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NONMINIMAL DESCRIPTION OF SPIN 2

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The nonminimal description (with the help of 3-rd and 4-th rank Lorentz tensor) of the spin 2, equivalent to the Pauli-Fierz theory, is given. The variational principle is formulated.

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1. *Introduction*

To describe a massive boson with the definite spin (and parity) we use the field with the transformation property of a Lorentz tensor. Generally, the field has to obey some supplementary conditions projecting it on the definite value of spin because

(*i*) the spin spectrum of the Lorentz group representation $(k, l) \oplus (l, k)$ is $s = k+l, k+l-1, \ldots, |k-l|$,

(*ii*) certain number of such representations is needed to construct a field equation of the second order from which supplementary conditions result [1–4].

We can follow an uneconomical way and obtain an infinite number of descriptions of the spin s using Lorentz tensors with $s_{\max} > s$. As an example of such theory we can regard the Kemmer description of the spin 0 where the field is the Lorentz vector [5]. On the other hand, we usually deal with the economical description using Lorentz tensors with $s_{\max} = s$. In this case, the highest representations being contained in tensors are those with $k+l = s$. There are $s+1$ such representations

$$\left(\frac{s}{2}, \frac{s}{2}\right), \quad \left(\frac{s-1}{2}, \frac{s+1}{2}\right) \oplus \left(\frac{s+1}{2}, \frac{s-1}{2}\right), \ldots, \quad (0, s) \oplus (s, 0). \tag{1.1}$$

So, $s+1$ descriptions of the spin s are possible.

The formalism of Fierz [6] is an example of a realisation of such description. The Fierz tensors

$$A^{\mu_1 \ldots \mu_s}, \quad A^{\mu_1 \ldots \mu_{s-1}[\nu_1 \sigma_1]}, \ldots, A^{[\nu_1 \sigma_1] \ldots [\nu_s \sigma_s]} \tag{1.2}$$

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($[\nu_i\sigma_i]$ denotes antisymmetric pair of indices, the tensors are symmetric with respect to any permutation of indices μ_i and pairs $[\nu_i\sigma_i]$) contain the representations (1.1) (or their linear combinations) as the highest ones. The description using the symmetric tensor $A_{\mu_1\ldots\mu_s}$ including the highest representation $(s/2, s/2)$ we call the minimal one. We will refer to the others as to the nonminimal ones.

In the Fierz formalism supplementary conditions do not follow from field equations. This can be regarded as a defect of the formalism. The theory in which supplementary conditions are the consequence of the field equation, is more consistent. It allows one to formulate the variational principle and it can form the starting point for studing interactions. As an example of such a theory we can regard the Proca equation for the spin 1:

$$\Box A^\mu - \partial^\mu\partial_\nu A^\nu + m^2 A^\mu = 0 \tag{1.3}$$

from which there follows the condition $\partial_\mu A^\mu$, or the Pauli-Fierz equation for the spin 2 [1]. Both these theories make use of the minimal description. Ogievetsky and Polubarinov [7] have given the nonminimal theory for the spin 1. Their equation reads

$$\partial^\mu\partial_\sigma F^{\sigma\nu} - \partial^\nu\partial_\sigma F^{\sigma\mu} + m^2 F^{\mu\nu} = 0, \tag{1.4}$$

where $F_{\mu\nu} = -F_{\nu\mu}$. From Eq. (1.4) the condition $\varepsilon_{\alpha\beta\mu\nu}\partial^\beta F^{\mu\nu} = 0$ follows. The equivalence of the Eqs. (1.3) and (1.4) has been proved, too.

In the present paper we discuss the nonminimal descriptions of the spin 2, equivalent to the Pauli-Fierz theory. The Pauli-Fierz equation is favoured in the family of the spin 2 equations, because:

(*i*) it is free of ghosts and tachyons [8, 9],

(*ii*) in $m^2 \to 0$ limit it turns into the linearized Einstein equation.

