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THE NEWTONIAN FORM OF WAVE MECHANICS

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Following the general principles of both Newton's mechanics and quantum mechanics a new formulation of wave mechanics is proposed. The new basic equations do not contain physical parameters and admit a different interpretation of the Planck constant.

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1. *Introduction*

It is well-known that the Newtonian form of classical mechanics provides the most general theoretical framework for the description of mechanical phenomena [1]. At its base lie such fundamental notions as the positions of bodies, their velocities, their moments and forces acting on them [2]. Although all these notions are usually represented in the formalism by some functions of time, they have different geometrical and physical meaning [3]. The laws of mechanics partially relate the functions representing the fundamental notions in the form of universal equations of motion

$$\frac{d\vec{x}_\alpha}{dt} = \vec{v}_\alpha(\vec{x}(t)), \tag{1.1a}$$

$$\frac{d\vec{p}_\alpha}{dt} = \vec{F}_\alpha(t; \vec{x}(t); \vec{v}(t)) \tag{1.1b}$$

($\alpha = 1, 2, \ldots N =$ number of bodies) which do not contain any particular parameters of the considered bodies. In order to obtain a well defined system of differential equations for the trajectories $\vec{x}_\alpha(t)$ we must specify the explicit form of the force functions in (1.1b) and use the constitutive material equations

$$\vec{p}_\alpha = m_\alpha \vec{v}_\alpha. \tag{1.2}$$

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In quantum mechanics the situation is slightly different. The basic equations of motion in the algebraic formulation of quantum mechanics [4] have the form

$$\frac{i}{\hbar}[\hat{H}, \hat{x}_\alpha] = \hat{v}_\alpha, \tag{1.3a}$$

$$\frac{i}{\hbar}[\hat{H}, \hat{p}_\alpha] = \hat{F}_\alpha, \tag{1.3b}$$

where $\hat{x}_\alpha$, $\hat{v}_\alpha$, $\hat{p}_\alpha$ and $\hat{F}_\alpha$ are the quantum mechanical observables of position, velocity, momentum and force, respectively, while $\hat{H}$ is the Hamiltonian of the system. In spite of the fact that equations (1.3) were obtained from the classical equations (1.1) through the correspondence principle the domain of applicability of (1.3) is narrower than that of (1.1) since the time evolution of all quantum mechanical observables must be determined by the Hamiltonian. Quantum mechanics deals therefore only with Hamiltonian systems.

A similar situation exists in the Schrödinger wave mechanics. Here the basic equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi \tag{1.4}$$

also uses the Hamiltonian and again describes only Hamiltonian systems.

Of course, all that is perfectly well known and is caused by quantum mechanics gemmating from the canonical formalism of classical mechanics.

Apart from the narrower generality of all known formulations of quantum mechanics with respect to Newton's mechanics there is another difference between the two, which at first sight may seem to be of minor importance: we have already pointed out that in classical mechanics all parameters characterizing particular bodies are introduced into the theory by means of the constitutive equations and specification of the force, and not by the equations of motion (1.1). Looking at (1.3) or (1.4) we see that in quantum mechanics all basic equations from the outset do contain the parameters of particular bodies in the explicit form of the Hamiltonian. In our opinion this is foreign to the spirit of Newton's mechanics. The presence of particular parameters in the basic equations of Newton's mechanics shows that we have already used the constitutive equations and therefore restricted the theory to a particular case.

The basic quantum mechanical equations (1.3) or (1.4) always contain the fundamental Planck constant $\hbar$. Although this is commonly understood just as a manifestation of quantum character of the theory, we regard it as being an indication that we have to do with a special case of some more general formalism. Here, of course, we cannot appeal to classical mechanics because it does not deal with the phenomena characterized by some fundamental constant. Since our problem is of methodological nature, we may freely appeal to another perfect theory where such kind of problems arises. The obvious candidate is classical electrodynamics, which describes all possible electrodynamical phenomena and for which the velocity of light in vacuum is the fundamental constant. But the basic Maxwell equations

$$\operatorname{rot} \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0; \quad \operatorname{div} \vec{B} = 0$$

$$\operatorname{rot} \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}; \quad \operatorname{div} \vec{D} = \varrho \tag{1.5}$$

with an appropriate choice of units do not contain this constant at all. It appears in the theory by means of the constitutive equations of the vacuum medium

$$\vec{D} = \varepsilon_0 \vec{E}, \quad \vec{B} = \mu_0 \vec{H} \tag{1.6}$$

provided we consider the wave solutions of the equations (1.5) and then

$$c = (\varepsilon_0 \mu_0)^{-1/2}. \tag{1.7}$$

For static solutions of (1.5) we shall never see any fundamental constant of the dimension of velocity!

