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BACKGROUND TO THE LOW MASS DILEPTON PRODUCTION FROM QCD PLASMA IN ION-ION COLLISIONS*

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The production of low-mass dileptons in ion-ion collisions is estimated under the assumption that this process is a "sum" of nucleon-nucleon collisions. The resulting estimate of dilepton production can be considered as a background for dileptons originated by a true quark-gluon plasma.

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1. *Introduction*

The quark-gluon plasma when discovered will provide both a confirmation of basic ideas of quantum chromodynamics (QCD) and a new state of matter worth of detailed study. Since the QCD plasma can be formed only in ion-ion collisions [1–4] leading anyway to complicated final states, the question of signatures of the plasma formation is a nontrivial one. One of the expected signatures is the production of low mass (less than about 2 GeV/c^2) dileptons [5–11]. In order to understand whether the dileptons from the QCD plasma are really observed it is necessary to have some understanding of the expected background corresponding to collisions with no QCD plasma being formed. At present we are still far from a complete theory of the collision of heavy ions at high energies but it seems that there are models [12–15] which describe successfully the multiparticle production in heavy ion collisions without assuming that the QCD plasma is formed in the intermediate stages. In these models the hadrons observed in the final state originate from "strings" which have properties similar to strings formed in nucleon-nucleon and e^+e^- collisions. In order to estimate the low mass dilepton in "no-plasma" heavy ion collisions we shall take one of these models, namely [12] and assume that the dilepton production from a single string is the same as in a nucleon-nucleon collision. Summing up contributions from different strings we shall obtain in this way the estimate of the background to plasma formation

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in ion-ion collisions. We believe that this is the lower estimate. The following argument provides perhaps some support for this opinion. Suppose that we are increasing both the nucleon number of the colliding ions and their energy. At the beginning the ion-ion collision can be viewed as a "sum" of strings originated by individual nucleon-nucleon collisions. The density of strings gradually increases and the strings start to overlap. In the overlap regions the energy density becomes higher and the matter approaches the stage of plasma formation. Even before the plasma is formed the production of dileptons increases because of higher densities of quarks and antiquarks. When the plasma forms in larger regions the dilepton production increases further. In our estimates of the dilepton production we shall take into account no overlap effects of the strings and in that sense the estimate is rather a lower bound.

In order to estimate the dilepton production as coming from separated strings similar to those in nucleon-nucleon collision we have to start from a pragmatically successful model of low mass dilepton production. We shall consider two models [16, 17]. The former [16] is based on the idea [18] that low mass dileptons are originated by annihilations of quarks (Q's) and antiquarks ($\overline{Q}$'s) created during the evolution of the collision. The latter assumes[1] that the quark-gluon plasma is formed already during the hadronic collision [17].

The paper is organized as follows. In Sect. 2 we discuss briefly both models [16, 17] of dilepton production in hadron-hadron collisions and try to find out why they lead to similar conclusions, although starting from different pictures of the collision. The estimate of the background for dilepton production in heavy ion collisions will be presented in Sect. 3 where we shall also compare the background with estimates of the dilepton production from the QCD plasma. Section 4 contains comments to what features of the dilepton spectrum are most useful as a signal of the plasma formation and some conclusions.

2. *Models of low mass dilepton production in nucleon-nucleon collisions*

We shall discuss here the soft annihilation [16] and the thermodynamic [17] models.

The soft annihilation model

We shall describe only the qualitative features of the model using pictures, partly borrowed from Bjorken's article [19] on the space-time evolution of hadronic collisions. Technicalities of the soft annihilation model can be found in [16]. In this model the low mass dileptons are produced by annihilations of quarks and antiquarks created during the hadronic collision. The creation of Q's and $\overline{Q}$'s follows the space time evolution of the collision. In Fig. 1a we have the two hadrons A and B just before the collision and in Fig. 1b just when the interaction of wee partons around $y = 0$ (in the c.m.s.) starts. The interaction of wee partons reaches at $t_0 \approx 1$ fm/c the region with $\Delta y \approx 1$ shown in Fig. 2a. With the increasing time the "burning region" shifts to higher rapidities (Fig. 2b). The space-time evolution of the "burning region" is shown in Fig. 3.