We assume that the field equations in the nonminimal formulation are, as in the Pauli--Fierz case, of the second order.

In Section 2 it is shown that the description using tensor containing (1/2, 3/2) $\oplus$ (3/2, 1/2) as the highest representation, and the description with the help of tensor containing a combination (0,2) $\oplus$ (2,0) $\oplus$ (1,1) as the highest representation, are possible. The admixture of the (1,1) representation in the last case is necessery, since there exists no second-order equation based on the (0, 2) $\oplus$ (2,0) representation only[1]. In Section 3 the variational principle is formulated. It is shown that the different equivalent descriptions are connected by the Legendre transformation.

2. *The field equations*

Let us start with the Pauli-Fierz equation for the symmetric tensor $h^{\alpha\beta}$ transforming under the Lorentz group as (1, 1) $\oplus$ (0, 0) representation

$$\Box h^{\alpha\beta} - (\partial^\alpha h^\beta + \partial^\beta h^\alpha) + g^{\alpha\beta}\tilde{h} - (g^{\alpha\beta}\Box - \partial^\alpha\partial^\beta)h$$
$$+ m^2(h^{\alpha\beta} - g^{\alpha\beta}h) = 0, \tag{2.1}$$

[1] Equations of higher order have been proposed by Weinberg [10].

where $h \equiv h^\alpha_\alpha$, $h^\alpha \equiv \partial_\beta h^{\beta\alpha}$, $\tilde{h} \equiv \partial_\alpha \partial_\beta h^{\alpha\beta}$. From Eq. (2.1) the supplementary conditions result

$$h = 0, \tag{2.2}$$

$$h^\alpha = 0. \tag{2.3}$$

Putting these conditions into Eq. (2.1) we get

$$(\Box + m^2)h^{\alpha\beta} = 0.$$

So, the field $h^{\alpha\beta}$ has the mass m. The supplementary conditions restrict the number of spin variables to 5. Indeed, in the momentum space in the rest system ($p = (m, 0, 0, 0)$), the field $h^{\alpha\beta}(\vec{p})$ has only 5 independent components: $h^{ij} = h^{ji}$, $h^i_i = 0$ $(i, j = 1, 2, 3)$. So, the field $h^{\alpha\beta}$, obeying Eq. (2.1), carries the spin 2.

Let us describe the Pauli-Fierz equation in the form of a set of the two first-order equations

$$-m(h^{\beta\nu} - g^{\beta\nu}h) = \frac{1}{\sqrt{2}}(S^{\beta\nu} + S^{\nu\beta} + \partial^\beta S^\nu + \partial^\nu S^\beta - 2g^{\beta\nu}S) \tag{2.4a}$$

$$mS^{\alpha\beta\nu} = \frac{1}{\sqrt{2}}(\partial^\alpha h^{\beta\nu} - \partial^\beta h^{\alpha\nu}), \tag{2.4b}$$

where $S^{\alpha\beta\nu} = -S^{\beta\alpha\nu}$, $S^\alpha \equiv S^{\alpha\beta}{}_\beta$, $S^{\beta\nu} \equiv \partial_\alpha S^{\alpha\beta\nu}$. It follows from Eq. (2.4b) that $\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta\nu} = 0$, i.e. $S^{\alpha\beta\nu}$ transforms as (1/2, 3/2) ⊕ (3/2, 1/2) ⊕ (1/2, 1/2). The set (2.4) is unique up to the point transformation

$$S^{\alpha\beta\nu} \to S^{\alpha\beta\nu} + a(g^{\alpha\nu}S^\beta - g^{\beta\nu}S^\alpha), \quad a \neq -\tfrac{1}{3}$$

and to the scaling

$$S^{\alpha\beta\nu} \to \lambda S^{\alpha\beta\nu}.$$

Excluding $h^{\alpha\beta}$, we get from the set (2.4) the second-order equation for the field $S^{\alpha\beta\nu}$

