FRACTIONAL BARYON NUMBERS OF CHIRAL BAGS*

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Results on the fractional baryon numbers of the fermionic vacua in chiral bag models are reviewed. The simple case of the one-dimensional bag is presented in order to give an analogue for the much more complicated calculations in three dimensions. The baryonic number distribution inside the one-dimensional bag has been explicitly calculated. It exhibits logarithmic singularities.

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1. Introduction

In the standard MIT bag model each hadron is associated with a cavity in the physical vacuum. The vacuum outside the cavity is a very complicated state containing $q\bar{q}$ and gluon condensates. By definition this vacuum state minimizes the energy density. Therefore, it tends to spread everywhere and in particular excerpts a pressure from outside on the cavity. Inside the cavity are the valence quarks and/or antiquarks. They are free in the sense that inside the bag they satisfy the free-particle Dirac equation. They exert an outward pressure on the surface of the cavity. The pressures from the outside and from the inside on the cavity surface compensate each other.

An analogue of this system can be found in the theory of superconductivity. The superconducting state corresponds to the vacuum. Cooper pairs in the superconductor correspond to the condensates in the physical vacuum. When a small but strong magnet is put inside the superconductor, the magnetic field destroys superconductivity in a small region around the magnet. Similarly in the MIT bag model it is believed that the strong chromoelectric field of the valence quarks (and/or antiquarks) destroys the physical vacuum in a small region of space and thus the cavity, also known as bag, is formed.

In the chiral bag model, which was proposed in 1975 [1] by members of the same group which had suggested the MIT bag model in 1974, the cavity is surrounded by a pion field. We consider the simplest version of the model, where the cavity is exactly spherical and

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neither the pions, nor the condensates can penetrate inside. The boundary condition for the quarks is

$$-i\gamma \boldsymbol{n}\psi(\boldsymbol{r}) = e^{i\theta \boldsymbol{\tau}\cdot\boldsymbol{n}\gamma_5}\psi(\boldsymbol{r}) \quad \text{for} \quad r = |\boldsymbol{r}| = R. \tag{1}$$

Here ψ is a Dirac bispinor with an additional isospin index. Under rotations in isospin space ψ transform as members of an irreducible doublet (we consider u and d quarks only). The unit vector $\boldsymbol{n}$ is parallel to the external normal to the surface of the bag. Thus for a spherical cavity, $\boldsymbol{n}$ at point $\boldsymbol{r}$ of the surface is parallel to $\boldsymbol{r}$. The components of $\boldsymbol{\tau}$ are Pauli matrices acting on the isospin indices of ψ. Real number θ, known as the chiral angle, is a constant.

For $\theta = 0$ condition (1) reduces to that for the standard MIT bag model [2]. Note that the boundary condition $\psi(r = R) = 0$, which would be natural for the Schrödinger equation, is unacceptable for the Dirac equation (which is first order), because it would imply $\psi = 0$ everywhere inside the bag. Condition (1) guarantees that the component normal to the bag surface of the vector current $\bar{\psi}\gamma^{\mu}\psi$ vanishes on the surface of the bag [2].

The chiral angle θ changes under chiral transformations $\psi \to \exp(i\alpha\gamma_5)\psi$. It is fixed by the convention that $\theta(r \to \infty) = 0$. The values of θ for $r = R$ and $r \to \infty$ are correlated by the equations describing the pion field. Thus θ may be interpreted as $\theta(R) - \theta(\infty)$. This is invariant under chiral transformations and can be a well defined physical parameter.

In 1983 Rho, Goldhaber and Brown discovered [3] that boundary condition (1) implies that the fermionic vacuum in the bag contributes a fractional baryon number. This result would have been very bad for the chiral bag model in 1975. In 1983, however, it was already well known that the pion field outside the bag can also contribute to the baryon number of the system [4]. Thus, it should be possible to formulate the model so that the fractional parts of the two contributions cancel [3]. Integer parts of these contributions, if any, can of course be compensated by adding or subtracting quarks. In the following we will assume that quarks have baryonic numbers one. This simplifies the wording and introduces no error, because in order to keep the hadron white, quarks with baryon number 1/3 must be added or subtracted in sets qqq, $\bar{q}\bar{q}\bar{q}$, or $q\bar{q}$.

