SOME REMARKS ON THE FOLDY-WOUTHUYSEN TRANSFORMATION FOR THE DIRAC HAMILTONIAN WITH AN EXTERNAL YANG-MILLS FIELD*

By K. Golec

Institute of Physics, Jagellonian University, Cracow**

(*Received June 28, 1985*)

The Foldy-Wouthuysen transformation for the Dirac Hamiltonian with an external Yang-Mills field up to the order $\left(\frac{1}{mc}\right)^4$ is performed. The color operators in the F–W representation are considered.

PACS numbers: 11.10.Qr

1. Introduction

The Foldy–Wouthuysen transformation (F–W) has greatly contributed to the understanding of the physical content of the Dirac theory of electron [1] and theories of particles with the spin 0 and 1 [2]. Quite recently the F–W transformation has been applied to the study of the classical limit of the colored Dirac particle in an external Yang–Mills field [3]. The classical equations obtained in [3] from the Heisenberg equations of motion have a nonrelativistic form. Their relativistic generalization was proposed in [4]. It was obtained by means of an operatorial gauge transformation. Both the Heisenberg equations and the operatorial gauge transformation were calculated in [3, 4] up to the order to which the F–W transformation was performed, i.e. $\left(\frac{1}{mc}\right)^2$.

In order to check the consistency of the relativistic generalization proposed in [4] the higher order F–W transformation has to be known. In this paper we present the calculation of the F–W transformation for the Dirac Hamiltonian with an external Yang–Mills field up to the order $\left(\frac{1}{mc}\right)^4$. Also, we discuss the color operators in the F–W representation. Such a discussion has not been given in [3, 4].

---

* Paper supported in part by the Polish Ministry of Higher Education and Technology, project MR.I.7.

** Address: Zakład Teorii Pola, Instytut Fizyki UJ, Reymonta 4, 30-059 Kraków, Poland.

This paper is organized as follows. In Section 2 we derive the corrections of the order $\left(\frac{1}{mc}\right)^3$ and $\left(\frac{1}{mc}\right)^4$ to the F–W Hamiltonian. In Section 3 we consider the color operators transformed to the F–W representation and their transformation law under the gauge group. In Section 4 we estimate the significance of the higher order corrections to the F–W Hamiltonian in the electromagnetic case.

2. *Hamiltonians H_3 and H_4*

Let us consider a colored particle in an external SU(N) gauge field described by the Dirac equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{1}$$

with the following Hamiltonian

$$H = c\vec{\alpha}\left(\vec{p}-\frac{g}{c}\hat{\vec{A}}\right)+g\hat{A}_0+\beta mc^2, \tag{2}$$

where $\vec{\alpha}$, β are Dirac matrices in the Dirac representation, $\vec{p} = -i\hbar\vec{\nabla}$; $\hat{\vec{A}} = \vec{A}^a\hat{T}^a$, $\hat{A}_0 = A_0^a\hat{T}^a$, $a = 1, 2, \ldots N^2-1$ are Yang–Mills potentials, $\hat{T}^a$ are generators of SU(N) group in the fundamental representation and g is a colored coupling constant. The generators $\hat{T}^a$ are called the color operators.

The Foldy–Wouthuysen transformation, [1]

$$\psi' = e^{iM}\psi$$

$$H' = e^{iM}\left(H-i\hbar\frac{\partial}{\partial t}\right)e^{-iM} \tag{3}$$

is performed in order to remove odd powers of $\vec{\alpha}$ matrices from the Hamiltonian (2). It is not possible to perform an exact transformation (3) for an arbitrary Yang–Mills field. Actually, H' can be obtained only in the form of an expansion in powers of $\frac{1}{mc}$. In the paper [1] the Hamiltonian H' was found up to the order $\left(\frac{1}{mc}\right)^2$. We want to obtain corrections to H' of the order $\left(\frac{1}{mc}\right)^3$ and $\left(\frac{1}{mc}\right)^4$. We follow the procedure of Bjorken and Drell [5]. For simplicity we put $c = 1$. The calculation is a standard one but rather lengthy so we present here only main points.