$$\tfrac{1}{2}[\partial^\alpha(S^{\beta\nu} + S^{\nu\beta}) - \partial^\beta(S^{\alpha\nu} + S^{\nu\alpha})$$
$$+ \partial^\nu(\partial^\alpha S^\beta - \partial^\beta S^\alpha) + \tfrac{2}{3}(g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)S] + m^2 S^{\alpha\beta\nu} = 0, \tag{2.5}$$

where $S \equiv \partial_\alpha S^\alpha$. From Eq. (2.5), the following supplementary conditions result

$$\varepsilon_{\mu\nu\alpha\beta}S^{\alpha\beta\nu} = 0, \tag{2.6}$$

$$S = 0, \tag{2.7}$$

$$\varepsilon_{\mu\lambda\alpha\beta}\partial^\lambda S^{\alpha\beta\nu} = 0. \tag{2.8}$$

Inserting these conditions to Eq. (2.5), one gets

$$(\Box + m^2)S^{\alpha\beta\nu} = 0.$$

The tensor $S^{\alpha\beta\nu}$ obeying the conditions (2.6) and (2.7) transforms as (1/2, 3/2) ⊕ (3/2, 1,2). So, it has 16 components. The condition (2.8) gives 11 restrictions (15 equations minus 4 identities of Bianchi type). Therefore, only 5 components are independent. In the momentum space in the rest system these independent components are: $S^{0ij} = S^{0ji}$, $S^{0i}{}_i = 0$ $(i, j = 1, 2, 3)$. So, the field obeying Eq. (2.5), carries the spin 2.

Eq. (2.5) can be described in alternative way as the set

$$-mS^{\alpha\beta\nu} = \frac{1}{\sqrt{2}}(\partial^\alpha R^{\beta\nu}-\partial^\beta R^{\alpha\nu})+\frac{1}{6\sqrt{2}}(g^{\alpha\nu}\partial^\beta-g^{\beta\nu}\partial^\alpha)R, \tag{2.9a}$$

$$mR^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{2}}(\partial^\mu S^{\alpha\beta\nu}-\partial^\nu S^{\alpha\beta\mu}+\partial^\alpha S^{\mu\nu\beta}-\partial^\beta S^{\mu\nu\alpha}), \tag{2.9b}$$

where $R^{\beta\nu} \equiv R^{\alpha\beta}{}_\alpha{}^\nu$, $R \equiv R^\alpha{}_\alpha$, or as the set

$$-mS^{\alpha\beta\nu} = \sqrt{2}\,[2\theta\partial_\mu B^{\mu\nu\alpha\beta}+(1-\theta)(\partial^\alpha B^{\beta\nu}-\partial^\beta B^{\alpha\nu})$$
$$+\theta(g^{\alpha\nu}B^\beta-g^{\beta\nu}B^\alpha)+\tfrac{1}{9}(2-3\theta)(g^{\alpha\nu}\partial^\beta-g^{\beta\nu}\partial^\alpha)B], \tag{2.10a}$$

$$mB^{\mu\nu\alpha\beta} = \frac{1}{4\sqrt{2}}[g^{\mu\nu}(S^{\nu\beta}+S^{\beta\nu}+\partial^\nu S^\beta+\partial^\beta S^\nu)$$
$$+g^{\nu\beta}(S^{\mu\alpha}+S^{\alpha\mu}+\partial^\mu S^\alpha+\partial^\alpha S^\mu)-g^{\mu\beta}(S^{\nu\alpha}+S^{\alpha\nu}+\partial^\nu S^\alpha+\partial^\alpha S^\nu)$$
$$-g^{\nu\alpha}(S^{\mu\beta}+S^{\beta\mu}+\partial^\mu S^\beta+\partial^\beta S^\mu)], \tag{2.10b}$$

where $B^{\beta\nu} \equiv B^{\alpha\beta}{}_\alpha{}^\nu$, $B \equiv B^\alpha{}_\alpha$, $B^\alpha \equiv \partial_\beta B^{\beta\alpha}$ and θ is a parameter.

