RADIATION OF A SUPERCHARGED SYSTEM

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A simple supersymmetric system of noninteracting photons and photinos at finite values of temperature and supercharge is studied. Particular attention is drawn to properties of photons. Analogs of the Planck formula and the Stefan-Boltzmann law are considered.

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Supersymmetry [1] is widely believed to be a key for solving two fundamental problems of physics: the construction of quantum gravity and consequently the unification of all interactions we know.

In this paper, however, we intend to consider quite a tiny problem being far from the above ambitious program. Namely, we study a very simple supersymmetric system of noninteracting photons and photinos in thermal equilibrium. We discuss properties of the system at finite values of supercharge and temperature. In particular, we focus our attention on properties of photons. We consider analogs of the Planck formula and the Stefan-Boltzmann law. Thermodynamics of a supercharged system has been studied in Refs. [2] and [3]. In our considerations we exploit many results of Kapusta, Pratt and Višnjić [2], and we present some of them for completeness.

We use space-time metric of the form $g_{11} = g_{22} = g_{33} = -g_{00} = 1$ and units where $c = \hbar = k = 1$.

The supersymmetric Lagrangian density of the system of photons, photinos (and antiphotinos) is [4]

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{i}{2} \bar{\psi}\partial^{\mu}\gamma_{\mu}\psi + \frac{1}{2} D^2, \tag{1}$$

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where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, ψ is a Majorana spinor, and D is an auxiliary field finally eliminated due to a trivial equation of motion $D = 0$. The Lagrangian (1) is invariant (up to a total divergence) under infinitesimal supersymmetry transformation [4]

$$\delta A_\mu = i\bar{\varepsilon}\gamma_\mu\psi,$$

$$\delta\psi = -\tfrac{1}{2} F_{\mu\nu}\gamma^\mu\gamma^\nu\varepsilon + iD\gamma_5\varepsilon,$$

$$\delta D = -\bar{\varepsilon}\gamma_5\partial^\mu\gamma_\mu\psi,$$

where ε is an infinitesimal Majorana spinor and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. This invariance leads to the existence of conserved supercharge current

$$j_\mu = -\tfrac{1}{2} F_{\nu\varrho}\gamma^\nu\gamma^\varrho\gamma_\mu\psi$$

and consequently conserved supercharge

$$Q = \int d^3x j_0(x)$$

being the Majorana spinor. Extending fields A_μ and ψ in plane waves, one finds

$$Q = \int d^3p Q(\bar{p}),$$

where $Q(\bar{p})$ can be chosen in the following form [3]

$$Q_1(\bar{p}) = \sqrt{\omega}\,(a_+(\bar{p})b_+^\dagger(\bar{p}) + a_+^\dagger(\bar{p})b_+(\bar{p})),$$

$$Q_2(\bar{p}) = \sqrt{\omega}\,(a_-(\bar{p})b_+^\dagger(\bar{p}) + a_-^\dagger(\bar{p})b_+(\bar{p})),$$

$$Q_3(\bar{p}) = \sqrt{\omega}\,(a_+(\bar{p})b_-^\dagger(\bar{p}) + a_+^\dagger(\bar{p})b_-(\bar{p})),$$

$$Q_4(\bar{p}) = \sqrt{\omega}\,(a_-(\bar{p})b_-^\dagger(\bar{p}) + a_-^\dagger(\bar{p})b_-(\bar{p})),$$

where $\omega = |\bar{p}|$ and $a_\pm^\dagger(\bar{p})$, $b_\pm^\dagger(\bar{p})$ are creation operators of photons with definite polarization (denoted by $+$ and $-$) and photinos with ± 1 helicity, respectively. Each component of supercharge is conserved, i.e. it commutes with the Hamiltonian

$$[H, Q_\alpha] = \int d^3p d^3q [H(\bar{p}), Q_\alpha(\bar{q})] = 0.$$

It has been observed in Ref. [3] that

$$[Q_\alpha(\bar{p}), Q_\alpha(\bar{q})] \neq 0.$$

Thus, Q is not an extensive quantity because it cannot be determined simultaneously at different space points of a system. We introduce another, extensive, conserved quantity, $\tilde{Q}$, which is related to the supercharge

