RELATIVISTIC TWO-BODY EQUATION FOR ONE DIRAC AND ONE DUFFIN-KEMMER-PETIAU PARTICLE, CONSISTENT WITH THE HOLE THEORY

By W. Królikowski and A. Turski

Institute of Theoretical Physics, Warsaw University*

(*Received May 24, 1985*)

An equal-time two-body wave equation consistent with the hole theory is found for one Dirac particle and one Duffin-Kemmer-Petiau particle (of spin 0 or 1) interacting through static potentials. A natural field of application of this work is the quark + diquark model of the nucleon.

PACS numbers: 11.10.Qr

1. *Introduction*

A couple of years ago an equal-time relativistic two-body wave equation was introduced for one spin-1/2 particle and one spin-0 or spin-1 particle which, if isolated from each other, could be described by the Dirac equation and the Duffin-Kemmer-Petiau equation, respectively [1]. In the centre-of-mass frame (where $\vec{p}_D = -\vec{p}_{DKP} \equiv \vec{p}$ and $\vec{r}_D - \vec{r}_{DKP} \equiv \vec{r}$) such an equation has the form

$$\{\beta^0[E-V-\vec{\alpha}\cdot\vec{p}-\beta(m+\tfrac{1}{2}S)]+\vec{\beta}\cdot\vec{p}-(M+\tfrac{1}{2}S)\}\Psi(\vec{r}) = 0, \tag{1}$$

if the interaction is given by a vector potential $V(\vec{r})$ and a scalar potential $S(\vec{r})$. Here, $(\beta^\mu) = (\beta^0, \vec{\beta})$ are Duffin-Kemmer-Petiau matrices [2] which can be formally represented as

$$\beta^\mu = \tfrac{1}{2}(\gamma_1^\mu + \gamma_2^\mu) \tag{2}$$

with $(\gamma_i^\mu) = (\beta_i, \beta_i\vec{\alpha}_i)$ $(i = 1, 2)$ denoting two mutually commuting sets of Dirac matrices. The formula (2) enables us to interpret the Duffin-Kemmer-Petiau particle as a formal limit of a tight system of two Dirac particles carrying equal momenta $\vec{p}_1 = \vec{p}_2 \equiv -\frac{1}{2}\vec{p}$ and equal effective masses $m_1 = m_2 \equiv \frac{1}{2}M$. We ascribe to an interacting Duffin-Kemmer-Petiau particle spin 0 or 1, if its "large-large" wave-function components correspond to such a spin.

* Address: Instytut Fizyki Teoretycznej, Uniwersytet Warszawski, Hoża 69, 00-681 Warszawa, Poland.

Recently, one of us (A.T.) derived from Eq. (1) with static potentials $V(r)$ and $S(r)$ (thus depending only on $r = |\vec{r}|$) the equivalent set of radial equations, and then discovered that for *any* potential $V(r)$ singular at $r = 0$ like r^{-b} with $b > 0$ (so even *less* singular than r^{-1}) there are too few solutions regular at $r = 0$ to describe the system of a Dirac particle and a Duffin-Kemmer-Petiau particle, *if* the latter has spin 1 [3]. In fact, instead of some expected regular solutions there appear in this case singular solutions displaying for $r \to 0$ rapid oscillations with infinitely rising phases. This phenomenon occurring in our system can be considered as a counterpart of the well known instability effect (the "continuum dissolution") observed previously in the system of three (or more) Dirac particles if the hole theory is not taken into account [4] (note that in both cases there are formally at least three spins 1/2). Thus, we meet here a *drastic* form of the Klein paradox which, as always, is a signal coming from the negative-energy virtual states as they appear in the one-particle version of Dirac theory, inconsistent with the hole theory. Recall that in the case of one or two Dirac particles with static interactions the Klein paradox at $r = 0$ appears *only* for too strong vector potentials, if the hole theory is not taken into account [5].