Before presenting quantitative results let us see on a simplified model, why the fermionic vacua give in this case nonvanishing contributions to the baryonic numbers.

2. *Simple model*

The boundary condition (1) breaks charge conjugation invariance. As a result the energy spectrum is not symmetric with respect to the change $E \to -E$ (see next section). In order to see how this asymmetry generates a non-zero baryon number, consider a system with the energy spectrum

$$E_n(\theta) = n + \theta; \qquad n = \pm 1, \pm 2, \ldots \tag{2}$$

For $\theta = 0$ this spectrum is symmetric. Otherwise it is not. Perhaps the simplest case is when $\theta = -N$, where N is a positive integer. This spectrum can be obtained by removing from a symmetric spectrum ($E_n(0)$ plus the $E = 0$ level) the level E_{-N}. Since by symmetry one can ascribe baryon number $B = 0$ to the symmetric spectrum, the system for $\theta = -N$, which has one sea quark less, should have $B = -1$. Consider now the case $0 < |\theta| < 1$, which is closer to what happens in a chiral bag.

Let us first define the baryonic number $B(\theta)$ for the system. With the sea filled by quarks and no positive energy quarks

$$B(\theta) = \sum_{E<0} 1 + C \tag{3}$$

where C is some (infinite) constant and the summation extends over all the negative energy levels. An equivalent description can be obtained by filling all the positive energy states with antiquarks. Then

$$B(\theta) = -\sum_{E>0} 1 - C. \tag{4}$$

Combining these two formulae

$$B(\theta) = \tfrac{1}{2}\Big(\sum_{E<0} 1 - \sum_{E>0} 1\Big). \tag{5}$$

Now it is necessary to give some meaning to the sums in (5). One way is to assume

$$B(\theta) = \lim_{L\to\infty} B_L(\theta), \tag{6}$$

where $B_L(\theta)$ is calculated according to formula (5), but including only the energy levels $|E_n| < L$. The L-dependence of $B_L(\theta)$ is shown in Fig. 1. Clearly the limit (6) does not exist as a standard limit. The limit in the mean

$$B(\theta) = \lim_{L\to\infty} \frac{1}{L}\int_0^L B_L(\theta)dL$$

is, however, well defined and gives

$$B(\theta) = \theta. \tag{7}$$

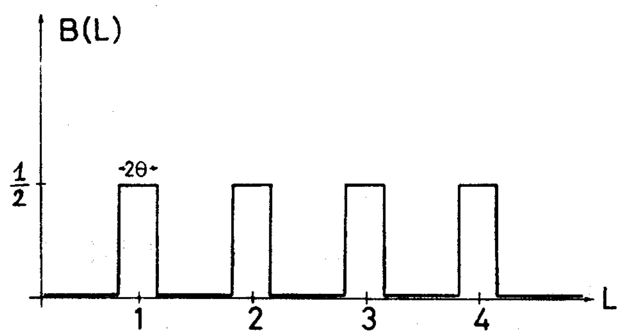

Fig. 1. L-dependence of the regularized baryon number $B_L(\theta)$

We have presented this clumsy calculation to stress that in order to calculate the baryonic numbers of interest it is not enough just to make a cut off. For this reason e.g. it is not obvious how to get $B(\theta)$ on a lattice.

A more elegant calculation of $B(\theta)$ uses formula (5) interpreted as

$$B(\theta) = \lim_{t\to+0} \tfrac{1}{2}\Big(\sum_{E<0} e^{-|E|t} - \sum_{E>0} e^{-Et}\Big). \tag{8}$$

This expression for the general case has been proposed by Goldstone and Jaffe [5]. For spectrum (2) the sums reduce to geometrical series and one finds again

$$B(\theta) = \lim_{t\to 0} \frac{\sinh \theta t}{1-e^{-t}} = \theta. \tag{8}$$

Note that however large is an integer N, one can omit the first N positive energy levels and the first N negative energy levels without affecting $B(\theta)$. Due to the divergence of series (5), the value of $B(\theta)$ is determined by only the high $|n|$ levels. This is a simple example of an anomaly. Note also that the limiting process used to give sense to relation (5) must be judiciously chosen. E.g. using prescription (6) with the additional condition that L must be an integer, one finds $B(\theta) = 1$. Substituting into (8) $\exp(-|E-\theta|t)$ instead of $\exp(-|E|t)$, which in the limit $t \to 0$ does not seem to make much change, gives $B(\theta) = 0$. Note finally that for $\theta = -N$ result (7) differs from that obtained from the discussion following formula (2). This reflects the fact that only the calculation of the fractional part of the baryonic number is unambiguous, while the integer part depends on which levels are assumed to be occupied in the vacuum state. Thus result (7) is noncontroversial only for $|\theta| < 1$.

