CHARGED ANALOGUE OF SCHWARZSCHILD'S INTERIOR SOLUTION

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A non-conformal, type one solution of Einstein–Maxwell equations for a static charged fluid sphere is obtained and joined smoothly to the Reissner–Nordstrom solution. The solution thus obtained is analysed numerically. The solution is reducible to Schwarzschild's interior solution in the absence of charge.

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1. Introduction

Certain solution to Einstein–Maxwell field equations representing charged analogues of the Schwarzschild's interior solution were considered by Buchdahl (1979), Banerjee et al. (1981) and Florides (1983). These solutions share at least one of the properties with the Schwarzschild's interior solution such as conformal flatness, constant energy density and constant non-gravitational energy density (Cooperstoch et al. (1978)), respectively. Also, all of these are reducible to Schwarzschild's interior solution in the uncharged case. In the present article the authors have worked out a non-conformal singularity-free charged fluid sphere joining smoothly the Reissner–Nordstrom solution at the pressure-free surface. The solution so obtained shares type one property with Schwarzschild's interior solution and reduces to the latter, when charge is taken away. Unlike the other cited authors we have studied the solution numerically as it has been carried out by Junovicus (1976) for Krori–Barua solution (1975). Consequently, the restrictions to be imposed on the model radius, mass and charge have been derived in order to accommodate the reality conditions such as positivity of pressure, energy density, electrostatic energy density etc.

2. Charged fluid of embedding type one

Karmarker's conditions (1948) so that a spherically symmetric space-time may be of embedding type one, can be expressed as

$$R_{1313}R_{2424}-R_{1224}R_{1334}+R_{2323}R_{1414} = 0, \tag{2.1}$$

provided $R_{2323} \neq 0$ (1982), (2.1) with reference to static metric in Schwarzschild coordinates

$$ds^2 = -a(r)dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 + c(r)dt^2, \tag{2.2}$$

gives rise to

$$\frac{a'}{a(1-a)} = \frac{-2c''}{c'} + \frac{c'}{c} + \frac{a'}{a}, \quad a \neq 1, \tag{2.3}$$

which on first integration gives

$$a = (1+KF'^2), \quad c = F^2, \tag{2.4}$$

K being non-zero arbitrary constant and dash denoting differentiation with respect to r.

For the said case, Einstein–Maxwell equation is given by

$$R^i_j - \tfrac{1}{2} R\delta^i_j = -8\pi T^i_j = -8\pi[M^i_j + E^i_j], \tag{2.5}$$

where M^i_j and E^i_j correspond to matter and electric charge respectively.

The non-vanishing components of the latter can be furnished as

$$M^1_1 = M^2_2 = M^3_3 = -p \quad \text{and} \quad M^4_4 = \varrho, \quad \text{Tolman (1962)}, \tag{2.6}$$

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = -\tfrac{1}{2} g^{11} g^{44} F^2_{14} \quad \text{Eddington (1960)}, \tag{2.7}$$

and also

$$J^4 = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial r}(-g^{11} g^{44} F_{14} \sqrt{-g}),$$

where p and ϱ are pressure and energy density, while F_{14} and J^4 are non-vanishing components of electromagnetic field tensor and current vector, respectively. J^4 as such is accepted to be charge-density, ($J^4 \cdot g^{44}$ being proper charge-density).

The field equations (2.5) with reference to (2.2), (2.4), (2.6) and (2.7) give rise to

$$8\pi T^1_1 = \frac{F'}{r^2(1+KF'^2)}\left(KF' - \frac{2r}{F}\right) = 8\pi p + 8\pi E^1_1, \tag{2.8}$$

$$8\pi T^2_2 = 8\pi T^3_3 = \frac{1}{F(1+KF'^2)}\left[\frac{KFF'F''}{r(1+KF'^2)} - \frac{F''}{1+KF'^2} - \frac{F'}{r}\right] = 8\pi p - 8\pi E^1_1, \tag{2.9}$$

$$8\pi T^4_4 = -\frac{2KF'F''}{r(1+KF'^2)^2} - \frac{KF'^2}{r^2(1+KF'^2)} = 8\pi\varrho + 8\pi E^1_1, \tag{2.10}$$

