It follows from the above equations that the vectors $t^{AB}_{(\lambda)}$ are orthogonal and nonvanishing. Number of them equal $\binom{N}{2}$ and they are linearly independent. So the number of generators t_λ must be not less than $\binom{N}{2}$. On the other hand the number of pure imaginary antisymmetric generators does not exceed $\binom{N}{2}$. Consequently it equals $\binom{N}{2}$ exactly and Q's span the space of selfrepresentation of the Lie algebra of SO(N). Because $[c, t_\lambda] = 0$ and $c^2 = \frac{1}{4} I$ it follows that $c = \frac{\eta}{2} I$ where $\eta = \pm 1$. But $\{t^5_\varepsilon, c\} = 0$ and consequently $t^5_\varepsilon = 0$ and $M^5_\varepsilon = 0$. Summarizing we have

$$a = d = e = 0, \quad b = \mp \frac{\eta}{2} I, \quad f = \pm \tfrac{1}{2} I, \quad c = \frac{\eta}{2} I,$$

$$B = C = F = S = 0, \quad A = I, \quad D = \mp I,$$

$$E = -\eta I, \quad R = \mp \eta I$$

$$N_\lambda = N^5_\varepsilon = M^5_\varepsilon = 0, \quad M_\lambda = \mp \eta \frac{2i}{\theta} t_\lambda, \quad t^5_\varepsilon = 0, \tag{5}$$

where $\theta = \mathrm{Tr}\,(t^2_\lambda)$ and t_λ — the generators of the selfrepresentation of SO(N). We have proved that the maximal internal symmetry group of the anti-de Sitter superalgebra is SO(N). The other possible supersymmetric extensions of kinematical groups are under investigation.

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