2. *New wave equation*

In the present paper we find a modified form of Eq. (1), consistent with the hole theory. To this end we replace in Eq. (1) the static vector potential $V(r)$ by its projection à la Salpeter [6]. In the case of realistic interactions of Dirac particles such a projection follows from the quantum field theory when the instantaneous static approximation is considered [6, 7]. In our case, when treating the Duffin-Kemmer-Petiau particle as a limit of a tight system of two Dirac particles (cf. Eq. (2)), we would substitute in Eq. (1):

$$\begin{aligned}
\beta^0 V &= \tfrac{1}{2}\beta_1 V + \tfrac{1}{2}\beta_2 V \\
&\to \tfrac{1}{2}\beta_1\left[\Lambda^{(+)}(\vec{p})\Lambda_1^{(+)}\left(-\frac{\vec{p}}{2}\right) - \Lambda^{(-)}(\vec{p})\Lambda_1^{(-)}\left(-\frac{\vec{p}}{2}\right)\right]V \\
&+\tfrac{1}{2}\beta_2\left[\Lambda^{(+)}(\vec{p})\Lambda_2^{(+)}\left(-\frac{\vec{p}}{2}\right) - \Lambda^{(-)}(\vec{p})\Lambda_2^{(-)}\left(-\frac{\vec{p}}{2}\right)\right]V \\
&= \left(\beta^0\,\frac{\vec{\alpha}\cdot\vec{p}+\beta m}{2\sqrt{\vec{p}^2+m^2}} - \frac{\vec{\beta}\cdot\vec{p}-M}{2\sqrt{\vec{p}^2+m^2}}\right)V,
\end{aligned} \tag{3}$$

where

$$\Lambda^{(\pm)}(\vec{p}) = \frac{\sqrt{p^2+m^2} \pm (\vec{\alpha}\cdot\vec{p}+\beta m)}{2\sqrt{\vec{p}^2+m^2}},$$

$$\Lambda_i^{(\pm)}\left(-\frac{\vec{p}}{2}\right) = \frac{\sqrt{\vec{p}^2+M^2} \pm (-\vec{\alpha}_i\cdot\vec{p}+\beta_i M)}{2\sqrt{\vec{p}^2+M^2}} \quad (i = 1, 2). \tag{4}$$

Thus, the modified form of wave equation (1), consistent with the hole theory, should be

$$\left\{\beta^0\left[E-(\vec{\alpha}\cdot\vec{p}+\beta m)\left(1+\frac{1}{2\sqrt{\vec{p}^2+m^2}}V\right)-\beta\tfrac{1}{2}S\right]\right.$$

$$\left.+(\vec{\beta}\cdot\vec{p}-M)\left(1+\frac{1}{2\sqrt{\vec{p}^2+M^2}}V\right)-\tfrac{1}{2}S\right\}\Psi(\vec{r})=0. \tag{5}$$

Here, we can write [8]

$$\frac{1}{2\sqrt{\vec{p}^2+m^2}}V(r)\Psi(\vec{r})=\frac{m}{(2\pi)^2}\int d^3\vec{r}'\,\frac{K_1(m|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}V(\vec{r}')\Psi(\vec{r}') \tag{6}$$

with $K_1(x) = -(\pi/2)H_1^{(1)}(ix)$. Note that $K_1(x) \simeq 1/x$ at $x \to 0$ and $K_1(x) \simeq \sqrt{\pi/2x}\exp(-x)$ at $x \to \infty$. So, the interaction in Eq. (5) related to the static vector potential $V(r)$ is nonlocal with the nonlocality extended over the Compton wave lengths $1/m$ and $1/M$ of two particles.

In the nonrelativistic limit, both our Eqs. (1) and (5) take the Schrödinger form

$$\left(\varepsilon-\frac{1}{2\mu}\vec{p}^2-V-S\right)\Psi(\vec{r})=0, \tag{7}$$

where $\varepsilon = E-m-M$ and $\mu = mM(m+M)^{-1}$.

3. *Absence of Klein paradox at $r = 0$*

Since Eq. (5), in contrast to Eq. (1), is consistent with the hole theory, one can expect that it is free of the Klein paradox at $r = 0$ which is troubling so much Eq. (1).