3. *Baryon number of a chiral bag with massless quarks*

Let us consider first a one-dimensional chiral bag. For $|x| < R$ the Dirac bispinors satisfy the Dirac equation

$$(\gamma^0 E + i\gamma^x \partial_x - m)\psi = 0. \tag{9}$$

Here and in the following we write for further reference formulae including the quark mass m, but only the case $m = 0$ is discussed in the present section. The boundary condition is

$$\mp i\gamma^x \psi = e^{\pm i\theta\gamma_5}\psi \quad \text{for} \quad x = \pm R. \tag{10}$$

Since σ_x commutes with all the operators acting on ψ in both (9) and (11), we can assume without loss of generality that

$$\sigma_x \psi = \lambda\psi; \qquad \lambda = \pm 1. \tag{11}$$

The two conditions (10) become equivalent, if $\psi(-x) = \pm\gamma^0\psi(x)$. Thus, it is enough to look for solutions in the form

$$\psi_s(x) = \begin{pmatrix} u \cos kx \\ v \sin kx \end{pmatrix}, \quad \psi_a(x) = \begin{pmatrix} u \sin kx \\ v \cos kx \end{pmatrix}; \quad k \geqslant 0 \tag{12}$$

and from (9) we find (for $m = 0$) $E = \varepsilon k$, where $\varepsilon = \pm 1$. Condition (10) for $x = +R$ consists of four equations. Because of (11), however, their number reduces to two and because of the identity

$$\mathrm{Det}\,(i\gamma^x - e^{i\theta\gamma_5}) = 0, \tag{13}$$

it is enough to consider one equation only. For $m = 0$ the energy eigenvalues are

$$E_{n,\eta,\lambda} = \frac{1}{2R}\left(2n\pi + \eta\frac{\pi}{2} - \lambda\theta\right); \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{14}$$

where $\eta = +1$ for ψ_s states, $\eta = -1$ for ψ_a states and for simplicity it has been assumed that

$$|\theta| < \pi/2. \tag{15}$$

Let us note that when both $\lambda = \pm 1$ are included, the energy spectrum (14) is symmetric with respect to $E \to -E$ and consequently the fermionic vacuum has zero baryon number. In order to get $B \neq 0$, we assume that

$$\lambda = -1. \tag{16}$$

The minus sign is chosen in order to get results closer to those for the three-dimensional case. Calculating like in the preceding section one finds

$$B(\theta) = \frac{\theta}{\pi}. \tag{17}$$

The corresponding calculation for the three-dimensional case has been performed first in Ref. [5]. The result is

$$B(\theta) = \frac{1}{\pi}(\theta - \sin\theta\cos\theta). \tag{18}$$

It includes all the states in a sphere without making an arbitrary choice like (16). Comparing (18) with the contribution from the pion field outside, one finds that for $|\theta| < \pi/2$ both contributions exactly cancel [5]. For other values of θ the sum is an integer, thus again there is no trouble, For a discussion of the various possibilities cf. [5]. The argument used in Ref. [5] assumes only very general topological properties of the bag surface. Formula (18) is found to hold for any surface, which can be continuously deformed into a sphere. Results for other surfaces also follow. E.g. for a torus $B(\theta) = 0$ for all values of θ. Another derivation, which can be extended to massive quarks, has been given by Jaroszewicz [6]. A brute force derivation by calculating high $|n|$ approximations to the energy eigenvalues and explicitly performing the summation (8) has been given in Ref. [7]. This derivation, applicable only for spherical bags, is certainly the least elegant of the three, but it is also the most flexible one. It can be used not only to derive the result for $m \neq 0$, but also to calculate the distribution of the baryon number inside the bag.