The set (2.9) is unique up to the point transformation

$$R^{\mu\nu\alpha\beta} \to R^{\mu\nu\alpha\beta}+A(g^{\mu\alpha}R^{\nu\beta}+g^{\nu\beta}R^{\mu\alpha}-g^{\mu\beta}R^{\nu\alpha}-g^{\nu\alpha}R^{\mu\beta})+B(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})R$$

with $1+2A \neq 0$ and $1+6A+12B \neq 0$, and to the scaling

$$R^{\mu\nu\alpha\beta} \to \lambda R^{\mu\nu\alpha\beta}.$$

The same is valid for the set (2.10)

(*i*) Let us discuss the set (12.9). From the definition of $R^{\mu\nu\alpha\beta}$ (Eq. (2.9b)) we see that it has the symmetry of the curvature tensor

$$R^{\mu\nu\alpha\beta} = -R^{\nu\mu\alpha\beta} = R^{\alpha\beta\mu\nu}, \quad \varepsilon_{\lambda\mu\nu\alpha}R^{\mu\nu\alpha\beta} = 0,$$

i.e. it transforms as (2, 0) ⊕ (0, 2) ⊕ (1, 1) ⊕ (0, 0).

From the set of Eqs. (2.9) we get the second-order equation for the field $R^{\mu\nu\alpha\beta}$

$$\partial^\mu\partial^\alpha R^{\nu\beta}+\partial^\nu\partial^\beta R^{\mu\alpha}-\partial^\mu\partial^\beta R^{\nu\alpha}-\partial^\nu\partial^\alpha R^{\mu\beta}$$
$$-\tfrac{1}{6}(g^{\mu\alpha}\partial^\nu\partial^\beta+g^{\nu\beta}\partial^\mu\partial^\alpha-g^{\mu\beta}\partial^\nu\partial^\alpha-g^{\nu\alpha}\partial^\mu\partial^\beta)R+m^2R^{\mu\nu\alpha\beta} = 0. \tag{2.11}$$

From Eq. (2.11) we obtain the supplementary conditions

$$\varepsilon_{\lambda\alpha\beta\nu}R^{\mu\nu\alpha\beta} = 0, \tag{2.12}$$

$$R = 0, \tag{2.13}$$

$$\partial_\beta R^{\beta\alpha} = 0, \tag{2.14}$$

$$\varepsilon_{\lambda\sigma\mu\nu}\partial^\sigma R^{\mu\nu\alpha\beta} = 0. \tag{2.15}$$

It can be verified that taking into account these conditions one gets from the field equation (2.11):

$$(\Box+m^2)R^{\mu\nu\alpha\beta} = 0. \tag{2.16}$$

In the momentum space in the rest system, the 5 independent components are: $R^{0i0j} = R^{0j0i}$ $R^{0i}{}_{0i} = 0$ $(i, j = 1, 2, 3)$. So, Eq. (2.11) describes the spin 2.

Let us discuss in some detail the supplementary conditions. The standard decomposition of the tensor $R^{\mu\nu\alpha\beta}$ is (see Appendix):

$$R^{\mu\nu\alpha\beta} = C^{\mu\nu\alpha\beta}+E^{\mu\nu\alpha\beta}+G^{\mu\nu\alpha\beta},$$

where $C^{\mu\nu\alpha\beta}$ is "the Weyl tensor" transforming as $(2, 0) \oplus (0, 2)$, $E^{\mu\nu\alpha\beta}$ is constructed from the traceless part of "the Ricci tensor" transforming as $(1, 1)$, $G^{\mu\nu\alpha\beta}$ describes "the scalar curvature" transforming as $(0, 0)$.