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[1] Note that this assumption is not essential for our purpose; it is only important that the model gives a good description of dilepton production in nucleon-nucleon collisions.

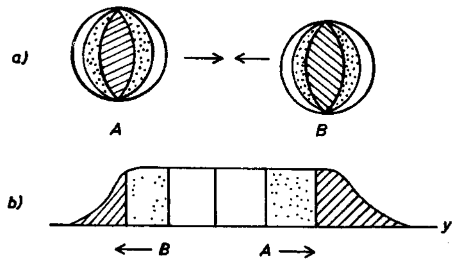

Fig. 1. Colliding hadrons A, B before the collision: a) in the x-space, b) in rapidity. Dashed regions contain hard partons, dotted regions somewhat softer partons and empty regions correspond to wee partons

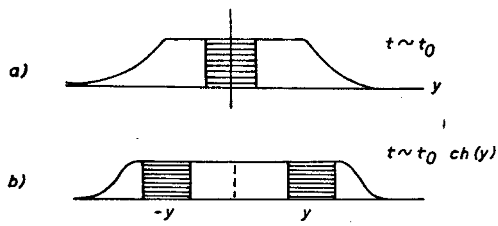

Fig. 2. The position of the "burning region": a) at $t_0 \approx 1$ fm/c, b) at $t \approx t_0\text{ch}(y)$

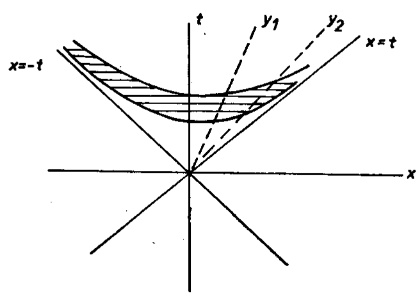

Fig. 3. The space time evolution of a collision in the $x-t$ diagram. Region with y_2 "burns" later than region around y_1

During the "burning stage" the gluons present in a given rapidity region are converted to Q's and $\overline{\text{Q}}$'s and they finally recombine to hadrons observed in the final state.

During the "burning" the density of Q's is increased by about a factor of 5. This is seen simply from the fact that gluons in a free nucleon carry about five times more momentum than sea quarks and both have rather similar x-distributions. Another argument for this factor comes from the comparison of rapidity density of quarks in colliding hadrons before the collision (about 0.6 per rapidity unit) with the rapidity density of quarks contained in final state hadrons. Since there are about three hadrons per rapidity unit, most of them being pions, we have about 3 quarks per rapidity unit (without taking into account the colour or flavour). The factor 5 is just the ratio of 3 to 0.6.

At a given time t quarks and antiquarks created during the collision are present in the

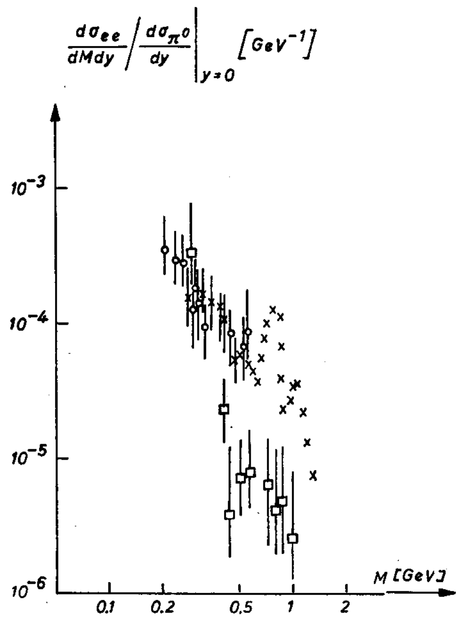

Fig. 4. The mass-spectrum of low mass dileptons produced in hadronic collisions, ○ — e^+e^- from π^-p collisions at 16 and 17 GeV/c [20, 21], × — $\mu^+\mu^-$ from π^-N collisions at 225 GeV/c [22], □ — e^+e^- from pN interactions at 13 GeV/c [23]