In order to demonstrate this statement we insert into Eq. (6) with $\Psi(\vec{r}) = \Psi_{lm_l}(\vec{r})$ the expansion

$$K_1(x)=\frac{1}{x}+\sum_{k=0}^{\infty}\frac{x^{2k+1}}{2^{2k+1}k!(k+1)!}\left[\ln\frac{x}{2}-\frac{\psi(k+1)+\psi(k+2)}{2}\right], \tag{8}$$

where

$$\psi(k+1)=\begin{cases}-C & \text{for}\quad k=0\\ -C+\sum_{n=1}^{k}\frac{1}{n} & \text{for}\quad k>0\end{cases} \tag{9}$$

with $C = 0.5772\ldots$. Then the term $1/x$ in Eq. (8) gives

$$\frac{1}{2\sqrt{\vec{p}^2}}V(r)\Psi_{lm_l}(\vec{r})=\frac{1}{(2\pi)^2}\int d^3\vec{r}'\,\frac{1}{(\vec{r}-\vec{r}')^2}V(r')\Psi_{lm_l}(\vec{r}')$$

$$=\frac{1}{2\pi}Y_{lm_l}(\vec{r})r\int_0^{\infty}\xi d\xi Q_l\left(\frac{1+\xi^2}{2\xi}\right)V(r\xi)\Psi_l(r\xi), \tag{10}$$

where $\Psi_{lm_l}(\vec{r}) = \Psi_l(r) Y_{lm_l}(\hat{r})$, $\quad \xi = r'/r$ and

$$Q_l(x) = \tfrac{1}{2} \int_{-1}^{1} d\cos\theta \, \frac{P_l(\cos\theta)}{x - \cos\theta}. \tag{11}$$

In the proof of Eq. (10) it is convenient to put $\hat{r} = (0, 0, 1)$ and $\hat{r} \cdot \hat{r}' = \cos\theta$. In particular, $Q_0\left(\frac{1+\xi^2}{2\xi}\right) = \ln\left|\frac{1+\xi}{1-\xi}\right|$. In our forthcoming argument we shall assume that $V(r) \sim r^{-b}$ at $r \to 0$ with $0 < b \leqslant 1$ and then use the ansatz $\Psi_l(r) \sim r^{l-a_l}$ at $r \to 0$ with $-b < a_l < 1$ (such a behaviour of $\Psi_l(r)$ will be verified a posteriori). In this case it is not difficult to show that the rest of terms in Eq. (8), when inserted into Eq. (6) with $\Psi(\vec{r}) = \Psi_{lm_l}(\vec{r})$, gives at $r \to 0$ the leading contribution $A_l r^l Y_{lm_l}(\hat{r})$, where A_l is a constant depending on m. So, we obtain

$$\frac{1}{2\sqrt{\vec{p}^2 + m^2}} V(r) \Psi_{lm_l}(\vec{r}) \underset{r \to 0}{\simeq} \frac{1}{2\sqrt{\vec{p}^2}} V(r) \Psi_{lm_l}(\vec{r}) + A_l r^l Y_{lm_l}(\hat{r}) + O(r^{l+1}) Y_{lm_l}(\hat{r}). \tag{12}$$

Further, in an analogical way as in Eq. (10) we get

$$2\sqrt{\vec{p}^2}\, \Psi_{lm_l}(\vec{r}) = \frac{2}{\sqrt{\vec{p}^2}} \vec{p}^2 \Psi_{lm_l}(\vec{r}) = \frac{1}{\pi^2} \int d^3\vec{r}' \, \frac{1}{(\vec{r} - \vec{r}')^2} \vec{p}'^2 \Psi_{lm_l}(\vec{r}')$$
$$= \frac{2}{\pi} Y_{lm_l}(\hat{r}) r \int_0^\infty \xi d\xi Q_l\left(\frac{1+\xi^2}{2\xi}\right) [p_l'^2 \Psi_l(r')]_{r' = r\xi}, \tag{13}$$