Eq. (2.16) (using the condition (2.13) and the dual properties of $C^{\mu\nu\alpha\beta}$ and $E^{\mu\nu\alpha\beta}$, see Appendix) reads:

$$(\Box+m^2)\begin{pmatrix}C^{\mu\nu\alpha\beta}\\ E^{\mu\nu\alpha\beta}\end{pmatrix} = 0.$$

The condition (2.15) can be rewritten as

$$\varepsilon_{\sigma\lambda\mu\nu}\partial^\lambda\partial_\alpha C^{\mu\nu\alpha\beta} = 0 \tag{2.17}$$

and

$$\partial_\sigma C^{\sigma\alpha\mu\nu} = \tfrac{1}{2}(\partial^\mu R^{\nu\alpha}-\partial^\nu R^{\mu\alpha}). \tag{2.18}$$

Eq. (2.17) restricts the number of independent components of $C^{\mu\nu\alpha\beta}$ to 5. The supplementary condition (2.18) connects these components to the ones of "the Ricci tensor". In the rest frame $C^{0i0j} = \frac{1}{2}R^{ij}$, $R^i{}_i = 0$. We see that it is impossible to describe the spin 2 with "the Weyl tensor" only. This result is general and invariant under the point transformations So, the representation $(2, 0) \oplus (0, 2)$ can be used as the highest one only in the combination with $(1, 1)$.

(*ii*) Let us study the system (2.10). Eleminating the variable $S^{\alpha\beta\nu}$ we obtain the equation for the field $B^{\mu\nu\alpha\beta} = -B^{\nu\mu\alpha\beta} = B^{\alpha\beta\mu\nu}$:

$$\theta(g^{\mu\alpha}\tilde{B}^{\nu\beta}+g^{\nu\beta}\tilde{B}^{\mu\alpha}-g^{\mu\beta}\tilde{B}^{\nu\alpha}-g^{\nu\alpha}\tilde{B}^{\mu\beta})$$
$$+\tfrac{1}{2}(1-\theta)\Box(g^{\mu\alpha}B^{\nu\beta}+g^{\nu\beta}B^{\mu\alpha}-g^{\mu\beta}B^{\nu\alpha}-g^{\nu\alpha}B^{\mu\beta})$$

$$-\tfrac{1}{2}(1-\theta)\,[g^{\mu\nu}(\partial^\nu B^\beta+\partial^\beta B^\nu)+g^{\nu\beta}(\partial^\mu B^\alpha+\partial^\alpha B^\mu)$$
$$-g^{\mu\beta}(\partial^\nu B^\alpha+\partial^\alpha B^\nu)-g^{\nu\alpha}(\partial^\mu B^\beta+\partial^\beta B^\mu)]$$
$$-\theta(g^{\mu\nu}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\tilde{B}-\tfrac{1}{9}(2-3\theta)\,(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\Box B$$
$$+\tfrac{1}{18}(5-3\theta)\,(g^{\mu\alpha}\partial^\nu\partial^\beta+g^{\nu\beta}\partial^\mu\partial^\alpha$$
$$-g^{\mu\beta}\partial^\nu\partial^\alpha-g^{\nu\alpha}\partial^\mu\partial^\beta)B+m^2B^{\mu\nu\alpha\beta}=0, \tag{2.19}$$

where $\tilde{B}^{\nu\beta}\equiv\partial_\mu\partial_\alpha B^{\mu\nu\alpha\beta}$, $\tilde{B}\equiv\tilde{B}^\alpha{}_\alpha$. From Eq. (2.19) we get the supplementary conditions:

$$\varepsilon_{\lambda\mu\nu\alpha}B^{\lambda\mu\nu\alpha}=0, \tag{2.20}$$

$$B=0, \tag{2.21}$$

$$B^\alpha=0, \tag{2.22}$$

$$B^{\mu\nu\alpha\beta}=\tfrac{1}{2}(g^{\mu\alpha}B^{\nu\beta}+g^{\nu\beta}B^{\mu\alpha}-g^{\mu\beta}\beta^{\nu\alpha}-g^{\nu\alpha}B^{\mu\beta})$$
$$-\tfrac{1}{6}(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})B \tag{2.23}$$

