SOLVABLE QUANTUM-MECHANICAL MODEL WITH EXPONENTIAL SPECTRUM AND GAUGE STRUCTURE

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(*Received January 9, 1985*)

An exactly solvable, one-dimensional quantum-mechanical model of a new kind is constructed. It gives an exponentially rising discrete energy spectrum bounded from below and displays an explicit gauge invariance. To our knowledge, such a type of spectrum is novel for mechanical systems.

PACS numbers: 03.65.Ge, 03.65.Bz

In this note we construct an exactly solvable, one-dimensional quantum-mechanical model giving an exponentially rising discrete energy spectrum bounded from below. To our knowledge, no mechanical model possessing such a spectrum was described so far in the literature [1]. Speaking in general terms, the model can be characterized as a harmonic oscillator, where some additional interaction is introduced in the form of an energy-dependent "minimal" self-coupling. Despite its pure mechanical character, the model displays an explicit gauge invariance [2], which by itself is an interesting feature and may be significant for understanding the physical origin and affiliation (if any) of the introduced self-coupling. But we shall not attempt to speculate about this point here. We shall treat our model as a formal construction (of some stimulating properties).

Our construction is based on the hamiltonian of the form

$$H = m_0 + \frac{1}{2\Lambda}[p + gA(q,p)]^2 + \frac{\Lambda\omega^2}{2}q^2 - \frac{\omega}{2}, \tag{1}$$

where q denotes a coordinate, p represents its conjugate momentum and $A(q,p)$ is a Hermitian operator built up of both q and p, while $m_0 > 0$, $\Lambda > 0$ and $\omega > 0$ are mass-dimensional constants and $g \neq 0$ a dimensionless coupling constant. We specify the function $A(q,p)$ by means of the following operator equation relating $A(q,p)$ to the energy H given in Eq. (1):

$$-i[q, A(q,p)] = gl(H - m_0), \tag{2}$$

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where $l > 0$ is a length-dimensional constant. We also impose the condition

$$A(-q, -p) = -A(q, p) \tag{3}$$

which guarantees that $H(-q, -p) = H(q, p)$.

Note that in the momentum space Eq. (2) takes the form

$$\frac{\partial}{\partial p} A\left(i\frac{\partial}{\partial p}, p\right) = gl\left[H\left(i\frac{\partial}{\partial p}, p\right) - m_0\right] \tag{4}$$

telling us that the gradient of $A(q, p)$ in momentum space in proportional to the excitation energy $H - m_0$. As it turns out, this assumption causes the energy spectrum to rise exponentially, starting from the ground-state energy $E_0 = m_0$. We will call the dynamical system specified as in Eqs. (1), (2) and (3) a "pseudo-harmonic" oscillator. It becomes the usual harmonic oscillator when $g \to 0$.

Although Eqs. (1) and (2) define the hamiltonian H only implicitly, the eigenvalue equation for H can be solved exactly by means of the following pseudo-annihilation and -creation operators:

$$\left.\begin{matrix} a \\ a^+ \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left\{\sqrt{\Lambda\omega}\, q \pm \frac{i}{\sqrt{\Lambda\omega}}\,[p + gA(q, p)]\right\}. \tag{5}$$

They satisfy the commutation relation

$$[a, a^+] = 1 + g^2 l(H - m_0) \tag{6}$$

and lead to the formula for H

$$H - m_0 = \frac{\omega}{2}(a^+ a + a a^+ - 1) = \omega[N + \tfrac{1}{2}\, g^2 l(H - m_0)], \tag{7}$$

where

$$N = a^+ a. \tag{8}$$

Solving Eqs. (6) and (7) with respect to $[a, a^+]$ and $H - m_0$, we obtain the following commutation relation:

$$[a, a^+] = 1 + (\lambda^2 - 1)N \tag{9}$$

(which can be rewritten also as $aa^+ - \lambda^2 a^+ a = 1$) and formula for H:

$$H = m_0 + \frac{\omega}{2}(\lambda^2 + 1)N, \tag{10}$$

where

$$\lambda^2 = \frac{1 + g^2\omega l/2}{1 - g^2\omega l/2}. \tag{11}$$

Note that $|\lambda^2| > 1$ (since $\omega > 0$ and $l > 0$) and $\lambda^2 > 0$ or < 0 if $g^2\omega l/2 < 1$ or > 1, respectively. Note also that $\lambda^2 \to 1$ for the harmonic oscillator. Henceforth, we will consider the case of $g^2\omega l/2 < 1$, thus we shall have $\lambda^2 > 1$.