Enlightened by these arguments we may formulate our problem as follows: We treat all existing formulations of basic equations of quantum mechanics as particular cases of some more general formulations of mechanics which we want to find. The new basic equations should reflect the basic laws of classical and quantum mechanics and be free from any physical constants. The necessary constants may appear only through some kind of constitutive relations and by considering some special solutions. Finally, the new equations should admit that solutions which satisfy the known equations of quantum mechanics (with the Planck constant $\hbar \neq 0$) and those which satisfy the Newton's equations (for $\hbar \neq 0$). For obvious reasons we may call such formalism to be the Newton's form of wave mechanics.

The aim of the present paper is to show that such a formulation of generalized mechanics exists and has a firm physical foundation, because it is based on general principles of both Newton's mechanics and quantum physics. Guided by those principles we construct a special case of Galilean field theory for which the basic equations possess a structure similar to that of Newton's equations of motions. The classical Newton's equations and the quantum Schrödinger equation are particular cases of our general field equations. In this way we get a new and non-trivial unification of classical and quantum mechanics.

2. *Basic equations for the generalized wave mechanics*

It is well known that for low energy phenomena we may replace, with a sufficient degree of accuracy, the true relativity group — the Lorentz-Poincarè group — by the approximate relativity group — the Galilei group. The quantum physics may then be constructed either on the basis of unitary projective representations of the Galilei group or on the basis of unitary representations of the one-parameter extension of the group [5]. For our purpose we must exclude the first possibility, since from the outset it contains in the phase factors some physical parameters of the particular systems considered. We must work therefore with the one-parameter extension of the Galilei group. This group acts in a five-dimensional extended space-time with an additional fifth coordinate θ, the meaning of which was clarified in Ref. [6].

Under the change of the inertial frame of reference the five space-time coordinates transform according to the Galilean rules

$$\vec{x} \to \vec{x}' = R\vec{x}+\vec{v}t+\vec{a},$$

$$t \to t' = t+b,$$

$$\theta \to \theta' = \theta+\vec{v}\cdot R\vec{x}+\tfrac{1}{2}v^2t+\omega, \tag{2.1}$$

where R denotes the rotation of axes, $\vec{v}$ — the relative velocity and $\vec{a}$, b, ω — the translations of origins of the inertial frames considered. Below, for shortness, we shall use the five-dimensional notation x^μ ($\mu = 1, 2, 3, 4, 5$) with $\vec{x} = (x^1, x^2, x^3)$, $t = x^4$ and $\theta = x^5$. Using this notation the transformations (2.1) may be written as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu+a^\mu, \tag{2.1a}$$

where the matrix elements $\Lambda^\mu{}_\nu$ of the matrix Λ can be read from (2.1). The covariant coordinates x_μ are defined as

$$x_\mu = g_{\mu\nu}x^\nu, \tag{2.2}$$

where $g_{\mu\nu}$ are the components of the covariant metric tensor, which for the Galilei group has the non-zero components

$$g_{ik} = -\delta_{ik}, \quad g_{45} = g_{54} = 1. \tag{2.3}$$

The contravariant metric tensor $g^{\mu\nu}$ defined by the relation

$$g^{\mu\nu}g_{\nu\lambda} = \delta^\mu{}_\lambda \tag{2.4}$$

has the non-zero components equal to

$$g^{ik} = -\delta_{ik}, \quad g^{45} = g^{54} = 1. \tag{2.5}$$

In accordance with the general principles of quantum physics we shall describe each physical system by some collection of fields defined over the five-dimensional Galilean space-time. The Galilean invariance of the theory requires that all the fields used should carry some representation of the Galilei relativity group but apart from economical reasons there is no general argument for restricting the number of fields needed. In the usual approach to quantum physics the unitary representations of the Galilei group are used as wave functions of physical objects. In spite of the fact that the wave functions contain all the information on the system, we treat them as a quantum mechanical counterpart of the classical trajectory only and following general principles of Newton's mechanics we enlarge the number of fields in order to have from the beginning the field theoretical counterparts of velocity, momentum and force functions. As a matter of fact, our procedure constitutes a new correspondence principle formulated between the Newtonian mechanics and the wave mechanics while the usual correspondence principle operates between the Newtonian mechanics and the algebra of quantum mechanical observables. In our formalism all basic fields are treated as independent, unless the equations of motion or the constitutive equations link them.