independently of the value of the parameter θ. It follows from Eq. (2.23) that the field variable is actually the symmetric tensor $B^{\nu\beta}$. The equation for $B^{\nu\beta}$ reads:

$$\Box B^{\nu\beta}-(\partial^\nu B^\beta+\partial^\beta B^\nu)-g^{\nu\beta}B+\tfrac{5}{4}\partial^\nu\partial^\beta B+\tfrac{1}{9}g^{\nu\beta}\Box B+m^2B^{\nu\beta}=0. \tag{2.24}$$

It turns into the Pauli-Fierz equation after the point transformation

$$B^{\nu\beta}\to B^{\nu\beta}+2g^{\nu\beta}B.$$

Let us make the final remark about the field equation (2.11). It can be obtained directly from the Pauli-Fierz equation written as the set:

$$-(h^{\nu\beta}-g^{\nu\beta}h)=2(R^{\nu\beta}-\tfrac{1}{2}g^{\nu\beta}R),$$
$$m^2R^{\mu\nu\alpha\beta}=\tfrac{1}{2}(\partial^\mu\partial^\alpha h^{\nu\beta}+\partial^\nu\partial^\beta h^{\mu\alpha}$$
$$-\partial^\mu\partial^\beta h^{\nu\alpha}-\partial^\nu\partial^\alpha h^{\mu\beta}).$$

3. *The variational principle*

We start with the action

$$I=\int dx\left\{-\frac{1}{\sqrt{2}}mS_{\alpha\beta\nu}[\partial^\alpha h^{\beta\nu}-\partial^\beta h^{\alpha\nu}\right.$$
$$-(g^{\alpha\nu}h^\beta-g^{\beta\nu}h^\alpha)+(g^{\alpha\nu}\partial^\beta-g^{\beta\nu}\partial^\alpha)h]$$
$$\left.+\tfrac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu}-h^2)+\tfrac{1}{2}m^2(S_{\alpha\beta\nu}S^{\alpha\beta\nu}-2S_\alpha S^\alpha)\right\}. \tag{3.1}$$

From $\delta I = 0$ we obtain the set (2.4) which is equivalent to Eq. (2.1) and to Eq. (2.5). To obtain the action in term of $h^{\alpha\beta}$ only, we aliminate the field $S^{\alpha\beta\nu}$ ($S^{\alpha\beta\nu}$ is a Lagrange multiplier) from the action (3.1) using Eq. (2.4b). We get the action for the Pauli-Fierz theory:

$$I = \int dx\{-\tfrac{1}{2}[(\partial^\lambda h^{\mu\nu})^2 + 2h_\mu \partial^\mu h - 2(h^\mu)^2 - (\partial^\mu h)^2] + \tfrac{1}{2} m^2[(h^{\mu\nu})^2 - h^2]\}. \tag{3.2}$$

Performing integration by parts in the action (3.1) we convert $h^{\alpha\beta}$ into Lagrange multiplier that can be removed using Eq. (2.4a). The action in term of the field $S^{\alpha\beta\nu}$ reads

$$\begin{aligned} I = \int dx[&-\tfrac{1}{2} S_{\beta\nu}(S^{\beta\nu}+S^{\nu\beta}) - \tfrac{1}{2}(\partial_\beta S_\nu)^2 + \tfrac{5}{6} S^2 - S_{\beta\nu}\partial^\nu S^\beta \\ &+\tfrac{1}{2} m^2(S_{\alpha\beta\nu}S^{\alpha\beta\nu} - 2S_\alpha S^\alpha)]. \end{aligned} \tag{3.3}$$