and then from Eqs. (12) and (13) we calculate

$$\sqrt{\vec{p}^2} \frac{1}{\sqrt{\vec{p}^2 + m^2}} V(r) \Psi_{lm_l}(\vec{r}) \underset{r \to 0}{\simeq} V(r) \Psi_{lm_l}(\vec{r}) + 2A_l \sqrt{\vec{p}^2}\, r^l Y_{lm_l}(\hat{r})$$
$$+ \sqrt{\vec{p}^2}\, O(r^{l+1}) Y_{lm_l}(\hat{r}) = V(r) \Psi_{lm_l}(\vec{r}) + O(r^l) Y_{lm_l}(\hat{r}), \tag{14}$$

where $\vec{p}^2 Y_{lm_l}(\hat{r}) = Y_{lm_l}(\hat{r}) p_l^2$ and

$$p_l^2 = -\frac{1}{r} \frac{d^2}{dr^2} r + \frac{l(l+1)}{r^2}. \tag{15}$$

Since

$$Q_l\left(\frac{1+\xi^2}{2\xi}\right) \simeq \frac{2^{l+1} l!}{(2l+1)!!} \begin{cases} \xi^{l+1} & \text{for} \quad \xi \to 0 \\ \xi^{-l-1} & \text{for} \quad \xi \to \infty \end{cases}, \tag{16}$$

we can introduce the limit $r \to 0$ under the integral in Eq. (13) *if* $\Psi_l(r) \sim r^{l-a_l}$ at $r \to 0$ with $-1 < a_l < 2l+1$ (the physical condition $r\Psi_l(r) \to 0$ at $r \to 0$ imposes on $\Psi_l(r)$ the more severe restriction $a_l < l+1$, while in our argument we operate with the even more restrictive ansatz $-b < a_l < 1$ with $0 < b \leqslant 1$, so that certainly $-1 < a_l < 2l+1$). Then

$$2\sqrt{\vec{p}^2}\,\Psi_{lm_l} \underset{r\to 0}{\simeq} D_l r\vec{p}^2\Psi_{lm_l}(\vec{r}), \tag{17}$$

where

$$D_l = \frac{2}{\pi}\int_0^\infty d\xi Q_l\left(\frac{1+\xi^2}{2\xi}\right)\xi^{l-1-a_l} \tag{18}$$

is a constant. In particular, we calculate $D_0 = 2\tan\frac{\pi a_0}{2}\Big/a_0$.

Now, we multiply our wave equation (5) from the left by the operator $2\sqrt{\vec{p}^2}$ and consider the limit $r \to 0$, assuming that $S(r) \to 0$ at $r \to 0$. Then, making use of Eq. (14) we get the following asymptotic form of this equation at $r \to 0$:

$$\left[\beta^0\left(E - V\frac{E}{2\sqrt{\vec{p}^2}+V} - \vec{\alpha}\cdot\vec{p} - \beta m\right) + \vec{\beta}\cdot\vec{p} - M\right]\chi(\vec{r}) = 0, \tag{19}$$

where

$$\chi(\vec{r}) = (2\sqrt{\vec{p}^2}+V)\Psi(\vec{r}). \tag{20}$$

Here, we took into account singular potentials $V(r) \sim r^{-b}$ at $r \to 0$ with $0 < b \leqslant 1$, omitting the mass-dependent terms $O(r^l)Y_{lm_l}(\hat{r})$ in Eq. (14) where $\Psi_l(r) \sim r^{l-a_l}$ at $r \to 0$ with $-b < a_l < 1$. The effective potential $VE(2\sqrt{\vec{p}^2}+V)^{-1}$ in Eq. (19) is nonsingular at $r \to 0$ giving effectively for $r \to 0$ a constant E_0 which is zero if $0 < b < 1$ and nonzero if $b = 1$. In both cases Eq. (19) becomes for $r \to 0$ a free wave equation of the type (1) and so we can deduce that at $r \to 0$

$$\vec{p}^2\chi(\vec{r}) = 0, \tag{21}$$

implying $\chi_l(r) \sim r^l$ at $r \to 0$. Thus, omitting consequently in Eq. (20) the terms $\chi_{lm_l}(\vec{r}) = O(r^l)Y_{lm_l}(\hat{r})$ we obtain the following asymptotic equation at $r \to 0$:

$$(2\sqrt{\vec{p}^2}+V)\Psi(\vec{r}) = 0. \tag{22}$$

For $0 < b < 1$, the ansatz $\Psi_l(r) \sim r^{l-a_l}$ applied to Eq. (22) gives $\sqrt{\vec{p}^2}\Psi(\vec{r}) = 0$ at $r \to 0$ so that $\Psi_l \sim r^l$ at $r \to 0$ and hence $a_l = 0$, in consistency with our hypothesis $-b < a_l < 1$. For $b = 1$, i.e. for asymptotically Coulombic potentials $V(r) \simeq \mp\alpha/r$ at $r \to 0$, we can also solve Eq. (22) by the ansatz $\Psi_l(r) \sim r^{l-a_l}$. In fact, with this ansatz Eqs. (17) and (22) give

$$D_l[l(l+1)-(l+1-a_l)(l-a_l)] \mp \alpha = 0 \tag{23}$$

and hence for $l = 0$

$$a_0 \pm \frac{\alpha}{2}\cot\frac{\pi a_0}{2} = 1 \quad \text{or} \quad a_0 = \pm\frac{\alpha}{\pi} + O(\alpha^2), \tag{24}$$

where we used the explicit form of D_0. Also this result is consistent with our hypothesis $-b < a_l < 1$. As can be shown from Eqs. (18) and (23), analogical solutions exist also for $l > 0$, satisfying the hypothesis $-b < a_l < 1$ (cf. Eq. (26)). Thus, our original ansatz is fully verified. So, we can conclude that Eq. (5) is free of the Klein paradox at $r = 0$ for singular potentials $V(r) \sim r^{-b}$ at $r \to 0$ with $0 < b \leqslant 1$. Of course, in the case of $b = 1$ the strength of $V(r)$ at $r \to 0$ must not be too great in order to keep a_0 real and lying in the range $-1 < a_0 < 1$ which is required for the integral D_0 to be convergent.

Note that Eq. (22) is formally *identical* with the asymptotic form at $r \to 0$ of the Schrödinger relativistic two-body wave equation [9–12],

$$(E - \sqrt{\vec{p}^{\,2} + m^2} - \sqrt{\vec{p}^{\,2} + M^2} - V)\Psi(\vec{r}) = 0, \tag{25}$$

describing two spinless particles whose spectral content is restricted to positive-energy virtual states. In particular, for $V(r) \simeq -\alpha/r$ at $r \to 0$ the asymptotic behaviour of the partial waves $\Psi_l(r)$ corresponding to Eq. (25) was found in Refs. [10] and [11]:

$$\Psi_l(r) \underset{r\to 0}{\simeq} r^{l-a_l}, \qquad a_l = \frac{\alpha}{\pi}\,\frac{(2l)!!}{(2l+1)!!} + O(\alpha^2). \tag{26}$$

This form is consistent with our result (24).

4. Outlook

A natural and actually interesting field of application for the relativistic wave equations described in this note is the quark + diquark model of the nucleon (and its excited states) where, previously, the relativistic two-body wave equations involving two Klein-Gordon particles were used [13] as an approximation. Note that in the case of a quark + diquark system the effective two-body potentials $V(\vec{r})$ and $S(\vec{r})$ are not singular at $r = 0$, if the finite spatial extension of the diquark is taken into account. So, in this case, Eq. (1) does not suffer from the Klein paradox at $r = 0$ and, therefore, it is (a priori) an open question which equation is more adequate: Eq. (1) with basic interactions smeared out by the finite diquark extension or, rather, Eq. (5) with Salpeter nonlocal interactions (additionally smeared out by this finite diquark extension). The latter equation is evidently consistent with the hole theory, but as far as the instantaneous nonstatic effects are concerned it is approximate in a different way than the former equation. For instance, if formally $m/M \to 0$, Eq. (5) in contrast to Eq. (1) does not approach the Dirac equation [14]. On the other hand, a quark + diquark wave equation might be expected to do it, since for $m \ll M$ the quark + diquark system could be compared (mutatis mutandis) with an electron moving in the Coulomb field of a nucleus of a finite size [15].