Now, let us take into account the eigenvalue equation for N:

$$N|n\rangle = N_n|n\rangle, \quad \langle n|n\rangle = 1 \quad (n = 0, 1, 2, \ldots), \tag{12}$$

where $|n\rangle$ and N_n are unknown (we only anticipate that the spectrum of N is discrete). Then, the commutation relation (9) implies

$$\begin{aligned} Na^+|n\rangle &= (\lambda^2 N_n+1)a^+|n\rangle, \\ Na|n\rangle &= \frac{1}{\lambda^2}(N_n-1)a|n\rangle. \end{aligned} \tag{13}$$

Hence we can conclude that [3]

$$\begin{aligned} a^+|n\rangle &= \sqrt{\lambda^2 N_n+1}\,|n+1\rangle, \\ a|n\rangle &= \sqrt{N_n}\,|n-1\rangle \end{aligned} \tag{14}$$

and

$$N_{n+1} = \lambda^2 N_n + 1. \tag{15}$$

Defining $|0\rangle$ through the condition $a|0\rangle = 0$, which gives $N_0 = 0$, and solving the spectral recurrence formula (15) we get the spectrum for N:

$$N_n = \frac{\lambda^{2n}-1}{\lambda^2-1} = \begin{cases} 0 & \text{for} \quad n = 0, \\ 1+\lambda^2+ \ldots +\lambda^{2n-2} & \text{for} \quad n \geqslant 1. \end{cases} \tag{16}$$

Then, Eq. (10) gives the following spectral recurrence formula and spectrum for H:

$$E_{n+1} - m_0 = \lambda^2(E_n - m_0) + \frac{\omega}{2}(\lambda^2+1) \tag{17}$$

and

$$E_n = m_0 + \frac{\omega}{2}(\lambda^2+1)\frac{\lambda^{2n}-1}{\lambda^2-1}. \tag{18}$$

Since $\lambda^2 > 1$, it is an exponentially growing spectrum $\sim \exp(2n \ln \lambda) + \text{const}$ (here $\lambda = \sqrt{\lambda^2} > 1$). To give a numerical example: with $\lambda = 4$ we have $E_n = m_0 + \frac{17}{2}\omega N_n$ where $N_n = 0, 1, 17, 273, \ldots$ for $n = 0, 1, 2, 3, \ldots$. In the case of $\lambda^2 \to 1$ we obtain the harmonic-oscillator formulae $N_n = n$ and $E_n = m_0 + \omega n$. It is interesting to remark that the energy spectrum (18) is independent of the mass scale Λ (appearing in the hamiltonian (1)) when ω and λ^2 are fixed. Such a property is well known to hold for the usual harmonic oscillator (corresponding to $g \to 0$ or $\lambda^2 \to 1$) when ω is fixed.

Note that the energy spectrum (18) satisfies also the three-term recurrence formula [4]

$$E_{n+2}-E_{n+1} = \lambda^2(E_{n+1}-E_n) \tag{19}$$

which is less informative than the two-term recurrence formula (17) equivalent to the spectrum (18) (if $E_0 = m_0$).

