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STOCHASTIC TIME SCALE FOR THE UNIVERSE

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An intrinsic time scale is naturally defined within stochastic gradient dynamical systems. It should be interpreted as a "relaxation time" to a local potential minimum after the system has been randomly perturbed. It is shown that for a flat Friedman-like cosmological model this time scale is of order of the age of the Universe.

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1. Introductory remarks

In the previous works [1, 2], random effects have been introduced to the Friedman-like evolution of the Universe, changing it into a stochastic process. It seems that there is no realistic cosmology without probabilistic elements. Immense degree of extrapolation, present in every cosmological research, may enlarge some instances of our "local lack of knowledge" to the rank of an important "random factor". Initial data, when known with unavoidable measurement errors, inevitably lead to a non-deterministic behaviour. Moreover, it is highly probable that quantum effects at Planck's threshold impressed random characteristics on all future evolution. Independently of this motivation, stochastic cosmological models form an interesting mathematical object worthwhile to be studied and

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developed. One of the most striking of its features is the fact that cosmological singularities, being properties of single phase trajectories and not properties of the entire phase space, are stochastically irrelevant. The present work is aimed at studying another interesting aspect of those stochastic world models which can be put into the form of gradient dynamical systems. There is an intrinsic time scale naturally defined within such models which can be interpreted as a "relaxation time" to a local potential minimum after the system has been perturbed by a random fluctuation. We effectively compute this time scale for both flat and non-flat Friedman cosmological models perturbed with a "white noise". In the case of a simple flat model the intrinsic time scale can be expressed in terms of fundamental physical constants, reproducing, in this way, one of the well known "large number coincidences" between fundamental constants and the age of the universe. It turns out that the intrinsic time scale is strictly connected with the structure of the phase plane and with the so-called energy conditions.

In Sec. 2 we give necessary preliminaries of Friedman gradient dynamical systems. In Sec. 3 we discuss their stochastic generalization with a special emphasis on two time scales involved in the Fokker-Planck equation. In Sec. 4 the stochastic time scale is effectively computed for Friedman-like world models filled with the perfect fluid. Sec. 5 contains a short comment upon the obtained results.

2. *Friedman's cosmological models as gradient dynamical systems*

As it is known (see, e.g. [3]), Friedman's equation can be reduced to the form of the dynamical system

$$\dot{H} \equiv \frac{dH}{dt} = -H^2 - \tfrac{1}{6}(E^2+3p) + \frac{\Lambda}{3}$$

$$\dot{E} \equiv \frac{dE}{dt} = -\tfrac{3}{2}\,\frac{E^2+p}{E}\,H \tag{1}$$

where $H \equiv \dfrac{\dot{R}}{R}$ is the Hubble parameter, $E = \varepsilon^{1/2}$, ε being the energy density, p — pressure, and Λ — the cosmological constant. The system has the first integral

$$E^2 - 3H^2 = -\Lambda + \frac{3k}{R^2}\,. \tag{2}$$

For the flat models ($k = 0$), system (1) can be given the form of a one-dimensional gradient dynamical system

$$\dot{H} = -\frac{\partial V}{\partial H} \tag{3}$$

with the potential function

$$V(H) = \tfrac{1}{2}\int_{H_0}^{H} [E^2(z)+p(z, E(z))]dz. \tag{4}$$

For the perfect fluid with the equation of state $p = (\gamma-1)\varepsilon$, $1 \leqslant \gamma \leqslant 2$, potential (4) assumes the form

$$V(H) = \tfrac{1}{2}\gamma H^3 - \tfrac{1}{2}\gamma \Lambda H + V_0 \tag{5}$$

V_0 being a constant. More generally, after rescaling $\tilde{H} = \alpha H$, where $\alpha^2 = \frac{9}{2}\gamma/(3\gamma-2)$, system (1) for the perfect fluid is also a gradient dynamical system

$$\dot{\tilde{H}} = -\frac{\partial H}{\partial \tilde{H}}$$

$$\dot{E} = -\frac{\partial V}{\partial E} \tag{6}$$

with the potential function

$$V(\tilde{H}, E) = \frac{\tilde{H}^3}{3\alpha} + \frac{3\gamma-2}{6}\alpha \tilde{H} E^2 + \frac{\Lambda\alpha}{3}\tilde{H} + V_0 \tag{7}$$

where V_0 is a constant. Potential function (5) is shown in Fig. 1. By using these functions one can give a new classification of cosmological models (see [4]).

