A VARIATIONAL APPROACH TO DEFORMABLE ELECTROMAGNETIC SOLIDS

By F. Bampi and A. Morro

DIBE, Università di Genova, Viale F. Causa 13, 16145 Genova, Italy

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Maxwell's equations and the mechanical balance equations for a deformable electromagnetic solid are obtained via a strictly variational approach. The Lagrangian (density) is taken as the sum of the electromagnetic field Lagrangian, the matter Lagrangian, and the interaction Lagrangian. The construction of the interaction Lagrangian is performed by modeling the solid as a set of charged particles and by defining the macroscopic quantities through a suitable average procedure: the interaction Lagrangian then takes the form of a multipole expansion. The Euler-Lagrange equations provide directly Maxwell's equations, the equations of motion, and the constitutive equations. The calculations are made explicitly up to the quadrupole approximation; the agreement with existing theories makes the approach applicable to multipole expansions of any order.

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1. Introduction

In the last two decades much attention has been devoted to the subject of electromagnetic interactions in deformable solids. Besides, being conceptually attractive, this subject is important because of the large extent of applications in technology and applied physical sciences [1]. However, in spite of this subject being so fundamental in nature, the literature bears evidence of many coexisting theories and results [2]. On the one hand such a variety of theories arises from the different models accounting for the involved phenomena occurring in electromagnetic solids: interactions between polarization, magnetization, and mechanical deformation. On the other, this paradoxical feature is due to the conceptual difficulty that electromagnetic fields inside matter are expressed in terms of field variables which cannot be measured in laboratories. That is why usually electromagnetic theories cannot be sorted out or proven to be incorrect by appropriate experiments. Hence, at this stage one can favour a theory merely through the mathematical rigour and the soundness and generality of the model.

Based on this observation and motivated by the conceptual advantages of variational procedures, in this paper we attempt to develop a systematic approach to electromagnetic

interactions in deformable solids by starting from a suitable variational formulation. As a preliminary step towards the determination of the Lagrangian density we consider a polarizable and magnetizable deformable body acted upon by an electromagnetic field; the body is taken as constituted by charged particles and then the macroscopic quantities are defined via suitable averages (Sect. 2). Next, in Section 3, we look for the Lagrangian (density) as the sum of three contributions, namely the electromagnetic field Lagrangian, the matter Lagrangian, and the interaction Lagrangian. While the electromagnetic field Lagrangian and the matter Lagrangian for the model adopted can be written in a direct way, the crucial point lies in the interaction Lagrangian. By following the average procedure we construct systematically the interaction Lagrangian which takes the form of a multipole expansion: here, for the sake of definiteness, we content ourselves with the electric quadrupole and the magnetic dipole terms. Once this is made, Maxwell's equations, the equations of motion, and the constitutive equations are just the corresponding set of Euler-Lagrange equations; their explicit form is exhibited in Section 4. Thus we are in a position to make a comparison with the customary theories of electromagnetism in matter (Sect. 5). The fact that our results agree with the corresponding results derived through different approaches emphasises the validity of our procedure. Furthermore, this fact gives value to the main feature of our variational approach, namely that we can obtain theories as accurate as we please by merely considering the appropriate multipole expansion in the interaction Lagrangian. This value is strengthened by the derivation of the pertinent equations being performed in a purely deductive way.

2. *A model of electromagnetic solids*

We consider a body constituted by charged particles, possibly gathered into stable groups such as atoms and ions. In order to derive macroscopic laws for the body we have recourse to averages over a number of particles contained in a mass element which is large enough so that smooth average fields can be defined but which is still small from a macroscopic point of view. So, for any volume element in the actual configuration of the body we describe the position of the particles by $\boldsymbol{x}_\alpha = \boldsymbol{x} + \boldsymbol{\xi}_\alpha$, $\boldsymbol{x}$ being the position of the center of mass of the system of particles under consideration which are labeled by the subscript α. The smooth function $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$ provides the position $\boldsymbol{x}$ of the center of mass at time t in terms of the position $\boldsymbol{X}$ of the center of mass in a suitable reference configuration; the Jacobian $J = \det \boldsymbol{F}$, $\boldsymbol{F} = \partial \boldsymbol{x}/\partial \boldsymbol{X}$, is taken to be strictly positive. The particle α is endowed with a charge e_α which is supposed to be constant in time. The electromagnetic fields acting upon the particles are assumed to be smooth functions so that suitable expansions about any center of mass may be written.