This action is equivalent to

$$\begin{aligned} I = \int dx[&-\sqrt{2}\, m(B_{\mu\nu\alpha\beta}\partial^\mu S^{\alpha\beta\nu} - \tfrac{2}{9} BS) \\ &-\sqrt{2}\, m(\tfrac{1}{2} R_{\nu\beta}S^{\nu\beta} + \tfrac{1}{2} R_{\nu\beta}\partial^\nu S^\beta - \tfrac{1}{3} RS) \\ &+\tfrac{1}{2} m^2(S_{\alpha\beta\nu}S^{\alpha\beta\nu} - 2S_\alpha S^\alpha) + \tfrac{1}{3} m^2(R_{\mu\nu\alpha\beta}B^{\mu\nu\alpha\beta} - \tfrac{2}{9} RB)], \end{aligned} \tag{3.4}$$

where $B^{\mu\nu\alpha\beta}$ and $R^{\mu\nu\alpha\beta}$ are Lagrange multipliers. The field equations are given by Eqs. (2.9b), (2.10b) and

$$\begin{aligned} -mS^{\alpha\beta\nu} = \sqrt{2}\,[&\partial_\sigma B^{\sigma\nu\alpha\beta} + \tfrac{1}{2}(g^{\alpha\nu}B^\beta - g^{\beta\nu}B^\alpha) \\ &-\tfrac{1}{18}(g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)B + \tfrac{1}{4}(\partial^\alpha R^{\beta\nu} - \partial^\beta R^{\alpha\nu}) + \tfrac{1}{24}(g^{\alpha\nu}\partial^\beta - g^{\beta\nu}\partial^\alpha)R]. \end{aligned} \tag{3.5}$$

Eliminating $B^{\mu\nu\alpha\beta}$ and $R^{\mu\nu\alpha\beta}$ from the action (3.4) (using Eqs. (2.9b) and (2.10b)) we obtain the action (3.3). Performing integration by parts in the action (3.4) and eliminating $S^{\alpha\beta\nu}$, we get

$$\begin{aligned} I = \int dx\{&-[(\partial_\sigma B^{\sigma\nu\alpha\beta})^2 + \tfrac{2}{9} B_\alpha \partial^\alpha B - (B^\alpha)^2 \\ &-\tfrac{1}{27}(\partial^\alpha B)^2 + \tfrac{1}{8}(\partial^\alpha R^{\beta\nu})^2 - \tfrac{1}{4}(R^\alpha)^2 \\ &+\tfrac{1}{6} R^\alpha \partial_\alpha R - \tfrac{1}{16}(\partial^\alpha R)^2 + \partial_\sigma B^{\sigma\nu\alpha\beta}\partial_\alpha R_{\beta\nu} \\ &-\tfrac{1}{6}\partial_\alpha R B^\alpha + \tfrac{1}{9} R^\alpha \partial_\alpha B - \tfrac{1}{18}\partial_\alpha R \partial^\beta B] \\ &+\tfrac{1}{2} m^2(R_{\alpha\beta\mu\nu}B^{\alpha\beta\mu\nu} - \tfrac{2}{9} RB)\}. \end{aligned} \tag{3.6}$$

From this action we obtain the system of the second-order equations, from which the relations

$$\begin{aligned} B^{\mu\nu\alpha\beta} = &\tfrac{1}{2}(g^{\mu\alpha}B^{\nu\beta} + g^{\nu\beta}B^{\mu\alpha} - g^{\mu\beta}B^{\nu\alpha} - g^{\nu\alpha}B^{\mu\beta}) \\ &-\tfrac{1}{6}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})B \end{aligned} \tag{3.7}$$

and

$$B^{\nu\beta} - \tfrac{1}{2} R^{\nu\beta} + \tfrac{1}{4} g^{\nu\beta} R = 0 \tag{3.8}$$

result. With the help of these relations one can reduce the system to two independent equations: (2.11) and (2.24). We note that owing to Eq. (3.8) the action (3.6) describes only one spin 2.