In order to obtain the Hamiltonian H' up to the desired order of accuracy we perform five succesive transformations of the type (3), with M of the form:

$$M_i = -\frac{i\beta}{2m}O_i, \quad i = 1, 2 \ldots 5. \tag{4}$$

O_i are operators which contain only odd powers of $\vec{\alpha}$ matrices. We list them in the Appendix. Thus, we obtain the Hamiltonian H' free of odd operators up to the order $\left(\frac{1}{mc}\right)^4$

$$H' = H_2 + H_3 + H_4 \tag{5}$$

where

$$H_2 = \beta m + g\hat{A}_0 + \frac{\beta}{2m} O^2 - \frac{1}{8m^2}[O, \Phi], \tag{6}$$

$$H_3 = \frac{\beta}{8m^3}(\Phi^2 - O^4), \tag{7}$$

$$H_4 = \frac{1}{32m^4}\left[\Phi, [\Phi, g\hat{A}_0] + i\hbar \frac{\partial \Phi}{\partial t}\right] + \frac{1}{12m^4}[O^3, \Phi] + \frac{1}{384m^4}[[[\Phi, O], O], O] \tag{8}$$

and

$$O = \vec{\alpha} \cdot (\vec{p} - g\vec{\hat{A}}), \quad \Phi = [O, g\hat{A}_0] + i\hbar \frac{\partial O}{\partial t}.$$

The final step we perform is to evaluate the operator products in (6)–(8). The Hamiltonian H_2 is the same as in [3]

$$H_2 = \beta m + g\hat{A}_0 + \frac{\beta}{2m}\vec{\pi}^2 - \frac{g\hbar\beta}{m}\vec{\hat{B}} \cdot \vec{S}$$
$$+ \frac{g\hbar}{4m^2}\varepsilon_{ijk}S^k(\pi^i\hat{E}^j + \hat{E}^j\pi^i) - \frac{g\hbar^2}{8m^2}(D_i\hat{E}^i), \tag{9}$$

where $\vec{\pi} = \vec{p} - g\vec{\hat{A}}$, $\vec{S} = \frac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$ is the spin operator, $\vec{\sigma}$ are Pauli matrices, $D_i = \frac{\partial}{\partial x^i} + \frac{ig}{\hbar}[\hat{A}_i, \cdot]$ is the covariant derivative and $\vec{\hat{E}} = \vec{E}^a\hat{T}^a$, $\vec{\hat{B}} = \vec{B}^a\hat{T}^a$ are colored electric and magnetic fields defined as follows:

$$\hat{E}^i = \hat{F}_{0i}, \quad \hat{B}^i = -\tfrac{1}{2}\varepsilon^{ikr}\hat{F}_{kr}, \quad i = 1, 2, 3,$$

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + \frac{ig}{\hbar}[\hat{A}_\mu, \hat{A}_\nu], \quad \mu, \nu = 0, 1, 2, 3. \tag{10}$$

The higher order corrections to H' have the form

$$H_3 = -\frac{\beta}{8m^3}\{\vec{\pi}^4 - 2g\hbar((\vec{\hat{B}} \cdot \vec{S})\vec{\pi}^2 + \vec{\pi}^2(\vec{\hat{B}} \cdot \vec{S}))$$
$$+ g^2\hbar^2(\vec{\hat{E}}^2 + \vec{\hat{B}}^2 + 2i\varepsilon_{klm}S^m(\hat{E}^k\hat{E}^l + \hat{B}^k\hat{B}^l))\} \tag{11}$$

and

$$H_4 = -\tfrac{30}{384}\frac{g\hbar}{m^4}[2\varepsilon_{klm}\{(\vec{S}\cdot\vec{\pi})(\pi^k\hat{E}^l+\hat{E}^l\pi^k)\pi^m+\pi^m(\pi^k\hat{E}^l+\hat{E}^l\pi^k)(\vec{S}\cdot\vec{\pi})$$

$$-S^n\vec{\pi}(\pi^k\hat{E}^l+\hat{E}^l\pi^k)\vec{\pi}\}+i\{\pi^m(\pi^n\hat{E}^m+\hat{E}^m\pi^n)\pi^n-\pi^m(\pi^m\hat{E}^n+\hat{E}^n\pi^m)\pi^n\}]$$

$$-\tfrac{66}{384}\frac{g\hbar}{m^4}\varepsilon_{klm}S^m\{\vec{\pi}^2(\pi^k\hat{E}^l+\hat{E}^l\pi^k)+(\pi^k\hat{E}^l+\hat{E}^l\pi^k)\vec{\pi}^2)\}$$

$$+\frac{g\hbar^2}{m^4}\{\tfrac{33}{384}((D_i\hat{E}^i)\vec{\pi}^2+\vec{\pi}^2(D_i\hat{E}^i))+\tfrac{30}{384}(\pi^k(D_i\hat{E}^i)\pi^k+2i\varepsilon_{klm}S^m\pi^k(D_i\hat{E}^i)\pi^l\}$$