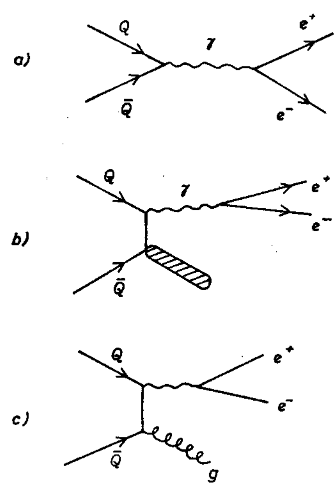

Fig. 5. a) The simple annihilation diagram for $Q\overline{Q} \to e^+e^-$; b) the annihilation with "something else" in the final state; c) the annihilation with the production of a "gluon"

"burning region" (Fig. 2) within a volume $V = V_0/\text{ch}\, y$ and the region "burns" for a time $t = t_0 \,\text{ch}\, y$. Since the annihilation rate of quarks and antiquarks is proportional to the product of densities $n_Q n_{\bar{Q}}$ and they are both by a factor of 5 higher than in free nucleons

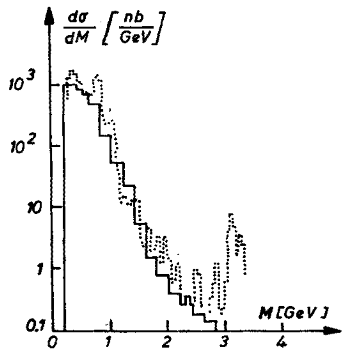

Fig. 6. The mass dependence of the dilepton production cross-section calculated according to the soft annihilation model [16] (solid histogram) compared with the data [22]

we expect that the low mass dilepton production will be higher by a factor of about 25 than the extrapolation of Drell-Yan cross-section to low dilepton masses.

This picture of low-mass dilepton production leads to a testable consequence: The shape of the mass spectrum of low mass dileptons is independent of energy of the collision. This is due to the assumed origin of dileptons from a "burning" region and the assumed independence of the properties of this region (in its rest frame) on the energy of colliding hadrons. The data (Fig. 4) seem to confirm that, although more accurate data would be very valuable. The p_T-distributions of low mass dileptons and single leptons are given by the p_T-distributions of quarks and antiquarks and by the annihilation mechanism. The former are chosen in a way which leads to empirical p_T-distributions of hadrons produced by the recombination of quarks and antiquarks. We have taken [16] the p_T-distributions of quarks and antiquarks as proportional to $\exp(-p_T^2/R_t^2)$ with $R^2 = 0.20$ GeV/c^2. The increase of lepton/pion ratio at low p_T [24] shows that apart of simple annihilation $Q\overline{Q} \to l^+l^-$ there is present also the annihilation with something else in the final state (Fig. 5a, b). Faute de mieux we have replaced something else by a gluon (Fig. 5c).

The model leads to a reasonable agreement [16] with the data on low mass dilepton production. For details of the model and for comparison with the data the reader is referred to [16]. In Fig. 6 we reproduce the comparison with the data for the mass dependence of the dilepton production cross-section.

Hydrodynamical model

The first quantitative estimate of the dilepton production within the hydrodynamical model with quark-gluon plasma was performed by Shuryak [17]. According to this model, the cross section for the dilepton production is

$$\sigma_{l^+l^-} = \sigma_{\text{inel}} \int_{T_i}^{T_f} dT\, W_{l^+l^-}(T)\phi(T), \tag{1}$$

where σ_{inel} is the inelastic hadronic cross-section, T_i and T_f are the initial and the final temperatures, $W_{l^+l^-}(T)$ is the rate per unit time and unit volume of the dilepton production and $\phi(T)$ is the size of the space-time region with temperature T during the whole collision

$$\phi(T) = \int d^4x\delta[T(x, t)-T]. \tag{2}$$

For an isentropic expansion it holds

$$\phi(T) \sim \frac{1}{T^7}. \tag{3}$$

Since Eq. (3) will be important for the following discussion we shall sketch its derivation. Let us consider the central region of a pp collision in the c.m. system: $y \approx 0$. Instead of variables x_L, t specifying a given point in the evolution of the collision we use the rapidity y and the proper time τ:

$$t = \tau \operatorname{ch} y \qquad x_L = \tau \operatorname{sh} y. \tag{4}$$

For $y \approx 0$ $x_L \approx \tau y$. In an isentropic expansion the rapidity density of entropy is constant $dS/dy \approx$ const. Because of $x_L \approx \tau y$ we have $dS/dx_L \approx (1/\tau)dS/dy$. Hence

$$\frac{dS}{dx_L d^2x_T} \approx \frac{1}{\tau}\frac{1}{\pi R^2}\frac{dS}{dy}, \tag{5}$$

where πR^2 is transverse dimension of the region with plasma. The spatial density of the entropy of relativistic gas is proportional to T^3, what is easiest to see on a black-body radiation. Because of that we have from Eq. (5)

$$\frac{1}{\tau}\left[\frac{1}{\pi R^2}\frac{dS}{dy}\right] \sim T^3. \tag{6}$$

The term in brackets is constant and therefore

$$T(y, \tau) \sim \tau^{-1/3}, \tag{7}$$

where we assumed y-independent character of initial conditions and of the whole evolution of the collision. In the expression (2) for $\phi(T)$ we first write $d^4x = d^2x_\perp \tau dy d\tau$ and insert (7) into the argument of the δ-function. In this way we obtain

$$\phi(T) = \int d^2x_\perp \int dy \int \tau d\tau \delta[b\tau^{-1/3}-T] = \frac{1}{B}\pi R^2 Y T^{-7} = AT^{-7}, \tag{8}$$

where $A = B^{-1}$ is a constant and Y is the total rapidity length. The constant A fixes the cross-section for the dilepton production, the rest of the Eq. (1) contains no free parameters. As follows from the preceding discussion the parameter A is calculable and can be expressed in terms of the rapidity density of entropy dS/dy. Shuryak [17] found a very good agreement of the mass dependence of the dilepton production cross-section with the data (Fig. 7).

At the first sight one might be surprised that both the soft-annihilation and hydrodynamical models describe the data reasonably well, although both start from different assumptions. In fact the analogy of the results is not very surprising. In the Shuryak's hydrodynamical model the factor T^{-7} in $\phi(T)$ leads to the dominance of low temperatures in the dilepton production. The rate of the dilepton production $dW_{l^+l^-}(T)/dM$ contains the factor [17] $T^{3/2} \exp(-M/T)$ where M is the dilepton mass and $\exp(-M/T)$ is essentially the Boltzmann factor. The expression $T^{3/2-7} \exp(-M/T)$ has the maximum at $T_M = M/(7-3/2) = M/5.5$. For low dilepton masses around 1 GeV we obtain $T_M \approx 0.18$ GeV.

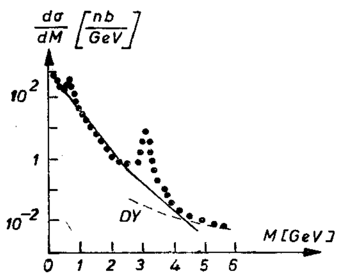

Fig. 7. The comparison of Shuryak's calculations (solid line) for $d\sigma/dM$ with the data

On the other hand it is well known that the experimental p_T spectra of the secondary hadrons in pp collisions are reasonably described by a Boltzmann factor $\exp(-E_\perp/T)$ with $T \approx 0.15$ GeV. In the soft annihilation model [16] we have used p_T-distributions of quarks and antiquarks which lead to these p_T-distributions of hadrons formed by recombination. Because of that these distributions are also reasonably described by Boltzmann factors corresponding to similar temperatures. In both models thus dominates the break-up stage, in the hydrodynamical one it is because of the factor T^{-7} and in the soft-annihilation model only the break-up stage is considered because the distributions of quarks and antiquarks are such that distributions of hadrons formed by recombination describe well the data.

3. *Dilepton production in heavy ion collisions*

It is frequently assumed that the dilepton production may serve as a signature for the plasma formation in heavy ion collisions. This is correct, provided that we understand the dilepton production in events where no plasma is formed and see a signal which is larger than this mundane background. In this section we shall discuss just this background.