Our main steps are as follows:

For each physical system we implement the notion of its localization in space-time by means of some collection of fields $\psi_\alpha(x)$ ($\alpha = 1, 2, \ldots, N$), which under the Galilean transformations (2.1) behave according to the rule

$$\psi_\alpha(x) \to {}^G\psi_\alpha(x') = \sum_{\beta=1}^{N} D_\alpha^\beta(R, \vec{v})\psi_\beta(x) + R_\alpha(G; x), \tag{2.6}$$

where G denotes the set of all parameters which specify the Galilean transformation. It is clear that the formula (2.6) defines the representation of the Galilean group if the matrix $D(R, \vec{v})$ and the N-tuple $R(G; x)$ satisfy the following composition laws

$$D(R_1, \vec{v}_1)D(R_2, \vec{v}_2) = D(R_1R_2, \vec{v}_1 + R_1\vec{v}_2) \tag{2.7}$$

and

$$R(G_1; G_2x) + D(R_1, \vec{v}_1)R(G_2, x) = R(G_1 \circ G_2; x), \tag{2.8}$$

where $G_1 \circ G_2$ denotes the composition of two Galilei transformations G_1 and G_2, while Gx denotes the transformed coordinates under the transformation specified by G.

In the case of the usual quantum mechanics we have always

$$R_\alpha(G; x) = 0, \tag{2.9}$$

but the more general formula (2.6) is needed mainly for Newton's classical mechanics. The principle of Galilean relativity admits the presence of R_α in (2.6) and we cannot exclude it on the ground of any general argument.

The physical meaning of the fields ψ_α depends on the adopted interpretation of the formalism. Here, as usual, we must distinguish between two cases: the classical and the quantum one.

In the classical case we consider only real valued fields and we compare directly the values of these fields with the values of some physical quantities. In particular, in the case of Newton's mechanics we identify the values of the fields with the components of the radius vectors of the considered system of mass points.

In the case of quantum mechanics we work with complex valued fields for which the matrix D in (2.6) defines unitary representations of the Galilei group, and as usual, we shall interpret the real function

$$\sum_{\alpha=1}^{N} |\psi_\alpha(x)|^2 \tag{2.10}$$

as the probability density of finding the object at the position with coordinate x.

Apart from the classical Newtonian mechanics and ordinary quantum mechanics our formalism has many other realizations where the fields ψ_α may have more general meaning with possible still unknown interpretations. In particular, we may treat our formalism as the basis for a second quantized theory of fields. In this case we should interpret the fields ψ_α in accordance with the principles of quantum field theory.

In addition to the fields $\psi_\alpha(x)$ we shall define fields $\varphi_{\mu,\alpha}(x)$ ($\mu = 1 \dots 5$; $\alpha = 1 \dots N$), which under Galilean transformations (2.1) change like

$$\varphi_{\mu,\alpha}(x) \to {}^G\varphi_{\mu,\alpha}(x') = \sum_{\nu=1}^{5} \sum_{\beta=1}^{N} \tilde{\Lambda}_\mu{}^\nu(R, \vec{v}) D_\alpha^\beta(R, \vec{v}) \varphi_{\nu,\beta}(x) + W_{\mu,\alpha}(G; x), \tag{2.11}$$

where $\tilde{\Lambda}_\mu{}^\nu(R, \vec{v})$ is the matrix describing the transformation rules for a covariant five-vector. Of course, we have the relation

$$\Lambda^\mu{}_\nu(R, \vec{v}) = g^{\mu\varrho} \tilde{\Lambda}_\varrho{}^\sigma(R, \vec{v}) g_{\sigma\nu}. \tag{2.12}$$