$E(= 8\pi E^1_1)$ can be termed as "electrostatic energy density". Therefore (2.8)–(2.10) contain three equations with four unknowns viz. $D(= 8\pi\varrho)$, $P(= 8\pi p)$, E and F. In order to have a unique solution as a charged analogue of Schwarzschild's interior solution, let us consider

$$F = A+\sqrt{B+Cr^2}, \quad B > 0,\ C,\ A\ (\neq 0). \tag{2.11}$$

Owing to (2.7)–(2.11), we get

$$D = \frac{KC^2}{2B}\cdot\frac{Y}{\alpha}\,[Y(5\alpha-\beta)+\alpha+\beta], \tag{2.12}$$

$$P = \frac{KC^2}{2B}\cdot\frac{Y}{\alpha}\cdot[Y(\beta-\alpha)+3\beta-\alpha], \tag{2.13}$$

$$E = -\frac{KC^2Y^2}{2B^2\alpha}(\beta-\alpha)(KC+1)r^2, \tag{2.14}$$

$$J^4 = \sqrt{\frac{-C^2(1+KC)}{8\pi\beta}}\cdot(KCY)^{3/2}\cdot\left[3+\frac{r^2\alpha^2}{2K^2C}-\frac{2CrY(1+KC)}{B}\right], \tag{2.15}$$

where

$$\alpha = KC(B+Cr^2)^{-1/2}, \quad \beta = (A+\sqrt{B+Cr^2})^{-1}$$

$$Y = B[B+(1+KC)Cr^2]^{-1}.$$

The constants A, B, C, and K can be utilized to join the model to the Reissner–Nordstrom metric

$$ds^2 = -\left(1-\frac{2M}{r}+\frac{4\pi\varepsilon^2}{r^2}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2+\left(1-\frac{2M}{r}+\frac{4\pi\varepsilon^2}{r^2}\right)dt^2. \tag{2.16}$$

In this process, the continuity of g_{11}, g_{44}, F_{14} and vanishing of pressure p at the surface $r = R$ yield the following values of the constants:

$$A = e\left[1-\frac{R^2(1-m)-4\pi\varepsilon^2}{2R^2(m-x)}\right]\sqrt{m}, \tag{2.17}$$

$$B = \frac{x[R^2(1-m)-4\pi\varepsilon^2]^2}{4R^4(m-x)^2}, \tag{2.18}$$

$$C = \frac{[R^2(1-m)-4\pi\varepsilon^2]^2}{4R^6(m-x)}, \tag{2.19}$$

$$K = \frac{4R^6(1-m)}{[R^2(1-m)-4\pi\varepsilon^2]^2}, \tag{2.20}$$

where $e = 1$ or -1 according to $(m-x) > 0$ or < 0 and

$$x = 1 - \frac{8x\varepsilon^2}{MR}, \qquad m = 1 - \frac{2M}{R} + \frac{4\pi\varepsilon^2}{R^2}.$$

It is very easy to observe that the vanishing of charge ε makes the model reducible to Schwarzschild's interior solution.

The regularity conditions, viz. $p_{,r} = \varrho_{,r} = g_{11,r} = 0$ and $\varrho > 3p > 0$ are satisfied at $r = 0$, provided

$$MR > 8\pi\varepsilon^2 \text{ and } 3 > \frac{4C\sqrt{B}}{KC^2(A+\sqrt{B})} > 2. \tag{2.21}$$

Still, the behaviour of ϱ, p, E and J^4 is not known in the required region $0 \leqslant r \leqslant R$. To study it, the analytical methods are not that handy. Therefore, the said physical quantities have been computed numerically and thereafter the limitations on R, M and ε have been fixed.

3. *Numerical values of the physical quantities*

A careful survey of the expressions (2.12)–(2.15) along with (2.17)–(2.20) reveals that the expressions for PXR^2, DXR^2, EXR^2 and J^4XR^4 are functions of three dimensionless quantities, viz. $S \equiv (4\pi\varepsilon^2)/MR$, $Y = 2M/R$ and $X = r/R$. Now, let us investigate the limitations on S, Y and X:

(*i*) It is very clear that from the origin $r = 0$ to the surface $r = R$, X occupies the interval $0 \leqslant X \leqslant 1$.

(*ii*) The reality of g_{44} at the origin requires $B > 0$ and hence from (2.18) we get $x > 0$ and therefore $0 \leqslant S \leqslant .5$.

(*iii*) To ensure the reality of m in (2.17) and knowing that S cannot exceed the value 0.5, we immediately get $0 < Y < 1$.