4. *Case of quarks with non-zero mass*

When non-zero quark masses are introduced two changes occur in the argument. Again let us begin with the one-dimensional case. Firstly the high $|n|$ eigenvalues get shifted as compared to the massless case. This shift cannot be calculated analytically, because the corresponding equation, which for $\lambda = -1$ reads

$$\mathrm{tg}\,(\sqrt{E^2-m^2}\,R) = \varepsilon \sqrt{\frac{|E|-\varepsilon m}{|E|+\varepsilon m}}\,\mathrm{tg}\left(\frac{\pi}{4}+\frac{\theta}{2}\right), \tag{19}$$

is too complicated. It is easily seen, however, that for $|E| \to \infty$ this shift is smaller than the shift corresponding to an arbitrarily small shift in θ. Consequently, result (17) for the anomalous contribution to $B(\theta)$ is not changed by giving non-zero masses to quarks. In the three-dimensional case the effects of the quark masses are somewhat more difficult to estimate, but this has been done [6, 8] and the result is the same — the anomalous contribution from the fermionic vacuum to the baryon number of the bag does not depend on the quark masses.

Another effect is that new solutions

$$\psi_s(x) = \begin{pmatrix} u\cosh kx \\ v\sinh kx \end{pmatrix}; \qquad \psi_a(x) = \begin{pmatrix} u\sinh kx \\ v\cosh kx \end{pmatrix} \tag{20}$$

may appear. Such solutions occur in solid state theory, where they are known as Tamm levels and correspond to states localized near the surface of a cristal. In solution (20) $0 \leqslant k \leqslant m$ and the corresponding energies are $E = \varepsilon\sqrt{m^2-k^2}$. Their dependence on the quark mass is shown in Fig. 2. The crucial parameter is in this case

$$m_+ = -\mathrm{tg}\left(\frac{\pi}{4}-\frac{\theta}{2}\right). \tag{21}$$

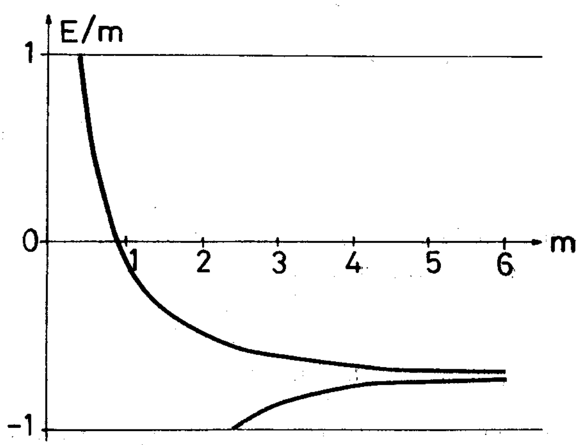

Fig. 2. Dependence of the energy levels $|E| < m$ on quark mass

If this is negative, there are no solutions of type (20). For $m_+ > 0$ and m increasing from $m = m_+$, one positive energy level gets into the zone $|E| < m$. If $\sin\theta > 0$, it reaches $E = 0$ at

$$m_0 = 2 \operatorname{artgh} m_+ \tag{22}$$

and tends to the limiting value $E_\infty = -m \sin\theta$. At $m = m_-$, where

$$m_- = -\operatorname{tg}\left(\frac{\pi}{4} + \frac{\theta}{2}\right), \tag{23}$$

a negative energy level gets into the zone $|E| < m$ and tends to the same limit E_∞.

A level crossing $E = 0$ changes the baryon number at most by an integer and consequently is of little interest for our discussion. The case $E = 0$ is, however, of some historical interest. Making this level empty, or full, one finds a contribution $\mp 1/2$ to the baryon number. This is how fractional baryon numbers have been first found [9]. By a suitable redefinition of the baryon number, e.g. in our case by taking

$$B = \frac{1}{2\Delta} \int_{m_0 - \Delta}^{m_0 + \Delta} B(\theta, m) dm; \qquad \Delta \ll m_0 - m_+$$

one easily checks that e.g. $B = 1/2$ is an eigenvalue of B and not an average over quantum fluctuations between $B = 0$ and $B = 1$. For discussions of such problems cf. Refs. [10, 11].