Finally, one may remark that the equal-time relativistic two-body wave equation for two Klein-Gordon particles of masses m and M [13, 16],

$$\left\{\left(\frac{E-V}{2}\right)^2-\vec{p}^2-\frac{(m+\frac{1}{2}S)^2+(M+\frac{1}{2}S)^2}{2}+\left[\frac{(m+\frac{1}{2}S)^2-(M+\frac{1}{2}S)^2}{2(E-V)}\right]^2\right\}\Psi(\vec{r})=0, \tag{27}$$

can be considered as a spinless approximation both to the Breit equation for two Dirac particles as well as to our Eq. (1).

Unlike Eq. (27), the Breit equation and our Eq. (1) are many-component and so more difficult to solve (none the less, two sets of eight radial equations [17], equivalent to the Breit equation, have been solved numerically in the case of quark + antiquark system with "realistic" QCD-suggested potentials, cf. Ref. [3] and references therein). This difficulty in handling increases considerably in the case of the Salpeter equation for two Dirac particles [6, 7],

$$\left[E-(\vec{\alpha}_1\cdot\vec{p}+\beta_1 m_1)\left(1+\frac{1}{\sqrt{\vec{p}^2+m_1^2}}V\right)-\beta_1\frac{S}{2}\right.$$
$$\left.+(\vec{\alpha}_2\cdot\vec{p}-\beta_2 m_2)\left(1+\frac{1}{\sqrt{\vec{p}^2+m_2^2}}V\right)-\beta_2\frac{S}{2}\right]\Psi(\vec{r})=0, \tag{28}$$

and of our Eq. (5) because of their nonlocality [8]. We hope to turn back in future to the task of solving Eqs. (1) and/or (5) in the case of quark + diquark system.

A part of this work was done while one of us (W.K.) stayed in the III Physikalisches Institut, RWTH, Aachen.

REFERENCES

[1] W. Królikowski, *Acta Phys. Pol.* **B14**, 97 (1983).
[2] Cf. e.g. H. Umezawa, *Quantum field theory*, North-Holland, Amsterdam 1956, p. 85.
[3] A. Turski, Warsaw University Ph. D. thesis, 1985 (to be published in an abridged form).
[4] Cf. e.g. G. Hardekopf, J. Sucher, *Phys. Rev.* **A30**, 703 (1984) and references therein.
[5] For an early discussion of the Klein paradox at $r \to 0$ in the Breit equation for two Dirac particles cf. W. Królikowski, A. Turski, J. Rzewuski, *Z. Phys.* **C1**, 369 (1979).
[6] E. E. Salpeter, *Phys. Rev.* **87**, 328 (1952).
[7] G. Feldman, T. Fulton, J. Townsend, *Phys. Rev.* **A8**, 1149 (1973).
[8] W. Królikowski, *Acta Phys. Austr.* **51**, 127 (1979).
[9] P. Cea, P. Colangelo, G. Nardulli, G. Paiano, G. Preparata, *Phys. Rev.* **D26**, 1157 (1982).
[10] B. Durand, L. Durand, *Phys. Rev.* **D28**, 396 (1982).
[11] P. Castorina, P. Cea, G. Nardulli, G. Paiano, *Phys. Rev.* **D29**, 2660 (1984).
[12] J. L. Nickisch, L. Durand, B. Durand, *Phys. Rev.* **D30**, 660 (1984).
[13] D. B. Lichtenberg, W. Namgung, E. Predazzi, J. G. Wills, *Phys. Rev. Lett.* **48**, 1653 (1982); *Z. Phys.* **C19**, 19 (1983).
[14] For the analogical problem in the case of two Dirac particles cf. e.g. W. Królikowski, *Acta Phys. Austr.* **53**, 223 (1981) and references therein.
[15] For the latter case cf. e.g. J. Rafelski, L. P. Fulcher, A. Klein, *Phys. Rep.* **38**, 227 (1978).
[16] W. Królikowski, *Acta Phys. Pol.* **B11**, 387 (1980).
[17] For a comment cf. W. Królikowski, *Phys. Rev.* **D30**, 660 (1984).