From the construction of the actions (3.1) and (3.4), it follows that the different but equivalent descriptions are connected by the Legendre transformation.

4. *Final remarks*

We have obtained two nonminimal descriptions equivalent to the minimal one of Pauli and Fierz. It is well known that theories equivalent for $m \neq 0$ need not be equivalent in the $m = 0$ limit [7, 11, 12]. The analysis of the zero mass limit of the equations obtained in the present paper will be given elsewhere.

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APPENDIX

The decomposition of the tensor $R^{\mu\nu\alpha\beta}$ in the irreducible parts is:

$$R^{\mu\nu\alpha\beta} = C^{\mu\nu\alpha\beta} + E^{\mu\nu\alpha\beta} + G^{\mu\nu\alpha\beta}$$

where

$$C^{\mu\nu\alpha\beta} = R^{\mu\nu\alpha\beta} - \tfrac{1}{2}(g^{\mu\alpha}R^{\nu\beta} + g^{\nu\beta}R^{\mu\alpha} - g^{\mu\beta}R^{\nu\alpha} - g^{\nu\alpha}R^{\mu\beta}) + \tfrac{1}{6}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})R,$$

$$E^{\mu\nu\alpha\beta} = \tfrac{1}{2}(g^{\mu\alpha}R^{\nu\beta} + g^{\nu\beta}R^{\mu\alpha} - g^{\mu\beta}R^{\nu\alpha} - g^{\nu\alpha}R^{\mu\beta}) - \tfrac{1}{4}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})R,$$

$$G^{\mu\nu\alpha\beta} = \tfrac{1}{12}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})R.$$

The dual properties of these parts are:

$$^{\sim}C^{\mu\nu\alpha\beta} = C^{\sim\mu\nu\alpha\beta},$$

$$^{\sim}E^{\mu\nu\alpha\beta} = -E^{\sim\mu\nu\alpha\beta},$$

$$^{\sim}G^{\mu\nu\alpha\beta} = G^{\sim\mu\nu\alpha\beta},$$

where left-handed and right-handed dual tensors are respectively:

$$^{\sim}A^{\mu\nu\alpha\beta} \equiv \tfrac{1}{2}\varepsilon^{\mu\nu}{}_{\sigma\lambda}A^{\sigma\lambda\alpha\beta},$$

$$A^{\sim\mu\nu\alpha\beta} \equiv \tfrac{1}{2}\varepsilon^{\alpha\beta}{}_{\sigma\lambda}A^{\mu\nu\sigma\lambda}.$$

REFERENCES

[1] M. Fierz, W. Pauli, *Proc. Roy. Soc.* **173A**, 211 (1939).
[2] R. Rivers, *Nuovo Cimento* **34**, 386 (1964).
[3] V. Ogievetsky, I. Polubarinov, *Ann. Phys.* (USA) **35**, 167 (1965).
[4] J. Rembieliński, *J. Phys. A* **13**, 3619 (1980).
[5] N. Kemmer, *Helv. Phys. Acta* **33**, 829 (1960).
[6] M. Fierz, *Helv. Phys. Acta* **12**, 1 (1939).
[7] V. Ogievetsky, I. Polubarinov, *Yad. Fiz.* **4**, 216 (1966).
[8] P. van Nieuwenhuizen, *Nucl. Phys.* **B60**, 478 (1973).
[9] P. van Nieuwenhuizen, *Phys. Rep.* **68**, 189 (1981).
[10] S. Weinberg, *Phys. Rev.* **133B**, 1318 (1964).
[11] T. Cukierda, J. Lukierski, Preprint 192, Inst. Theor. Phys., University of Wrocław 1969.
[12] S. Deser, P. K. Townsend, W. Siegel, *Nucl. Phys.* **B184**, 333 (1981).