5. Distribution of the baryon number density inside the bag

In the chiral bag model the distribution of the baryon number density consists of three contributions:

i) Contribution from the valence quarks.

ii) Contribution from the fermion vacuum.

iii) Contribution from the pion field.

We will discuss here only the contribution from the fermionic vacuum. The corresponding formula is a simple generalization of formula (8):

$$B(x) = \lim_{t \to +0} \tfrac{1}{2} \Big(\sum_{E<0} \psi_E^+(x)\psi_E(x) e^{-|E|t} - \sum_{E>0} \psi_E^+(x)\psi_E(x) e^{-Et} \Big). \tag{24}$$

For the one-dimensional bag one finds

$$\psi^+(x)\psi(x) = 1 + \varepsilon\eta m \frac{k \cos 2kx - \sin k}{|E|k + \varepsilon\eta m \sin k}. \tag{25}$$

Thus for $m = 0$ the distribution $B(x)$ is flat. For $m \neq 0$, after substitution of (25) into (24) the summation over E has to be performed in three steps.

i) The part divergent for $t = 0$ is extracted and summed just like in the preceding section. The resulting contribution to the baryonic number is

$$B_1(x) = B(\theta) = \frac{\theta}{\pi}\,. \tag{26}$$

ii) In the remaining series it is legitimate to put $t = 0$, but it is necessary to extract a slowly convergent series (n-th term of order $1/n$) and to sum it analytically. The result is (for $R = 1/2$):

$$B_2(x) = -\frac{m}{\pi}\sin \pi x \sin 2\theta x \ln |2 \sin 2\pi x|. \tag{27}$$

Note that for $|x| \to 1/2$ this sum actually diverges. Therefore, attempts to sum the series (24) numerically without previous extraction of the slowly convergent part can be misleading.

iii) The remaining series is rapidly convergent and can be summed on a pocket calculator yielding the third contribution B_3.

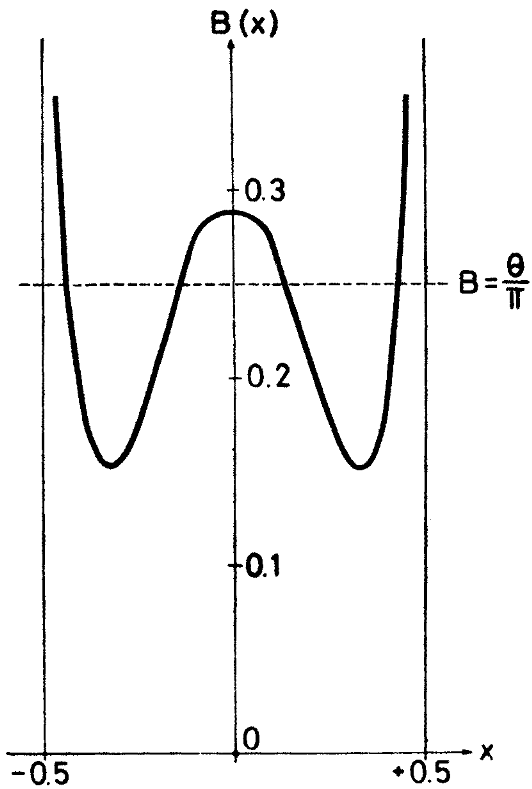

Fig. 3. Fermionic vacuum contribution to the baryon number distribution inside a one-dimensional chiral bag (for $m = 1$, $R = 1/2$, $\theta = \pi/4$)

The total contribution of the fermionic vacuum to the baryon number distribution inside the bag ($B_1+B_2+B_3$) is shown for $R = 1/2$, $m = 1$ and $\theta = \pi/4$ in Fig. 3. It is seen that this contribution varies rapidly with x and that it exhibits logarithmic singularities at $x = \pm 1/2$. Since these singularities are integrable, this is not obviously bad for the model.

The calculation in the three-dimensional case is much more difficult. Up to now only the second moment

$$\langle r^2 \rangle_{I=0} = \int r^2 B(\boldsymbol{r}) d^3 r \tag{28}$$

has been calculated [12]. This quantity is measurable experimentally. Thus in particular one finds a relation between the parameters R, θ of the bag and the parameters f_π, e of the skyrmion (pion field) outside the bag [12].

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