$$\tilde{Q}_\alpha = i \int d^3p c_\alpha(\bar{p}) Q_\alpha(\bar{p}),$$

where $c_\alpha(\bar{p})$ are the operators satisfying the anticommuting relations

$$\{c_\alpha(\bar{p}), c_\varrho(\bar{q})\} = 2\delta_{\alpha\varrho}\delta^{(3)}(\bar{p} - \bar{q})$$

and

$$c_\alpha(\bar{p}) = c_\alpha^\dagger(\bar{p}).$$

In addition, $c_\alpha(\bar{p})$ commutes with all bosonic and anticommutes with all fermionic operators.[1] Thus, we transform the fermionic supercharge Q into bosonic one what is, in fact, needed.

Among the four charges $\tilde{Q}$, two commute and can be simultaneously diagonalized. This means that the system can be decomposed into two subsystems, each of them containing one component (one polarization state) of the A_μ and ψ fields connected (mixed) by supersymmetry transformation. One can choose $\tilde{Q}_1$ and $\tilde{Q}_4$ as two commuting charges denoted later on by $\tilde{Q}_+$ and Q_-.

The partition function for our system is

$$Z = \operatorname{Tr} \exp\left[-\beta(H - \mu_+\tilde{Q}_+ - \mu_-\tilde{Q}_-)\right],$$

where β is the inverse temperature, $\beta \equiv T^{-1}$; μ_+ and μ_- are chemical potentials of conserved supercharges $\tilde{Q}_+$ and $\tilde{Q}_-$. The Hamiltonian density (in momentum space) of the system can be decomposed into two parts related to both subsystems

$$H(\bar{p}) = \tilde{Q}_+^2(\bar{p}) + \tilde{Q}_-^2(\bar{p}).$$

Because

$$[\tilde{Q}_+(\bar{p}), \tilde{Q}_-(\bar{p})] = 0$$

and

$$[\tilde{Q}_\alpha(\bar{p}), \tilde{Q}_\alpha(\bar{p})]_{\bar{p} \neq \bar{q}} = 0$$

the partition function factorize into modes — systems of photons and photinos in one polarization state with the same momentum, $\bar{p}$. In each mode there is an unlimited number of photons while the number of photinos is 0 or 1. The factorized partition function is

$$Z = \prod_{\bar{p}} Z_{\bar{p}}(\beta, \mu_+) Z_{\bar{p}}(\beta, \mu_-),$$

where

$$Z_{\bar{p}}(\beta, \mu_\pm) = \operatorname{Tr} \exp\left[-\beta(\tilde{Q}_\pm^2(\bar{p}) - \mu_\pm \tilde{Q}_\pm(\bar{p})\right].$$

Later on we assume that

$$\mu_+ = \mu_- \equiv \mu, \tag{2}$$

what makes

$$\langle\tilde{Q}_+\rangle = \langle\tilde{Q}_-\rangle \equiv \frac{\langle Q'\rangle}{2}.$$

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[1] In Ref. [3] the supercharge was modified with the help of one-component $c(p)$ field. In this case however contrary to what was stated in [3], $[\tilde{Q}_1, \tilde{Q}_4] \neq 0$ even though $\{Q_1, Q_4\} = 0$.