The new fields $\varphi_{\mu,\alpha}$ determine the space-time evolution of the fields ψ_α in exactly the same way as the classical velocity field $\vec{v}_\alpha(t)$ determines the time evolution of the trajectory $\vec{x}_\alpha(t)$. This means that we should have the equations

$$\frac{\partial \psi_\alpha(x)}{\partial x^\mu} = \varphi_{\mu,\alpha}(x) \tag{2.13}$$

as our field theoretical counterpart of (1.1a). Obviously, these $5N$ equations may be considered as the definition of the fields $\varphi_{\mu,\alpha}$ in terms of the fields ψ_α just like in classical mechanics the equations (1.1a) may be considered as the definition of the velocity. The knowledge of the fields ψ_α at different space-time points allows one to calculate the fields $\varphi_{\mu,\alpha}(x)$. Conceptually, however, the game is the opposite, just as it is in classical mechanics. They are the fields $\varphi_{\mu,\alpha}$ that determine the fields ψ_α through the equations (2.13).

Now we pass to the most essential part of our considerations by which we mean the formulation of the field theoretical counterpart of the Newton's second law of mechanics. For this purpose we need the field theoretical counterparts of the classical momentum and force. Again, guided by the classical relation of velocity to momentum as well as by the geometrical meaning of momentum as a covector field, we introduce into our theory further new fields $\pi_\alpha{}^\mu(x)$ with the following transformation rules under Galilean transformations:

$$\pi_\alpha{}^\mu(x) \to {}^G\pi_\alpha{}^\mu(x') = \sum_{\nu=1}^{5} \sum_{\beta=1}^{N} \Lambda^\mu{}_\nu(R, \vec{v}) D_\alpha^\beta(R, \vec{v}) \pi_\beta{}^\nu(x) + P_\alpha{}^\mu(G; x). \tag{2.14}$$

For each dynamical problem the fields $\pi_\alpha{}^\mu$ are responsible for the description of the dynamics. This means that for the free motion we should have

$$\frac{\partial \pi_\alpha{}^\mu(x)}{\partial x^\mu} = 0 \tag{2.15}$$

while for any other motion there should exist fields $\varrho_\alpha(x; \psi, \varphi_{\mu,\alpha})$ implementing the notion of force such that

$$\frac{\partial \pi_\alpha{}^\mu(x)}{\partial x^\mu} = \varrho_\alpha(x; \psi, \varphi_{\mu,\alpha}) \tag{2.16}$$

and which under the Galilean transformations always behave in the homogeneous way

$$\varrho_\alpha(x; \psi, \varphi_{\mu,\alpha}) \to {}^G\varrho_\alpha(x'; {}^G\psi, {}^G\varphi_{\mu,\alpha}) = \sum_{\beta=1}^{N} D_\alpha^\beta(R, \vec{v})\varrho_\beta(x; \psi, \varphi_{\mu,\alpha}). \tag{2.17}$$

This formulation of dynamics should be regarded as an exact field theoretical analogue of Newton's second law of mechanics [1]. Observe that the Galilean covariance uniquely fixes the shape of the left-hand sides in (2.15) and (2.16). This basic equations (2.16) here also do not contain any physical parameter.

To make the above formulation more precise we must specify two things: the notion of a free motion and the relation of the momentum field $\pi^\mu{}_\alpha$ to the velocity field $\varphi_{\mu,\alpha}$. Obviously, by free motion we shall mean such cases for which the fields carry the energy-momentum relation

$$E = \frac{\vec{p}^{\,2}}{2m} \tag{2.18}$$

characteristic for free motion. Observe that we did not include in (2.18) the rest energy Ω because in our formulation it is of dynamical origin.

The relation of momentum field $\pi^\mu{}_\alpha$ to the velocity field $\varphi_{\mu,\alpha}$ is taken from the direct analogue of the classical constitutive equations (1.2) in the form

$$\pi^\mu{}_\alpha(x) = \sum_{\nu=1}^{5} \sum_{\beta=1}^{N} M^{\mu\nu}{}_\alpha{}^\beta(x)\varphi_{\nu,\beta}(x), \tag{2.19}$$

where $M^{\mu\nu}{}_\alpha{}^\beta(x)$ are arbitrary functions with the dimension of mass. As a matter of fact the relation (2.19) is a part of our correspondence principle. This relation together with (2.15) uniquely determines the physical meaning of the fields $\pi^\mu{}_\alpha$. In particular, the relation (2.19) determines the dimension of $\pi^\mu{}_\alpha$ and by means of the matrix M it introduces into the theory the masses of considered objects. At this point we would like to turn the reader's attention to the fact that in general the mass matrix M in (2.19) may be nondiagonal and its matrix elements may be arbitrary functions of space-time variables. This enables us to describe objects with masses not well defined [7] and objects with variable masses. In this way we get a very important generalization of quantum mechanics for quantal description of such systems. This is a valuable result of our work, since no other known formulation of quantum mechanics can deal with such cases.

