ON THE FERMIONIC PATH INTEGRAL

By S. Giler, P. Kosiński, J. Rembieliński

Institute of Physics, University of Łódź*

and P. Maślanka

Institute of Mathematics, University of Łódź**

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The definition of the fermionic path integral is given which allows one to introduce the path integral representation of Feynman propagation function for theories with the fermionic degrees of freedom.

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1. *Introduction*

One of the most important problems in supersymmetry is to find the workable mechanism for breaking it. The supersymmetry differs significantly from ordinary symmetries. For example, it is difficult to break it spontaneously at the tree level and if it is unbroken at this level it remains unbroken to all orders of perturbation theory as a result of the peculiar boson-fermion cancellations. Therefore one has to search for some nonperturbative effects which break supersymmetry. However, it is usually very difficult to investigate such effects. To study them some simple models were invented. In particular Witten [1] introduced the supersymmetric version of ordinary quantum mechanics (SSQM). SSQM is invariant under some kind of supersymmetry transformations and consequently it must contain "fermions". To represent the fermionic degrees of freedom within the "coordinate representation" one introduces the Grassman variable $\zeta(\zeta^2 = 0)$ as an additional argument of wave function [2]

$$\Phi \equiv \Phi(x, \zeta; t).$$

Expanding with respect to ζ one obtains

$$\Phi(x, \zeta; t) = \Phi_1(x, t) + \zeta\Phi_2(x, t).$$

* Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149, 90-236 Łódź, Poland.

** Address: Instytut Matematyki, Uniwersytet Łódzki, Banacha 22, 90-238 Łódź, Poland.

The theory can be developed along the same lines as ordinary QM. One introduces the Feynman propagation function

$$K(x, \zeta; t \,|\, x', \zeta'; t') = K_0 + K_1\zeta + K_2\zeta' + K_3\zeta\zeta'.$$

The wave function evolves according to the equation

$$\Phi(x, \zeta; t) = \int K(x, \zeta; t \,|\, x', \zeta'; t')\Phi(x', \zeta', t')dx'd\zeta'$$

where the integration rule $\int d\zeta \cdot \zeta = 1$ was adopted. The fermion-number conservation law implies $K_0 = K_3 = 0$.

One would like to represent the Feynman kernel in terms of functional integral. To obtain such representation we make the Wick rotation to Euclidean time and write

$$K(x, \zeta; t \,|\, x', \zeta'; t') = \int \mathscr{D}x(\tau)\mathscr{D}\zeta(\tau)\mathscr{D}\eta(\tau) \exp\{-\int_{t'}^{t} d\tau L\}. \tag{1}$$

We restrict ourselves to the (Euclidean) lagrangians of the form

$$L = L_B(x, \dot{x}) + \eta\left(\frac{d}{d\tau} + W(x)\right)\zeta.$$

Here $L_B(x, \dot{x})$ is the bosonic part of the lagrangian, η and ζ are Grassman variables and η is canonically conjugate to ζ. In other words, Eq. (1) represents the "mixed" functional integral: we had started from the functional integral over paths in phase space and then integrated over bosonic momentum variables.

In connection with the Eq. (1) the question arises how to define the integral

$$\int \mathscr{D}\zeta\mathscr{D}\eta \exp \eta A\zeta$$

where A is the first-order differential operator

$$A \equiv -\partial_\tau + F(\tau).$$

Sometimes it is quite simple. If one uses the definition of measure in Grassman path integrals given by Berezin [3] one obtains

$$\int \mathscr{D}\zeta\mathscr{D}\eta \exp \eta A\zeta = \frac{\mathrm{Det}\,[-\partial_\tau + F(\tau)]}{\mathrm{Det}\,[-\partial_\tau]}. \tag{2}$$

To define the operator A one has to specify the boundary condition. If, for example, we are going to calculate the partition function, we impose the antiperiodic boundary conditions. Then the eigenvalues of A can be calculated in a simple way and one obtains closed expression for the integral under consideration [4]. Similarly with the periodic boundary conditions we would get the expression for the regularized Witten index [5]. However, in the case under consideration the situation is more complicated. In analogy with the usual case we should integrate over the paths $\zeta(t)$ with the boundary conditions $\zeta(t') = \zeta'$,

$\zeta(t) = \zeta$ and the paths $\eta(t)$ with free boundary conditions (because $\eta(t)$ is the momentum canonically conjugated to $\zeta(t)$). But the former conditions are incompatible with the form of A: we cannot impose such boundary conditions on the first order differential operator.

The most straightforward approach to this problem is that based on time discretization procedure [2]. On the other hand Leinaas and Olaussen [6] attempted to solve it in the following way. They defined the integral for the simplest case $A = -\partial_\tau$ by the suitable normal-mode decomposition and then changed the integration variables to obtain the answer in general case. The Jacobian of this change was read off from the composition law for Feynman kernel. However, we feel that simple and concise treatment of this problem is still lacking. Such formulation seems to be helpful in the discussion of instanton effects in SSQM. There are some subtle problems concerning the fermionic zero modes and their contribution to various matrix elements (cf. [2, 7–9]). To understand those problems fully we have to have the clear understanding of the notions used.