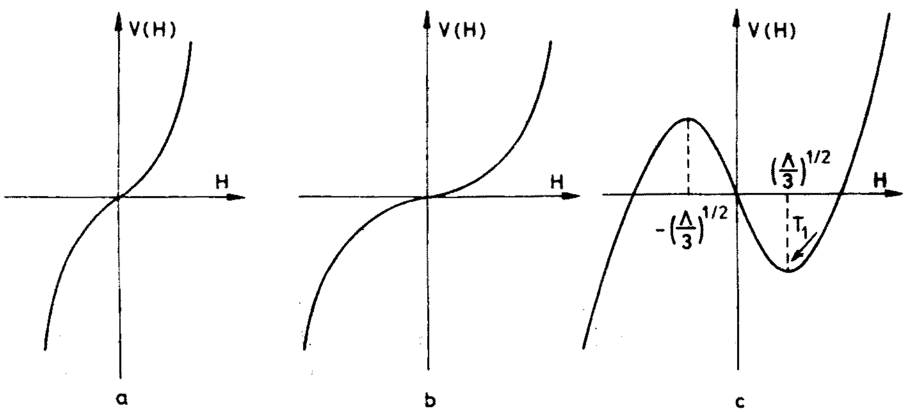

Fig. 1. Potential functions for the flat cosmological models: with (a) $\Lambda < 0$, (b) $\Lambda = 0$, (c) $\Lambda > 0$

3. *Stochastic cosmological equations*

Random effects influencing the evolution of the universe can be included into the cosmological scheme by the Friedman dynamical system (like that given by (3) or (6)) with a "white noise"

$$\dot{x} = -\frac{\partial V}{\partial x} + \eta_1(t)$$

$$\dot{y} = -\frac{\partial V}{\partial y} + \eta_2(t) \tag{8}$$

where η_i are "white noise corrections", namely $\langle\eta_i(t)\rangle = 0$ and $\langle\eta_i(t), \eta_i(t+\tau)\rangle = 2\theta\delta(\tau)$, $\theta = \text{const}$; $i, j = 1, 2$. Such world models were studied in [1, 2]. Their evolution is described by the Fokker-Planck equation

$$\frac{\partial}{\partial t} P = \nabla(P\nabla V)+\nabla^2(DP) \tag{9}$$

determining the probability density (or density distribution) P corresponding to a given potential V. D is here a diffusion constant (which for a process described by system (8) equals θ). The stationary solution of the process $\frac{\partial P}{\partial t} = 0$ can be easily found

$$P(x) = N_0 e^{-V(x)/D} \tag{10}$$

where N_0 is a normalization factor.

As analysed by Gilmore [5, 6], the first term, on the right-hand side of the Fokker--Planck equation ($\nabla P\nabla V$), the so-called transport term, describes a tendency for a probability density function P to move towards the nearest local minimum of the potential function V. The second term ($\nabla^2(DP)$), known as the diffusion term, expresses both a certain fuzziness of the potential centred around a local minimum, and a probability with which a stochastic fluctuation can lead the system from a metastable state to a distant global minimum. One can see, therefore, that two time scales are involved in any stochastic process described by Fokker-Planck equation (9), namely: (a) a time scale T_1 associated with the relaxation toward a local minimum after the system has been perturbed, and (b) a time scale T_2 associated with the transition from a local minimum to the global one.

The time scale T_1 is given by

$$T_1 = \max_i \{1/\lambda_i\} \tag{11}$$

where λ_i are eigenvalues of the linearization matrix of the considered gradient dynamical system, that is to say, of the matrix $V_{ij}(x_0)$ where x_0 is a critical point of the system (see [6]). For potential functions given in Sec. 2, there is no T_2 time scale since they have no global minimum.