For complex-valued fields which satisfy (2.9) we may construct a five-vector field

$$j^\mu(x) = \sum_{\alpha=1}^{N} (\pi^{\mu *}_\alpha(x)\psi_\alpha(x) - \psi^*_\alpha(x)\pi^\mu{}_\alpha(x)) \tag{2.20}$$

which is conserved if the mass matrix M in (2.19) is Hermitian and if

$$\sum_{\alpha=1}^{N} \psi_\alpha^* \varrho_\alpha = \sum_{\alpha=1}^{N} \varrho_\alpha^* \psi_\alpha. \tag{2.21}$$

In particular, the above condition is satisfied if

$$\varrho_\alpha(x; \psi, \varphi) = F(x; \psi, \varphi)\psi_\alpha(x), \tag{2.22}$$

where F is a scalar real valued function of its arguments. The systems for which the current j^μ is conserved will be called conservative.

A physical interpretation of the current (2.20) depends on the adopted interpretation of the formalism. In the case of usual quantum mechanics (2.20) represents the usual probability current. For other cases the meaning of (2.20) must be investigated separately.

Finally we would like to indicate one obvious generalization of our formalism. In the case of complex-valued fields all our equations are invariant under phase transformations of the considered fields. It is not difficult to generalize the formalism to the case of gauge transformations of the second kind with arbitrary variable phase factors, since it is sufficient everywhere to replace the usual space-time derivatives by the "covariant" ones given by

$$D_\mu = \partial_\mu - iA_\mu(x), \tag{2.23}$$

where the five gauge fields A_μ form a five-vector field. The complete Galilean field theory may then be obtained by the method described in Ref. [8].

3. *The reduction of the general scheme to the case of classical mechanics and to the usual formulation of wave mechanics*

In order to shed some light on the physical content of the general formalism developed till now we shall discuss three particular examples.

As the first example let us consider the real valued fields which depend on the time variable only. In particular, let us consider three-component fields $\psi_\alpha(t)$ ($\alpha = 1, 2, 3$) which transform according to the following representation of the Galilei group:

$$^G\psi_\alpha(t') = \sum_{\beta=1}^{3} R_{\alpha\beta}\psi_\beta(t) - v_\alpha t + a_\alpha. \tag{3.1}$$

From equations (2.13) the velocity fields have components

$$\varphi_{i,\alpha} = 0; \quad \varphi_{t,\alpha} = \frac{d\psi_\alpha(t)}{dt}; \quad \varphi_{\theta,\alpha} = 0. \tag{3.2}$$

The constitutive relation (2.19) with diagonal mass matrix $M^{\mu\nu}{}_\alpha{}^\beta = m\delta^\mu_\nu\delta^\beta_\alpha$ gives the following components of the momentum fields:

$$\pi^i{}_\alpha = 0; \quad \pi^t{}_\alpha = m\frac{d\psi_\alpha(t)}{dt}; \quad \pi^\theta{}_\alpha = 0 \tag{3.3}$$

and the equations (2.16) take the form

$$m \frac{d^2\psi_\alpha}{dt^2} = \varrho_\alpha. \tag{3.4}$$

Needless to say, the considered example is just the classical mechanics of the mass point. The components of the field $\psi_\alpha(t)$ are interpreted as Cartesian components of the radius vector, the non-zero components of the velocity field are the components of the usual velocity and, similarly, the non-zero components of the momentum field are the components of the usual momentum. The three components of the force field are the components of the mechanical force and equation (3.4) is just the Newtonian equation of motion.