We do not pretend to mathematical rigour. We think, however, that our approach is internally consistent and simple enough to be useful.

2. *The fermionic path integral*

We are going to define the integral

$$\int D\zeta(\tau)D\eta(\tau) \exp \int_{t'}^{t} d\tau\eta(\tau)\left[-\frac{\partial}{\partial\tau} + F(\tau)\right]\zeta(\tau), \tag{3}$$

$$\zeta(t) = \zeta,\ \eta(t) \text{— free b.c.},\ \zeta(t') = \zeta',\ F(\tau) \text{— real.}$$

Let $A = -\dfrac{\partial}{\partial\tau} + F(\tau)$ and let $A^\dagger = \dfrac{\partial}{\partial\tau} + F(\tau)$ be the formal adjoint of A. Then $A^\dagger \cdot A$ is the second order differential operator. Together with the boundary conditions

$$\varphi(t') = 0, \quad \varphi(t) = 0 \tag{4}$$

it defines the selfadjoint operator acting on the Hilbert space of square integrable functions on the interval (t', t). Denote by $\{\varphi_n\}_1^\infty$ and $\{\lambda_n\}_1^\infty$ the sets of eigenfunctions and eigenvalues of $A^\dagger \cdot A$, respectively. The operator $A^\dagger \cdot A$ is positive definite at least as long as $t' > -\infty$ or $t < \infty$. Indeed, if $\lambda_1 = 0$ then

$$A^\dagger \cdot A\varphi_1 = 0$$

and consequently

$$A\varphi_1 = 0.$$

The above equation together with the boundary condition implies

$$\varphi_1 = 0.$$

Remark now that $\{\lambda_n^{-\frac{1}{2}} A\varphi_n\}$ is also an orthonormal set $(\lambda_n^{-\frac{1}{2}} A\varphi_n, \lambda_k^{-\frac{1}{2}} A\varphi_k) = (\lambda_n \lambda_k)^{-\frac{1}{2}}(\varphi_n, A^\dagger \cdot A\varphi_k) = \delta_{nk}$. Moreover, if χ is such a function that $(\chi, A\varphi_n) = 0$ for all n, then

$$A^\dagger \chi = 0.$$

But the space of the solutions of this equation is onedimensional. This in turn implies that if we pick up the solution $\chi \equiv \varphi_0$ such that $(\varphi_0, \varphi_0) = 1$ then the set $\{\varphi_0, \{\lambda_n^{-\frac{1}{2}} A\varphi_n\}_1^\infty\}$ will be complete and orthonormal $((A\varphi_n, \varphi_0) = (\varphi_n, A^\dagger \varphi_0) = 0)$. As a result we can expand[1]

$$\eta(\tau) = \eta_0 \varphi_0(\tau) + \sum_{n=1}^{\infty} \lambda_n^{-1/2} A\varphi_n(\tau) \cdot \eta_n \tag{5}$$

with Grassman variables η_n. Now let $\zeta_c(\tau)$ be the unique solution of the boundary value problem

$$A^\dagger \cdot A\zeta_c = 0, \quad \zeta_c(t') = \zeta', \quad \zeta_c(t) = \zeta. \tag{6}$$

Remark that this problem is well defined in spite of the fact that ζ's are Grassman variables. One can simply solve the usual differential equation with the boundary conditions $a(t') = 1$, $a(t) = 0$, or $b(t') = 0$, $b(t) = 1$ and put $\zeta_c(\tau) = \zeta' \cdot a(\tau) + \zeta \cdot b(\tau)$. We expand the path $\zeta(\tau)$ according to the formula

$$\zeta(\tau) = \zeta_c(\tau) + \sum_{k=1}^{\infty} \varphi_k(\tau)\zeta_k \tag{7}$$

Using Eqs. (5) and (7) we get

$$\eta A\zeta \equiv (\eta, A\zeta) = \eta_0(\varphi_0, A\zeta_c) + \sum_{k=1}^{\infty} \lambda_k^{1/2} \eta_k \zeta_k. \tag{8}$$

Our definition of the integral is given by the equation

$$\int_{\text{b.c.}} D\zeta D\eta \exp \eta A\zeta = N^{-1}(t', t) \int \prod_{k=1}^{\infty} d\zeta_k d\eta_k d\eta_0 \exp \left[\eta_0(\varphi_0, A\zeta_c) + \sum_{k=1}^{\infty} \lambda_k^{1/2} \eta_k \zeta_k\right] \tag{9}$$

or, after integration over Grassman variables

$$\int_{\text{b.c.}} D\zeta D\eta \exp \eta A\zeta = N^{-1}(t', t)\, (\varphi_0, A\zeta_c)\, (\text{Det } A^\dagger \cdot A)^{1/2}. \tag{10}$$

Here N is the normalization factor which serves to cancel the infinities contained in Det $(A^\dagger \cdot A)$ (A is unbounded). N should not depend on the choice of $F(\tau)$. We show below that this is the case.