As was mentioned at the beginning, our mechanical model is gauge invariant. In fact, the wave equation

$$H\psi = E\psi \tag{20}$$

and the operator equation (2), where H is given in Eq. (1), are invariant in the position space under the following gauge transformation:

$$\begin{aligned} \psi(q) &\to \psi'(q) = e^{ig\alpha(q)}\psi(q), \\ A(q, p) &\to A'(q, p) = e^{ig\alpha(q)}A(q, p)e^{-ig\alpha(q)} - \frac{\partial\alpha(q)}{\partial q}. \end{aligned} \tag{21}$$

This follows from the relations

$$e^{ig\alpha(q)}[p+gA(q, p)]\psi(q) = [p+gA'(q, p)]\psi'(q) \tag{22}$$

and

$$e^{ig\alpha(q)}[q, A(q, p)]e^{-ig\alpha(q)} = [q, A'(q, p)]. \tag{23}$$

In order to satisfy the condition (3) in the new gauge (21) it is enough to require that $\alpha(-q) = \alpha(q)$.

It may be worth while to note that the same energy spectrum as given in Eq. (18) holds also for the "reciprocal" quantum-mechanical model, where [5]

$$H = m_0 + \frac{1}{2\Lambda}p^2 + \frac{\Lambda\omega^2}{2}[q+gB(q, p)]^2 - \frac{\omega}{2}, \tag{1'}$$

$$i[p, B(q, p)] = gl(H-m_0) \tag{2'}$$

and

$$B(-q, -p) = -B(q, p). \tag{3'}$$

In this case the pseudo-annihilation and -creation operators, now of the form

$$\left.\begin{matrix} a \\ a^+ \end{matrix}\right\} = \frac{1}{\sqrt{2}}\left\{\sqrt{\Lambda\omega}\,[q+gB(q, p)] \pm \frac{i}{\sqrt{\Lambda\omega}}\,p\right\}, \tag{5'}$$

lead to the same formulae as before: (6) and (7) as well as (9) and (10) (where λ^2 is given by Eq. (11)). Hence the same energy spectrum (18).

Finally, we should like to remark that the operators $\tilde{q} \equiv q$ and $\tilde{p} \equiv p+gA(q, p)$ in the original model and the operators $\tilde{q} \equiv q+gB(q, p)$ and $\tilde{p} \equiv p$ in the "reciprocal" model satisfy the commutation relation of *the same* form

$$[\tilde{q}, \tilde{p}] = i[1+g^2 l(H-m_0)], \tag{24}$$

where H is given by Eq. (1) and Eq. (1′), respectively. The noncanonical commutation relation of the type (24) was considered by Saavedra and Utreras [6] who made the bold conjecture that at small distances the usual Heisenberg canonical commutation relation $[q, p] = i$ should be thus modified [7]. This might be an approach alternative to that of postulating the novel kind of self-coupling as introduced in hamiltonian (1) or, eventually, the novel "reciprocal" self-coupling as introduced in hamiltonian (1′).

REFERENCES

[1] Cf. e.g. S. Flügge, *Practical Quantum Mechanics*, vol. 1 and 2, Springer-Verlag, Berlin-Heidelberg--New York 1971. For the case of singular potentials cf.: K. Case, *Phys. Rev.* **80**, 797 (1950); W. M. Frank, D. J. Land, R. M. Spector, *Rev. Mod. Phys.* **43**, 36 (1971).

[2] This explicit gauge invariance should be distinguished from the interesting implicit gauge structure discussed recently for mechanical systems in F. Wilczek, A. Zee, *Phys. Rev. Lett.* **52**, 2111 (1984).

[3] Cf. the classical reasoning for the harmonic oscillator in P. A. M. Dirac, *The Principles of Quantum Mechanics*, 4th ed., Oxford University Press, Oxford 1958, p. 136.

[4] This is an interesting formula by itself. Masses of charged leptons e^-, μ^- and τ^- satisfy it when $\lambda = 4$ (with a great accuracy!), if $E_0 = m_e$, $E_1 = m_\mu$ and $E_2 = m_\tau$. Then it implies that $E_3 = 28.5$ GeV. Would E_3 be the mass of charged lepton of the fourth generation?

[5] Cf. W. Królikowski, *Acta Phys. Pol.* **B14**, 629 (1983); **B15**, 861 (1984).

[6] I. Saavedra, C. Utreras, *Phys. Lett.* **98B**, 74 (1981).

[7] Cf. also H. S. Snyder, *Phys. Rev.* **71**, 38 (1947).