It is easy to generalize the above example to the case of n-particle systems. It is sufficient to choose the ψ-field composed of triplets $\psi_{\alpha_k}(t)$ ($k = 1, ..., n$; $\alpha_k = 1, 2, 3$) each of them behaving according to (3.1) under Galilean transformations. The mass matrix we take in the form

$$M^{\mu\nu}{}_{\alpha_k}{}^{\beta_j} = m_j \delta^\mu_\nu \delta^{\beta_j}_{\alpha_k} \delta^j_k, \tag{3.5}$$

where m_j is the mass of the j-th particle and instead of (3.4) we end up with the equations

$$m_k \frac{d^2\psi_{\alpha_k}}{dt^2} = \varrho_{\alpha_k}, \tag{3.6}$$

where the right-hand sides depend in general on all ψ_{α_k}. These are exactly Newton's equations of motion for the n-particle system.

In the second example we shall consider fields ψ_α which realize the passage from unitary representations of the extended Galilei group to projective unitary representations of the non-extended Galilei group used in the Schrödinger wave mechanics. According to the general group theoretical prescription [5] these fields have the form

$$\psi_\alpha(\vec{x}, t, \theta) = \exp\left(\frac{im\theta}{\hbar}\right) \phi_\alpha(\vec{x}, t), \tag{3.7}$$

where m is the mass of the particle described by the wave function $\phi_\alpha(\vec{x}, t)$. The presence of the Planck constant in (3.7) is caused by dimensional reasons and the de Broglie connection between the characteristics of the plane wave and the mechanical characteristics of the particle [9].

Putting in (2.6) $R_\alpha = 0$ we see that the wave functions $\phi_\alpha(\vec{x}, t)$ have the following Galilean transformation properties

$$^G\phi_\alpha(\vec{x}', t') = \exp\left\{\frac{im}{\hbar}(\vec{v}\cdot R\vec{x} + \tfrac{1}{2}\vec{v}^2 t + \omega)\right\} \sum_{\beta=1}^{N} D^\beta_\alpha(R; \vec{v})\phi_\beta(\vec{x}, t) \tag{3.8}$$

well-known from the Schrödinger theory.

For fields of the form (3.7), using the constitutive equation (219) with $M^{\mu\nu}{}_{\alpha}{}^{\beta} = mg^{\mu\nu}\delta^{\beta}_{\alpha}$ as mass matrix and the force fields of the type (2.22), we end up with the equation

$$\left(-\frac{\hbar^2}{2m}\Delta + i\hbar\frac{\partial}{\partial t}\right)\phi_\alpha(\vec{x}, t) = -V(\vec{x}, t; \phi)\phi_\alpha(\vec{x}, t), \tag{3.9}$$

where we have denoted

$$V(\vec{x}, t; \phi) = -\frac{\hbar^2}{2m^2}F(\vec{x}, t; \phi). \tag{3.10}$$

Obviously, equation (3.9) coincides with the Schrödinger equation. This, of course, is an expected result because we have used projective unitary representations of the Galilei group. The non-trivial part of our result is that we have derived in a consistent way the non--linear Schrödinger equation. This is in sharp contrast to the usual quantization procedure where we first get the linear Schrödinger equation and then we add to it some non-linear terms.

The non-linear Schrödinger equations are widely used for practical calculations [10] and our general approach supplies a firm foundation to such kind of quantum mechanics. Our approach shows that the non-linearities in the Schrödinger equation do not violate the principle of superposition because the basic equations (2.13) and (2.16) are linear. The force fields may non-linearly depend on the wave functions, just as in classical mechanics the force may be a non-linear function of the position.

The fact that our general field theoretical formulation of mechanics contains as particular cases both the classical mechanics (as solutions of basic equations without Planck constant) and the usual quantum mechanics (as solutions of the same basic equations for which the Planck constant is needed) is a sufficient justification for taking it seriously and for investigating all possible other solutions. We shall follow this extensive research program in future papers. At present we are satisfied having unified classical and quantum mechanics in a non-trivial and new way.

In order to show what prospects for new applications are opened by our formalism let us consider a third example in which we take the basic fields ψ_α in the form of (3.7) but with the Planck constant replaced by function $\hbar(\vec{x})$. Physically this corresponds to quantum mechanics with a variable Planck constant. The fact that we may consistently discuss such problems is the second big advantage from our work. For example, we may look what kind of effects we get if we take

$$\hbar(\vec{x}) = \begin{cases} \hbar_0 & \text{for} \quad |\vec{x}| < r_0 \\ \hbar_1 & \text{for} \quad |\vec{x}| > r_1 > r_0 \end{cases} \tag{3.11}$$

and in the intermediate region $r_0 < |\vec{x}| < r_1$, $\hbar(\vec{x})$ smoothly joins the values $\hbar_0$ and $\hbar_1$. Recently [11] it was argued that this kind of quantum mechanics is desirable in astrophysics.