4. *Examples*

(1) *Flat cosmological models described by system* (3) *perturbed with a white noise, having potential* (5). There are two critical points $H_0 = \pm\left(\frac{\Lambda}{3}\right)^{1/2}$ which represent the expanding and contracting de Sitter universes. One readily finds

$$T_1 = \pm\frac{1}{\gamma(3\Lambda)^{1/2}}\,. \tag{12}$$

If one agrees to interpret the cosmological constant Λ as being linked to the energy density of the vacuum (as it is a commonplace in contemporary grand unifying theories),

$\Lambda = \frac{8\pi G\varepsilon_{vac}}{c^4}$, where G is the gravitational constant, and $\varepsilon_{vac} = \frac{m_p^6 c^4 G}{\hbar^4}$, one immediately gets

$$T_1 = \frac{\hbar^2}{\gamma(24\pi)^{1/2} m_p^3 cG} \propto \frac{\hbar^2}{m_p^3 Gc} \tag{13}$$

where m_p is the proton mass. Surprisingly enough, this turns out to be well known "large number coincidence" giving the age of the universe in terms of fundamental physical constants. Numerical value of T_1 is $\sim 10^{10}$ years.

For models with the vanishing cosmological constant, $\Lambda \to 0$, one obtains $T_1 \to \infty$, which should be interpreted as a "relaxation time" for a perturbed model to go to the asymptotic de Sitter state.

(2) *Flat cosmological models described by system* (3) *perturbed with a white noise, having potential* (4). The linearization matrix is

$$V_{ij}(H_0) = \tfrac{1}{2}\frac{d}{dH}(E^2(H) + p(H, E(H))|_{H=H_0}$$

and consequently

$$T_1 = \frac{1}{\tfrac{1}{2}\frac{d}{dH}(E^2(H) + p(H, E(H))|_{H=H_0}}. \tag{14}$$

(3) *Cosmological models described by system* (6) *perturbed by a white noise, having potential function* (7). The linearization matrix reads

$$V_{ij} = \begin{pmatrix} \frac{2}{\alpha}\tilde{H}_0 & \frac{3\gamma}{2\alpha}E_0 \\ \frac{3\gamma}{2\alpha}E_0 & \frac{3\gamma}{2\alpha}\tilde{H}_0 \end{pmatrix}.$$

For static critical points ($H_0 = 0$), the eigenvalues are $\lambda_{1,2} = \pm\frac{3\gamma}{2\alpha}E_0$, and consequently the "relaxation time" T_1 to the Einstein static state

$$T_1 = \frac{1}{(\gamma\Lambda)^{1/2}}. \tag{15}$$

For non-static critical points, one has $\tilde{H}_0 = \pm\alpha\left(\frac{\Lambda}{3}\right)^{1/2} \neq 0$ which, as a "relaxation time" to the de Sitter state, gives

$$T_1 = \max\{T_1^1, T_1^2\} = \max\left\{\frac{\alpha}{2\tilde{H}_0}, \frac{2\alpha}{3\gamma\tilde{H}_0}\right\}. \tag{16}$$

It is interesting to notice that till the radiation era, i.e. for $\gamma > 4/3$, $T_1 = T_1^1$, and after the radiation era, i.e. for $\gamma < 4/3$, $T_1 = T_1^2$.

5. *Comment*

A possibility to obtain a large number coincidence of the order of the age of the Universe as an intrinsic time scale of a stochastic cosmological process is certainly a striking circumstance. Two factors are responsible for this result. First is that the world evolution has been assumed to be a stochastic process of a particular kind (white noise perturbed Friedman), and second that the dynamical system describing this process turned out to be a gradient system with explicitly known potentials. Is all this another coincidence? One suggestion could be made. If one agreed to treat "stochastic corrections" to the Friedman equation as vestiges of quantum inaccuracies in the initial data for the universe, one would have another interconnection between the effects of fundamental laws of physics and the global world structure.

Another, apparently more formal, interconnection is evident from the above considerations. Stochastic time T_1 turns out to be strictly related — through the linearization matrix — to the phase space structure of the underlying non-perturbed dynamical system. On the other hand, it has been demonstrated, in Ref. [7], that the situation and character of critical points is uniquely determined by strong energy condition $\varepsilon+p \geqslant 0$ and the Lorentz invariance condition $\varepsilon+3p-2\Lambda \geqslant 0$, and consequently there are energy conditions which determine the stochastic time scale. One might speculate that some global constraints, such as energy conditions, play no less a role in shaping the structure of the universe than does the Friedman equation itself.

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