$$-\tfrac{33}{384}\frac{g^2\hbar^2}{m^4}\{\hat{F}_{kl}(\pi^k\hat{E}^l+\hat{E}^l\pi^k)+(\pi^k\hat{E}^l+\hat{E}^l\pi^k)\hat{F}_{kl}$$

$$+2i\varepsilon_{ijk}S^l(\hat{F}_{kl}(\pi^i\hat{E}^j+\hat{E}^j\pi^i)-(\pi^i\hat{E}^j+\hat{E}^j\pi^i)\hat{F}_{kl})\}$$

$$+\frac{g^2\hbar^3}{m^4}\{\tfrac{33}{384}\varepsilon_{klm}S^m((D_i\hat{E}^i)\hat{F}_{kl}+\hat{F}_{kl}(D_i\hat{E}^i))$$

$$-\frac{i}{32}([\hat{E}^k, D_0\hat{E}^k]+2i\varepsilon_{mkl}S^l(\hat{E}^m(D_0\hat{E}^k)+(D_0\hat{E}^k)\hat{E}^m))\}. \qquad (12)$$

We have assumed the Einstein summation convention.

The Hamiltonians H_3 and H_4 together with H_2 may serve as a starting point in the analysis of the relativistic generalization of the classical equations for a colored, spinning particle obtained by means of the F–W transformation, [3, 4]. This application of H_3 and H_4 will be presented in another paper.

The Hamiltonians (9), (11), (12) may be reduced to the electromagnetic case if we replace the color fields $\hat{A}_\mu$, $\hat{\vec{E}}$, $\hat{\vec{B}}$ by the usual commuting electromagnetic fields A_μ, $\vec{E}$, $\vec{B}$ and the covariant derivative D_μ by $\frac{\partial}{\partial x^\mu}$. We also substitute the electric charge e for g. Thus, we obtain the Hamiltonian $H' = H_2+H_3+H_4$ where

$$H_2 = \beta mc^2+eA_0+\frac{\beta}{2m}\vec{\pi}^2-\frac{e\hbar\beta}{mc}\vec{B}\cdot\vec{S}+\frac{e\hbar}{4m^2c^2}(\vec{\pi}\times\vec{E}-\vec{E}\times\vec{\pi})\cdot\vec{S}-\frac{e\hbar^2}{8m^2c^2}\vec{\nabla}\cdot\vec{E}, \qquad (13)$$

$$H_3 = -\frac{\beta}{8m^3c^3}\vec{\pi}^4+\frac{e\hbar\beta}{4m^3c^3}\{\vec{\pi}^2(\vec{B}\cdot\vec{S})+(\vec{B}\cdot\vec{S})\vec{\pi}^2\}-\frac{e^2\hbar^2\beta}{8m^3c^4}(\vec{E}^2+\vec{B}^2) \qquad (14)$$

and $\vec{\pi} = \vec{p}-\frac{e}{c}\vec{A}$. We have completed (13) and (14) with the light speed c. The formula for H_4 is lengthy but easy to obtain from (12) so we do not write it here. The Hamiltonian

(13) supplemented by the first term of (14)

$$\tilde{H} = H_2 - \frac{\beta}{8m^3c^3}\vec{\pi}^4 \tag{15}$$

is well known, see e.g. [5]. We will make use of the Hamiltonians (13) and (14) in Section 4 where we will estimate their significance for the spectrum of the hydrogen atom.

3. Color operators in the F–W representation

The F–W transformation (3) is supplemented by a transformation rule for an arbitrary operator X acting in the space of bispinors ψ

$$X_{\mathrm{FW}} = e^{iM} X e^{-iM}. \tag{16}$$

The new operator X_{FW} is called the operator-representative of X in the F–W representation, [1].

In the nonabelian case new operators appear which do not exist in the abelian case — the color operators $\hat{T}^a$, $a = 1, 2, \ldots N^2-1$. They act in a space of bispinor functions $\psi = (\psi^a)$

$$(\hat{T}^a\psi)_b(x) = (\hat{T}^a)_{bc}\psi_c(x), \tag{17}$$

where $(\hat{T}^a)_{bc}$ are matrices of the fundamental representation of $\hat{T}^a$ and form a set of irreducible tensor operators under the local gauge group SU(N):

$$\Omega(x)\hat{T}^a\Omega^{-1}(x) = V^a{}_b(x)\hat{T}^b \tag{18}$$

where $V^a{}_b(x)$ is a matrix of the adjoint representation of the group element $\Omega(x) \in \mathrm{SU}(N)$.