A simple calculation shows that in the considered example instead of the Schrödinger equation (3.9) we now get the equation

$$-\frac{\hbar^2(\vec{x})}{2m}\Delta\phi+i\hbar(\vec{x})\frac{\partial\phi}{\partial t}+i\theta\left[\tfrac{1}{2}\Delta\hbar\cdot\phi-\frac{(\vec{\nabla}\hbar)^2}{\hbar}\phi+\vec{\nabla}\hbar\cdot\vec{\nabla}\phi\right]$$

$$+\frac{m\theta^2}{2\hbar^2}(\vec{\nabla}\hbar)^2\phi = -V(\vec{x},t,\theta;\phi)\phi, \tag{3.12}$$

where we have restricted ourselves to the case of conservative systems. Obviously, this equation can be solved only if

$$V(\vec{x},t,\theta;\phi) = V(\vec{x},t;\phi)+\theta V_1(\vec{x},t;\phi)+\theta^2V_2(\vec{x},t;\phi) \tag{3.13}$$

and then it splits into three equations

$$-\frac{\hbar^2(\vec{x})}{2m}\Delta\phi+i\hbar(\vec{x})\frac{\partial\phi}{\partial t} = -V(\vec{x},t;\phi)\phi, \tag{3.14}$$

$$\tfrac{1}{2}\Delta\hbar\cdot\phi-\frac{(\vec{\nabla}\hbar)^2}{\hbar}\phi+\vec{\nabla}\hbar\cdot\vec{\nabla}\phi = iV_1\phi, \tag{3.15}$$

$$(\vec{\nabla}\hbar)^2 = -\frac{2\hbar^2}{m}V_2. \tag{3.16}$$

Equation (3.14) formally coincides with the usual Schrödinger equation but in the place of the Planck constant we have here the function $\hbar(\vec{x})$. The full coincidence is to be obtained in spatial regions, where the function $\hbar(\vec{x})$ is constant. In these regions, obviously, $V_1 = V_2 = 0$ and (3.15) and (3.16) are tautologies. But in the regions where $\hbar(\vec{x})$ is variable a qualitatively new situation arises. First, for the description of the particle we need three potentials V, V_1 and V_2. Second, for given potentials we must simultaneously solve three equations for two unknown functions $\hbar(\vec{x})$ and $\phi(\vec{x},t)$ and a non-trivial solution may exist only for some special triplets V, V_1 and V_2. A full discussion of this problem and the numerical calculations will be presented elsewhere. Here we just wanted to convince the reader that our formalism enables us to treat problems of this kind. This has far-reaching consequences because we thus get a chance for a new interpretation of the Planck constant. It is not merely a constant but a property of the systems. There are some systems for which $\hbar(\vec{x})$ may be taken as a constant but there may also be systems for which the variability of $\hbar(\vec{x})$ may phenomenologically describe new effects. This resembles the situation in electromagnetics, where the passage from constant ε and μ to variable $\varepsilon(\vec{x})$ and $\mu(\vec{x})$ allows one to describe electromagnetic properties of non-homogeneous media. The variable $\hbar(\vec{x})$ means that we are dealing with quantally non-homogeneous media for which the quantum effects are stronger in some regions and possibly weaker in others.

4. *Conclusions*

The main purpose of our paper was to describe a specific Galilean invariant field theory which we called a Newtonian wave mechanics. For each physical object we have introduced four types of fields and following Newton's concept of mechanics we have obtained some basic field equations. Our equations are universal and do not contain any particular physical parameters.

We have shown that the general formalism contains as particular cases both classical and, separately, quantum mechanics. To our best knowledge this is the first case of such unification of these two theories.

Our formalism provides a quantum mechanics for general systems. We do not need to restrict the theory to hamiltonian systems, we get the foundation for non-linear theories without violating the quantum mechanical superposition principle and can describe systems with variable masses and variable Planck constant. The last property seems less heretical if we compare it with the situation of the velocity of light in electromagnetics. Only for homogeneous media it is constant and the nonhomogeneous media play so important a role in our life. Is it so with non-homogeneous quantum systems?

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