$\langle A \rangle$ denotes an expectation value of operator A. According to (2), both subsystems are symmetric in such a sense that average numbers of photons (photinos) with different polarization are equal at any temperature. In other words, photons (photinos) in the system are unpolarized. Thus,

$$Z = \prod_{\bar{p}} [Z_{\bar{p}}(\beta, \mu)]^2.$$

The partition function of a single mode found by Kapusta, Pratt and Višnjić [3] is

$$Z_{\bar{p}} = 1+2 \sum_{n=1}^{\infty} e^{n\beta\omega} \operatorname{ch} (\beta\mu \sqrt{n\omega}). \tag{3}$$

Numbers of photinos and photons with momentum $\bar{p}$ are, respectively, [3]

$$\langle \nu(\bar{p}) \rangle = 1-Z_{\bar{p}}^{-1}, \quad \langle n(\bar{p}) \rangle = \frac{\langle H(\bar{p}) \rangle}{\omega} - \langle \nu(\bar{p}) \rangle.$$

The supercharge and energy carried by particles with momentum $\bar{p}$ are the following

$$\langle Q'(\bar{p}) \rangle = 2T \frac{\partial}{\partial\mu} \ln Z_{\bar{p}},$$

$$\langle H(\bar{p}) \rangle = -2 \frac{\partial}{\partial\beta} \ln Z_{\bar{p}} + \mu \langle Q'(\bar{p}) \rangle.$$

In the above formulae contributions from the both subsystems are included. The total energy carried by photons is

$$U = \sum_{\bar{p}} \omega \langle n(\bar{p}) \rangle.$$

Changing

$$\sum_{\bar{p}} \rightarrow V \int \frac{d^3p}{(2\pi)^3},$$

where V is a volume of the system, we find

$$U = \frac{V}{2\pi^2} \int_0^{\infty} d\omega\omega^3 \langle n(\bar{p}) \rangle.$$

Thus, the energetic spectrum of photons is

$$U_{\omega} = \frac{V}{2\pi^2} \omega^3 \langle n(\bar{p}) \rangle, \tag{4}$$

where

$$\langle n(\bar{p}) \rangle = -\frac{2}{\omega} \frac{\partial}{\partial\beta} \ln Z_{\bar{p}} + \frac{2\mu}{\beta\omega} \frac{\partial}{\partial\mu} \ln Z_{\bar{p}} - 1 + Z_{\bar{p}}^{-1}.$$

The above expression is an analog of the Planck formula. Because the sum (3) cannot be expressed in terms of elementary functions, analytic calculations can be performed in some limits only. For $\mu \to 0$ and $T \neq 0$ one finds

$$Z_{\bar{p}} = \operatorname{cth}(\tfrac{1}{2}\beta\omega) + \frac{\beta^2\mu^2\omega}{4\operatorname{sh}^2(\frac{1}{2}\beta\omega)} + O(\beta^4\mu^4\omega^2).$$

For this case

$$\langle Q'(\bar{p})\rangle = 2\beta\mu\omega\left[\frac{1}{e^{\beta\omega}-1} + \frac{1}{e^{\beta\omega}+1}\right][1+O(\beta^2\mu^2\omega)]$$

and

$$\langle Q'\rangle = \frac{\pi^2}{120}\mu V T^3[1+O(\beta\mu^2)]. \tag{5}$$

It is seen that the expectation value of supercharge vanishes when $\mu = 0$.

$$U_\omega = \frac{V}{\pi^2}\omega^3\left[\frac{1}{e^{\beta\omega}-1} + \frac{\mu^2\beta^2\omega}{4}\frac{\operatorname{cth}\frac{1}{2}(\beta\omega)}{\operatorname{sh}(\beta\omega)} + O(\beta^4\mu^4\omega^2)\right].$$

The above formula changes to the Planck formula for $\mu = 0$. Pressure is defined by

$$p = T\frac{\partial}{\partial V}\ln Z.$$

For $\mu \to 0$ one can find an equation of state of supercharged photon-neutrino gas, namely

$$pV = \tfrac{1}{3}[\langle H\rangle - \tfrac{1}{2}\mu\langle Q'\rangle].$$

Using the formula (5), the equation of state can be rewritten in the form

$$pV = \tfrac{1}{3}\left[\langle H\rangle - \frac{60}{\pi^2}\frac{\langle Q'\rangle^2}{VT^3}\right].$$