Now, the values of PXR^2, DXR^2, EXR^2 and J^4XR^2 $(= J)$ have been computed numerically for the set cross product SXY by taking $S = 0$ (0.05).45 and $Y = 0(0.06)1.$ for the interval $X = 0.(0.1)1$. Besides this, some other values of S and Y have also been considered to get a better insight. Consequently, ϱ, p, E_1^1 and $\varrho - 3p$ were found positive and J^4 regular (reality conditions) in the following regions of the S–Y plane.

Behaviour of some physical quantities in the S–Y plane

S	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35
Y	0–0.54	0.12–0.6	0.24–0.63	0.36–0.66	0.48–0.72	0.6–0.78	0.72–0.84	0.9

On account of numerical computation of the values of various physical quantities such as p, ϱ, E and J^4 we come across the following observations:

TABLE I

Actual values of D, P, E and J for some values of S and Y

	$S = 0.0$ $Y = 0.54$				$S = 0.05$ $Y = 0.24$			
	DXR^2	PXR^2	EXR^2	$\mp J^4XR^2$	DXR^2	PXR^2	EXR^2	$\mp J^4XR^2$
0.0	1.620	0.594	0.000	0.000	0.780	0.036	0.000	0.085
0.2	1.620	0.482	0.000	0.000	0.774	0.034	0.0002	0.083
0.4	1.620	0.417	0.000	0.000	0.757	0.029	0.001	0.079
0.6	1.620	0.312	0.000	0.000	0.729	0.022	0.002	0.072
0.8	1.620	0.173	0.000	0.000	0.692	0.012	0.004	0.063
1.0	1.620	0.000	0.000	0.000	0.649	0.000	0.006	0.054
	$S = 0.1$		$Y = 0.42$		$S = 0.15$		$Y = 0.48$	
0	1.496	0.139	0.000	0.207	1.903	0.156	0.000	0.314
0.2	1.471	0.132	0.0009	0.200	1.848	0.147	0.002	0.298
0.4	1.398	0.110	0.004	0.179	1.697	0.119	0.008	0.255
0.6	1.289	0.782	0.008	0.149	1.485	0.813	0.016	0.199
0.8	1.157	0.399	0.014	0.116	1.253	0.396	0.026	0.144
1.0	1.016	0.000	0.021	0.083	1.030	0.000	0.036	0.968
	$S = 0.2$		$Y = 0.6$		$S = 0.25$		$Y = 0.72$	
0	2.700	0.308	0.000	0.530	3.780	0.635	0.000	0.918
0.2	2.580	0.284	0.004	0.490	3.535	0.573	0.006	0.819
0.4	2.268	0.222	0.015	0.392	2.935	0.423	0.024	0.595
0.6	1.867	0.143	0.029	0.277	2.241	0.250	0.048	0.369
0.8	1.469	0.650	0.046	0.177	1.633	0.105	0.072	0.205
1.0	1.128	0.000	0.060	0.104	1.170	0.000	0.090	0.104
	$S = 0.3$		$Y = 0.78$		$S = 0.35$		$Y = 0.9$	
0	4.973	0.771	0.000	1.355	7.425	2.164	0.000	2.901
0.2	4.501	0.679	0.013	1.159	6.394	1.790	0.010	2.275
0.4	3.444	0.469	0.044	0.759	4.373	1.066	0.054	1.217
0.6	2.395	0.256	0.079	0.419	2.697	0.490	0.110	0.539
0.8	1.604	0.986	0.104	0.213	1.643	0.162	0.146	0.228
1.0	1.076	0.000	0.117	0.105	1.030	0.000	0.158	0.098

1. Pressure p has maximum value at the centre and then decreases continuously to zero value at the surface.

2. Energy density ϱ has maximum value at the centre and decreases to a non-zero positive value at the surface. It is observed that, once $\varrho - 3p > 0$ at the centre, it continues to be so up to the surface.

3. Electrostatic energy density E has zero value at the centre and then increases continuously to a finite value at the surface. It is worth pointing out here that pressure equals electrostatic energy density only once in the region $0 \leqslant r \leqslant R$.

4. Charge density J^4 decreases continuously from a non-zero value to another non--zero value at the surface. Numerically, the charge density dominates pressure as well as electrostatic energy density throughout the region $0 \leqslant r \leqslant R$.

Besides the above observations, a better insight is achieved if one reads through the table of actual values of pressure, energy density etc. (Table I).

4. *Concluding remarks*

The authors have derived first ever charged analogue of Schwarzschild's interior solution, joining smoothly Reissner–Nordstrom solution. Moreover, the limitations on model radius, mass and charge have also been established successfully.

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