At present there is a couple of models [12–15] of heavy ion collisions which are qualitatively consistent with rather limited available data. These models describe the ion-ion collisions as a sum of individual nucleon-nucleon or quark-quark collisions. In collisions of constituents "strings" are formed with properties similar to "strings" in e^+e^- or pp collisions. As an illustration we show in Fig. 8 the formation of strings in a model of Białas, Czyż and Leśniak [12]. The average number of strings formed in the central region of an $^{16}O + ^{208}Pb$ collisions at 200 GeV/nucleon is about 15 [12].

In the soft annihilation model the cross-section for the dilepton production can be written in a schematic way as follows

$$\sigma_{ll}^{AB} = \sigma_{inel}^{AB} \int P_{AB}(M, y, p_T) dM dy d^2 p_T, \tag{9}$$

where σ_{inel}^{AB} is the cross-section for the AB collision and $P(M, y, p_T)$ is the probability density for the production of a dilepton with mass M, rapidity y and transverse momentum p_T. With N_{AB} independent strings we have

$$P_{AB}(M, y, p_T) = N_{AB} P_{pp}(M, y, p_T), \tag{10}$$

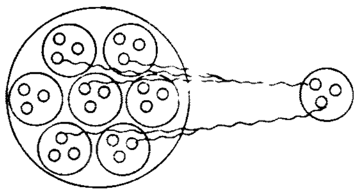

Fig. 8. The formation of strings in an ion-ion collision according to the model [12]

where $P_{pp}(M, y, p_T)$ is the probability for dilepton production from one string formed in a typical pp collision.

Taking the ratio of (9) to a similar equation with A = p, B = p we obtain after averaging over the number of strings

$$\sigma_{ll}^{AB} = \frac{\langle N_{AB} \rangle \sigma_{inel}^{AB}}{\sigma_{inel}^{pp}} \sigma_{ll}^{pp}. \tag{11}$$

This holds also for differential cross-sections of the dilepton production like $d\sigma/dy\, dM$ in the central region

$$\frac{d\sigma_{ll}^{AB}}{dM dy} = \frac{\langle N_{AB} \rangle \sigma_{inel}^{AB}}{\sigma_{inel}^{pp}} \frac{d\sigma_{ll}^{pp}}{dM dy}. \tag{12}$$

A crucial and simple property of these background dileptons is the same mass dependence of the cross-section as in a pp collision. Any deviation from this rule indicates that the dileptons are not produced from "strings" of the same type as in a pp collision. These deviations need not come from a thermalised plasma. They might be due to the overlapping of strings. To make this point in more detail, note that in the soft annihilation model [16] the fast decrease of the $d\sigma_{ll}^{pp}/dM$ with increasing M is due to the finite extension in rapidity of the "burning region". The mass of the dilepton obtained by annihilation of a quark with $y_1, \vec{p}_{T1}$ and an antiquark with $y_2, \vec{p}_{T2}$ is

$$M^2 = 2m_Q^2[1+\text{ch}\,(y_1-y_2)]-(\vec{p}_{T1}+\vec{p}_{T2})^2.$$

The cut on possible $|y_1-y_2|$ is also a cut on the possible dilepton mass. Such a cut is built into the soft annihilation model [16] and characterizes the space-time evolution of the process. If strings start to overlap (in events with a large number of strings) the space-time evolution might change, the strings might burn slower and a larger region in rapidity would be excited simultaneously. As a consequence the $d\sigma/dM$ would start increasing at large values of M.

The value of the numerical factor in Eq. (11) for $^{16}O + ^{208}Pb$ collision is roughly $\langle N_{AB} \rangle (A^{1/3}+B^{1/3})^2 \approx 15 \times 71 \approx 1070$ so that we have

$$\sigma_{ll}^{O+Pb} \approx 1070 \sigma_{ll}^{pp}. \tag{13}$$

In the low mass region σ_{ll}^{pp} is about 25 times larger than the Drell-Yan cross-section extrapolated to low masses. This is most natural since the densities of quarks and antiquarks increased by a factor of about 5. The Drell-Yan cross-section for an A+B collision is $\sigma_{DY}^{AB} = AB\sigma_{DY}^{pp}$.In this way we have

$$\sigma_{ll}^{O+Pb} \approx 1070 \times 25 \sigma_{DY}^{pp} \approx \frac{1075 \times 25}{16 \times 208} \sigma_{DY}^{O+Pb} \sim 8 \sigma_{DY}^{O+Pb}.$$

The background low mass dileptons are therefore expected to be about an order of magnitude above the Drell-Yan contribution extrapolated to low masses.