Operators $\hat{T}^a$ transformed to the F–W representation according to (16), where the transformation e^{iM} is listed in the Appendix, have rather complicated form:

$$\hat{T}^a_{\mathrm{FW}}[\hat{A}] = \hat{T}^a + \frac{g^2}{8m^2}(\vec{A}^a\cdot\vec{A}^b - \vec{A}^d\cdot\vec{A}^d\delta^{ab})\hat{T}^b$$

$$+\varepsilon^{abc}\left\{\frac{ig\beta}{2m}\vec{\alpha}\cdot\vec{A}^b + \frac{g\hbar}{4m^2}\vec{\alpha}\cdot\vec{E}^b - \frac{g\hbar}{8m^2}(\vec{\nabla}\cdot\vec{A}^b + 2i\vec{S}\cdot\vec{B}^b)\right\}\hat{T}^c \tag{19}$$

(19) is written only up to the order $\left(\frac{1}{m}\right)^2$. Note that $\hat{T}^a_{\mathrm{FW}}[\hat{A}]$ contain the odd operators $\vec{\alpha}$. It is obvious because the F–W transformation removes the odd operators only from the Hamiltonian (2).

The new color operators $\hat{T}^a_{\mathrm{FW}}[\hat{A}]$ do not have any reasonable transformation law under the gauge group. If we try to evaluate the gauge transformed operator $\hat{T}^a_{\mathrm{FW}}[\hat{A}']$ where

$\hat{A}'_\mu = \Omega \hat{A}_\mu \Omega^{-1} + \frac{i\hbar}{g}(\partial_\mu \Omega)\Omega^{-1}$ and $\Omega(x) \in SU(N)$ we will obtain

$$\begin{aligned} \hat{T}^a_{FW}[\hat{A}'] &= e^{iM[\hat{A}']}\hat{T}^a e^{-iM[\hat{A}']} \\ &= \Omega(x) e^{iM[\hat{A}]} V^a{}_b(x) \hat{T}^b e^{-iM[\hat{A}]} \Omega^{-1}(x). \end{aligned} \tag{20}$$

We utilized here (18) and the property $\Omega e^{iM[\hat{A}]}\Omega^{-1} = e^{iM[\hat{A}']}$ resulting from the fact that $M[\hat{A}]$ contains only operators transforming homogeneously under the gauge group. The matrix $V^a{}_b(x)$ does not commute with the operator $e^{iM[\hat{A}]}$ because $M[\hat{A}]$ contains also differential operators. Because of that we have

$$\hat{T}^a_{FW}[\hat{A}'] \neq V^a{}_b(x)\Omega(x)\hat{T}^a_{FW}[\hat{A}]\Omega^{-1}(x).$$

Such difficulties do not appear for the operators which do not involve free color indices and transform homogeneously under the gauge group. For example, the transformation rule for the operator-representative of the Yang-Mills field tensor $\hat{F}_{\mu\nu}[\hat{A}]$, (10) is the following:

$$\hat{F}^{\mu\nu}_{FW}[\hat{A}'] = \Omega(x)\hat{F}^{\mu\nu}_{FW}[\hat{A}]\Omega^{-1}(x). \tag{21}$$

The same rule is valid for the operator-representative of the rest of non-colored operators, e.g. $\vec{x}$, $\vec{S}$, $\vec{p}$.

4. The significance of H_3 and H_4 in the electromagnetic case

Having the Hamiltonians H_3 and H_4 it is interesting to estimate their significance for real physical systems. We will make it taking as an example the hydrogen atom. In this case we set $A_0 = -\frac{e}{r}$ and $\vec{A} = 0$ in (13) and (14).