An interesting limit is at $T = 0$ for nonzero supercharge, $\mu \neq 0$. In this case the partition function (3) can be approximated as follows [3]

$$Z_{\bar{p}} = \begin{cases} 1 & \text{for } |\mu| \leqslant \sqrt{\omega}, \\ e^{-\beta(\omega-|\mu|\sqrt{\omega})} & \text{for } \sqrt{\omega} < |\mu| \leqslant (1+\sqrt{2})\sqrt{\omega}, \\ \vdots & \\ e^{-\beta(n\omega-|\mu|\sqrt{n\omega})} & \text{for } (\sqrt{n-1}+\sqrt{n})\sqrt{\omega} < |\mu| \leqslant (\sqrt{n}+\sqrt{n+1})\sqrt{\omega}. \end{cases}$$

At $T = 0$ the photon spectrum is

$$U_\omega = \begin{cases} 0 & \text{for } |\mu| \leqslant \sqrt{\omega}, \\ \dfrac{V}{2\pi^2}\omega^3 & \text{for } \dfrac{|\mu|}{\sqrt{2}+1} \leqslant \sqrt{\omega} < |\mu|, \\ \vdots & \\ \dfrac{V}{2\pi^2}\omega^3(2n-1) & \text{for } \dfrac{|\mu|}{\sqrt{n}+\sqrt{n+1}} \leqslant \sqrt{\omega} < \dfrac{|\mu|}{\sqrt{n}+\sqrt{n-1}}. \end{cases}$$

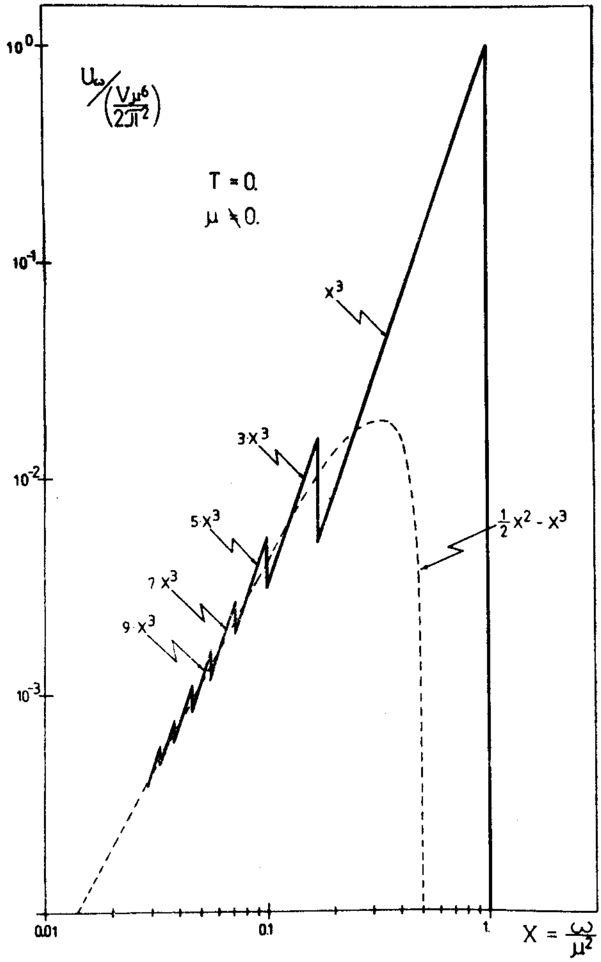

Fig. 1. The energetic photon spectrum at zero temperature

The above distribution is plotted in Fig. 1. Let us notice its unusual form with many sharp peaks. For $\omega \to 0$, U_ω approaches a smooth curve

$$\frac{V}{2\pi^2}\omega^3\left[\frac{\mu^2}{2\omega}-1\right].$$

In contrast to the photon one, the photino energetic spectrum, W_ω, at $T=0$ is quite familiar

$$W_\omega = \begin{cases} 0 & \text{for} \quad \mu \leqslant \sqrt{\omega}, \\ \dfrac{V}{2\pi^2}\,\omega^3 & \text{for} \quad \sqrt{\omega} < \mu. \end{cases}$$