To calculate $(\varphi_0, A\zeta_c)$ we find first φ_0. Recall that it fulfils the equation

$$\left(\frac{\partial}{\partial \tau} + F(\tau)\right) \varphi_0(\tau) = 0$$

[1] Note that in general the functions φ_0 and $A\varphi_n$ do not satisfy the boundary conditions (4).

together with the condition

$$(\varphi_0, \varphi_0) = 1.$$

The solution reads

$$\varphi_0(\tau) = C \exp\left(-\int_{t'}^{\tau} d\tau' F(\tau')\right),$$

$$C^{-2} = \int_{t'}^{t} d\tau \exp\left(-2\int_{t'}^{\tau} d\tau' F(\tau')\right). \tag{11}$$

Using the Eq. (11) we calculate $(\varphi_0, A\zeta_c)$

$$(\varphi_0, A\zeta_c) = \int_{t'}^{t} d\tau \varphi_0(\tau)\left[-\frac{\partial}{\partial\tau} + F(\tau)\right]\zeta_c(\tau)$$

$$= -C\{\zeta \exp\left(-\int_{t'}^{t} d\tau F(\tau)\right) - \zeta'\}. \tag{12}$$

Inserting the above expression into Eq. (10) we obtain finally

$$\int_{\text{b.c.}} D\zeta D\eta \exp \eta A\zeta = -N^{-1}(t', t)(\text{Det } A^{\dagger} \cdot A)^{1/2} C \exp\left(-\tfrac{1}{2}\int_{t'}^{t} d\tau F(\tau)\right)$$

$$\times \{\zeta \exp\left(-\tfrac{1}{2}\int_{t'}^{t} d\tau F(\tau)\right) - \zeta' \exp\left(\tfrac{1}{2}\int_{t'}^{t} d\tau F(\tau)\right)\}. \tag{13}$$

Let us now consider the expression

$$(\text{Det } A^{\dagger} \cdot A)^{1/2} C \exp\left(-\tfrac{1}{2}\int_{t'}^{t} d\tau F(\tau)\right) = (\text{Det } A^{\dagger} \cdot A)^{1/2}$$

$$\times \{\int_{t'}^{t} d\tau \exp\left(-2\int_{t'}^{\tau} d\tau' F(\tau')\right)\}^{-1/2} \exp\left(-\tfrac{1}{2}\int_{t'}^{t} d\tau F(\tau)\right). \tag{14}$$

Recall the following theorem [10]. Let $W = -\dfrac{\partial^2}{\partial\tau^2} + f(\tau)$ be the operator defined by the boundary condition $\varphi(t') = 0$, $\varphi(t) = 0$ and let $\psi(\tau)$ be the solution of the following initial-value problem

$$W\psi = 0, \quad \psi(t') = 0, \quad \psi'(t') = 1.$$

Then the ratio Det $W/\psi(t)$ does not depend on f. The solution to the initial-value problem

$$A^{\dagger} \cdot A\psi \equiv \left(\frac{\partial}{\partial\tau} + F(\tau)\right)\left(-\frac{\partial}{\partial\tau} + F(\tau)\right)\psi(\tau) = 0,$$

$$\psi(t') = 0, \quad \psi'(t') = 1$$

reads

$$\psi(t) = \exp \int_{t'}^{t} d\tau F(\tau) \int_{t'}^{t} d\tau \exp\left(-2 \int_{t'}^{\tau} d\tau' F(\tau')\right). \tag{15}$$

By comparing Eqs. (14) and (15) we conclude that the expression

$$-(\mathrm{Det}\, A^{\dagger} \cdot A)^{1/2} C \exp\left(-\tfrac{1}{2} \int_{t'}^{t} d\tau F(\tau)\right) \equiv N(t', t) \tag{16}$$

does not depend on $F(\tau)$. Eq. (16) serves as the definition of the normalization factor. In this way we obtain our final result

$$\int_{\mathrm{b.c.}} D\zeta D\eta \exp \eta A\zeta = \zeta \exp\left(-\tfrac{1}{2} \int_{t'}^{t} d\tau F(\tau)\right) - \zeta' \exp\left(\tfrac{1}{2} \int_{t'}^{t} d\tau F(\tau)\right). \tag{17}$$

Note that the inclusion of the normalisation factor here is completely analogous to the procedure used for bosonic Feynman integral. To make such integral convergent it is sufficient to include the normalisation factor $\left(\sqrt{\frac{m}{2\pi i\hbar}}\right)^{N=\infty}$ corresponding to the case of free particle.

The formal expression for N is easily obtained by putting $F = 0$

$$N(t', t) = \frac{1}{t'-t} \prod_{n=1}^{\infty} \frac{n\pi}{t-t'}. \tag{18}$$

3. *Conclusions*

We conclude with some remarks. It follows from Eq. (17) that by taking the trace we obtain the usual partition function. Indeed, taking the trace means for fermions putting $\zeta' = -\zeta$ and integrating over ζ. To obtain the regularised Witten index we have to put $\zeta' = \zeta$ instead.

Taking into account that the expression for the normalisation factor contains the determinant Det $(-\partial_\tau^2)$ we see the coincidence between the formula for the trace following from our approach and the one given by Eq. (2).

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