The energy levels stemming from the Dirac theory have the following form of the expansion in powers of the fine structure constant α, [5]:

$$E_{nj} = mc^2 - \frac{mc^2\alpha^2}{2n^2} - \frac{mc^2\alpha^4}{2n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) + O(\alpha^6) \tag{22}$$

and

$$O(\alpha^6) = \frac{mc^2\alpha^6}{n^3}\left\{\frac{1}{(2j+1)^3} - \frac{3}{2n(2j+1)^2} + \frac{3}{2n^2(2j+1)} - \frac{15}{48n^3}\right\} + \dots \tag{23}$$

where $n = 1, 2 \dots$ is the principal quantum number, $j = l \pm \frac{1}{2}$ and $l = 0, 1, \dots n-1$. The term $E^0_n = mc^2 - \frac{mc^2\alpha^2}{2n^2}$ in the formula (22) is the eigenvalue of the Hamiltonian $H_0 = mc^2 + eA_0 + \frac{\vec{p}^2}{2m}$ and $A_0 = -\frac{e}{r}$. If we add to H_0 as a perturbation the Hamiltonian

$V = \tilde{H} - H_0$ ($\tilde{H}$ is defined by (15)) the term in (22) proportional to α^4 may be obtained as the first order perturbative correction to E_n^0, [6]. This term is responsible for the fine structure of the hydrogen atom. It is obvious that the term of the order α^6 may be obtained as a perturbative correction to E_n^0 if we take into account the remaining terms proportional to $\left(\frac{1}{mc}\right)^3$ and $\left(\frac{1}{mc}\right)^4$ in the Hamiltonian $H' = H_2 + H_3 + H_4$. In this case the perturbative Hamiltonian $V = H' - H_0$.

The energy splitting due to the $O(\alpha^6)$ term is practically negligible. It is equal to 4.7×10^{-9} eV for the levels $2P_{3/2}$ and $2P_{1/2}$ whereas the magnitude of the Lamb shift for the levels $2P_{1/2}$ and $2S_{1/2}$ equals 4.4×10^{-6} eV. It means that the terms of the order $\left(\frac{1}{mc}\right)^3$ and $\left(\frac{1}{mc}\right)^4$ in H' $\left(\text{except for the term } \frac{\beta}{8m^3c^3}\vec{p}^4\right)$ are practically of no importance for the energy spectrum of the hydrogen atom.

The author wishes to thank Dr H. Arodź for stimulating discussion and constant encouragement.

APPENDIX

In order to obtain the Hamiltonian (2) free of odd operators up to the order $\left(\frac{1}{mc}\right)^4$ we perform the F–W transformation (3) where

$$e^{iM} = e^{iM_5}e^{iM_4} \dots e^{iM_1}.$$

The operators M_i have the form $M_i = -\frac{i\beta}{2m} O_i$, $i = 1, 2 \dots 5$ where

$$O_1 = \vec{\alpha} \cdot (\vec{p} - g\hat{\vec{A}}),$$

$$O_2 = \frac{\beta}{2m}\Phi - \frac{O_1^3}{3m^2} - \frac{\beta}{48m^3}[O_1, [O_1, \Phi]] + \frac{O_1^5}{24m^4},$$

$$O_3 = \frac{\beta}{2m}[O_2, \mathscr{E}'] + \frac{i\hbar\beta}{2m}\frac{\partial O_2}{\partial t},$$

$$O_4 = \frac{\beta}{2m}[O_3, \mathscr{E}''] + \frac{i\hbar\beta}{2m}\frac{\partial O_3}{\partial t},$$

$$O_5 = \frac{\beta}{2m}[O_4, \mathscr{E}''] + \frac{i\hbar\beta}{2m}\frac{\partial O_4}{\partial t},$$

and

$$\Phi = [O_1, g\hat{A}_0] + i\hbar \frac{\partial O_1}{\partial t} = ig\hbar\vec{\alpha} \cdot \hat{\vec{E}},$$

$$\mathcal{E}' = g\hat{A}_0 + \frac{\beta O_1^2}{2m} - \frac{1}{8m^2}[O_1, \Phi] - \frac{\beta O_1^4}{8m^3} + \frac{1}{384m^4}[O_1, [O_1, [O_1, \hat{\Phi}]]],$$

$$\mathcal{E}'' = \frac{\beta}{2m} O_2^2 + \mathcal{E}'.$$

It is not necessary to evaluate operator products in the formulae above in order to obtain the Hamiltonian (2).

REFERENCES

[1] L. L. Foldy, S. A. Wouthuysen, *Phys. Rev.* **78**, 29 (1950).
[2] K. M. Case, *Phys. Rev.* **95**, 1323 (1954).
[3] H. Arodź, *Acta Phys. Pol.* **B13**, 520 (1982).
[4] H. Arodź, K. Golec, *Phys. Rev.* **D**, (1986) in press.
[5] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Mechanics*, Mc Graw-Hill Book Company, New York 1965, Ch. 4.
[6] M. D. Scadron, *Advanced Quantum Theory*, Springer Verlag, New York, Berlin, Heidelberg 1979, Ch. 5.