It coincides with the spectrum of degenerated Fermi gas where the number of particles is conserved. μ^2 plays a role of Fermi energy. The total energy of photons at $T = 0$ is expressed by the formula

$$U = \frac{\lambda}{8\pi^2} V\mu^8,$$

where

$$\lambda \equiv \sum_{n=1}^{\infty} (2n-1) [(\sqrt{n}+\sqrt{n-1})^{-8} - (\sqrt{n}+\sqrt{n+1})^{-8}] = 1.002...$$

The value of λ has been found numerically. The expectation value of Q' at $T = 0$ is

$$\langle Q' \rangle = \frac{2\eta}{7\pi^2} V\mu^7,$$

where

$$\eta \equiv \sum_{n=1}^{\infty} \sqrt{n} [(\sqrt{n}+\sqrt{n-1})^{-7} - (\sqrt{n}+\sqrt{n+1})^{-7}] = 1.001...$$

In Fig. 2 we show the photon spectrum found numerically described by formula (4) for some values of μ. Because the particles are massless, there is no natural energy scale and arbitrary units of energy are used.

An analog of the Stefan-Boltzmann law can be found by integration of the formula (4) with respect to frequencies. In Fig. 3 we show U as a function of T. It is seen that at fixed T, U increases with $\langle Q' \rangle$, however, this increase becomes weaker when the temperature rises.

We conclude our considerations as follows.

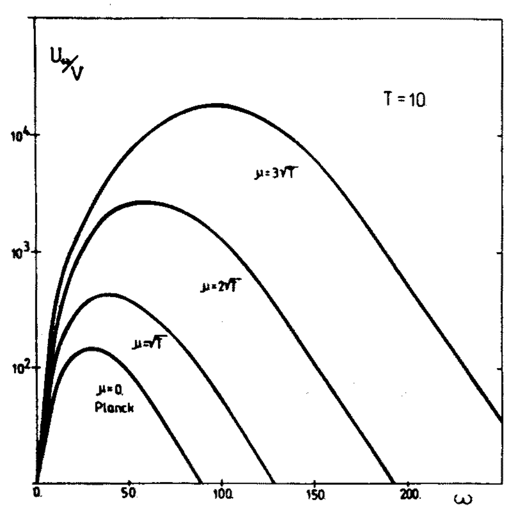

Fig. 2. The energetic photon spectrum for $T = 10$ at four values of chemical potential

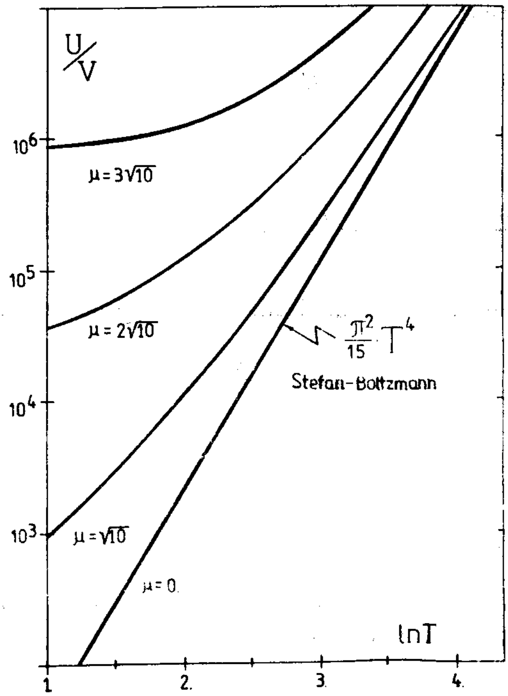

Fig. 3. The total photon energy versus temperature for four values of chemical potential

Even at zero temperature the photons radiated from the supercharged system carry finite energy which increases with the value of supercharge. The energy distribution of photons is not smooth, but it exhibits many sharp peaks.

At finite temperature the form of the photon spectrum is similar to that described by the Planck formula, although a maximum is shifted to higher frequencies. The total energy of photons increases with the value of supercharge, while the increase with temperature is weaker than T^4.

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