4. Comparison of the background dileptons with dileptons originated by the quark-gluon plasma

The most detailed calculation of properties of dileptons produced during the hydrodynamical evolution of the quark-gluon plasma has been recently performed by Hwa and

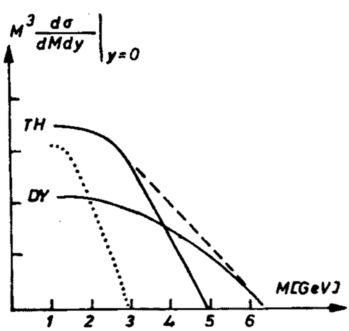

Fig. 9. Plasma dilepton rate (TH) compared with Drell-Yan extrapolated to lower masses (DY), dashed line represents the expected preequilibrium emission [12]. The "background" from no-plasma collisions — dotted line (present estimate)

Kajantie [10]. Their results are summarized in Fig. 9. The most important qualitative features of their results are:

(*i*) The dilepton continuum extends to masses of 5–6 GeV, at $M \approx 1$ GeV it is about an order of magnitude higher than the DY extrapolated to low masses and at $M \approx 4$–5 GeV it is comparable with the DY contribution.

(*ii*) The assumption of the isentropic expansion of the plasma leads to the correlation between the rapidity density of dileptons dN_{ll}/dy and of pions dN_π/dy:

$$\frac{dN_{ll}}{dy} \sim \left(\frac{dN_\pi}{dy}\right)^2. \tag{14}$$

These features of the plasma dileptons differ in some aspects from the background dileptons and these differences might permit to disentangle the signal due to plasma from the background.

The $d\sigma/dM$ for the background extends to lower masses (about 2 GeV, see Fig. 6) than for the plasma dileptons. The shape of this background is the same as $d\sigma/dM$ for a hadron-hadron collision. If the dileptons produced in heavy ion collisions will come from unseparated no-plasma and plasma events the shape of the $d\sigma/dM$ at low masses will be roughly as indicated in Fig. 9. Since the shape of the no-plasma contribution is known it can be subtracted. It may well happen that even the individual events will contain contributions both from the strings and from the plasma, but the separation can be performed in the same way even in this case.

The situation might be, however, more complicated if the events contain contributions from individual strings, mixed with contributions from overlapping strings (extending to higher values of M) and with true plasma events.

The relation (14) valid for isentropically expanding plasma is also valid for an individual string. The dilepton production is proportional to the product (dn_Q/dy) $(dn_{\bar{Q}}/dy)$ of quark and antiquark rapidity densities whereas the densities of mesons produced by recombination is proportional to dn_Q/dy (more details can be found in [25]). For many non-overlapping strings the relation (14) will not be exactly valid, but a detailed calculation has yet to be performed.

5. *Comments and conclusions*

The dilepton production remains one of the most suitable candidates for diagnosing the quark-gluon plasma. The background to low mass dileptons may cause however serious complications. It is about of the same order of magnitude as the expected signal from dileptons originated by the quark-gluon plasma. Fortunately the shapes of mass distributions of the plasma signal and the background are rather different what might permit to separate them.

If it turns out that the events with plasma formation can be distinguished by another signature, like large E_T, then the low mass dilepton production can bring an interesting information about the function $\phi(T)$ in Eqs (1) and (2).

The preliminary data [26] on the total E_T distributions in p+Pb collisions at 200 GeV/c can be interpreted in a model [27] in which hadrons in the final state are produced from individual strings. But that might hopefully change in the experiments with ^{16}O + ^{208}Pb collisions at the CERN SPS scheduled for autumn 1986.

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