Letting dv be the volume element in the actual configuration, as usual we define the charge density q, the polarization density vector $\boldsymbol{P}$, the electric quadrupole moment tensor $\boldsymbol{Q}$, and the magnetic moment vector $\boldsymbol{M}$ as

$$q dv = \Sigma_\alpha e_\alpha, \tag{2.1}$$

$$\boldsymbol{P} dv = \Sigma_\alpha e_\alpha \boldsymbol{\xi}_\alpha, \tag{2.2}$$

$$\boldsymbol{Q}dv = \tfrac{1}{2}\Sigma_\alpha e_\alpha \boldsymbol{\xi}_\alpha \otimes \dot{\boldsymbol{\xi}}_\alpha, \tag{2.3}$$

$$\boldsymbol{M}dv = \tfrac{1}{2}\Sigma_\alpha e_\alpha \boldsymbol{\xi}_\alpha \times \dot{\boldsymbol{\xi}}_\alpha, \tag{2.4}$$

a superposed dot denoting the material time derivative.

At any time t we may describe the fields under consideration in terms of the spatial position $\boldsymbol{x}$ (Eulerian — E for short — description) or in terms of the reference position $\boldsymbol{X}$ (Lagrangian — L for short — description). When dealing with external fields, as, for example, the electric field $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$, it is natural to apply the E-description. On the other hand, especially when treating deformable solids, the material properties are conveniently investigated in the L-description. The transformation between the E-description of any field and the corresponding L-description is accomplished via the function $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t)$. Within a variational formulation, the equivalence of E- and L-descriptions is made operative through a theorem formally proved in Ref. [3] although very often employed in previous papers by Lax and Nelson [4, 5]. Specifically, the theorem asserts that letting the Lagrangian density in the E-description equal to J times the Lagrangian density in the L-description makes the associated Euler-Lagrange equations equivalent.

3. The Lagrangian density

When dealing with physical systems which may be viewed as consisting of two separate subsystems it is customary to look at the Lagrangian of the system as the sum of the Lagrangians of the two subsystems and the interaction Lagrangian. In connection with the dynamical behavior of a set of particles of an electromagnetic solid, according to the scheme outlined in Section 2, we take the Lagrangian L as the sum of a field Lagrangian L_F, a matter Lagrangian L_M, and an electromagnetic field-matter interaction Lagrangian L_I, namely

$$L = L_F + L_M + L_I. \tag{3.1}$$

The field Lagrangian L_F is usually written in the E-description and its form is well-known; explicitly, on adopting rationalized MKS units we have

$$L_F = \left(\tfrac{1}{2}\varepsilon_0 E^2 - \frac{1}{2\mu_0} B^2\right) dv, \tag{3.2}$$

where ε_0, μ_0 are the dielectric constant and the magnetic permeability of the free space. With a view to the variational approach we have in mind, it is worth emphasizing that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the homogeneous Maxwell's equations

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{0}, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{3.3}$$

the symbol ∇ denoting the gradient operator (relative to E-description). Then we may regard as unknown functions, for the electromagnetic field, the vector potential $\boldsymbol{A}$ and the scalar potential Φ providing $\boldsymbol{E}$ and $\boldsymbol{B}$ through the relations

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t}, \quad \boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

The matter Lagrangian L_M is more easily expressed in the L-description. Letting V_α be the potential energy of the particle α we may write

$$L_M = \tfrac{1}{2}\Sigma_\alpha m_\alpha \dot{x}_\alpha^2 - \Sigma_\alpha V_\alpha$$

or, rather,

$$L_M = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2}\Sigma_\alpha m_\alpha \dot{\xi}_\alpha^2 - \Sigma_\alpha V_\alpha, \tag{3.4}$$

$m = \Sigma_\alpha m_\alpha$ being the total mass.

Owing to the well-known form of the Lagrangian for a charged particle in an electromagnetic field we are led to say that the interaction Lagrangian for a charge e with velocity v is $e(v \cdot A - \Phi)$. It is then natural to assume that the interaction Lagrangian for the system of particles (in the E-description) is

$$L_I = \Sigma_\alpha e_\alpha [v_\alpha \cdot A(x_\alpha, t) - \Phi(x_\alpha, t)]. \tag{3.5}$$

As it stands, the Lagrangian (3.5) would lead to cumbersome contributions to the Euler-Lagrange equations due essentially to the appearance of the unknown functions x_α which reflect the microscopic behaviour of the body. Then, taking advantage of the smoothness of the fields A, Φ, we consider suitable expansions involving terms which depend on the space variables through the position x of the center of mass only. Specifically, let

$$A(x_\alpha, t) = A(x, t) + (\xi_\alpha \cdot \nabla)A(x, t) + \tfrac{1}{2}[\ddot{\xi}_\alpha \cdot (\xi_\alpha \cdot \nabla)\nabla]A(x, t) + o(\xi_\alpha^2),$$

$$\Phi(x_\alpha, t) = \Phi(x, t) + \xi_\alpha \cdot \nabla\Phi(x, t) + \tfrac{1}{2}\xi_\alpha \cdot (\xi_\alpha \cdot \nabla)\nabla\Phi(x, t) + o(\xi_\alpha^2).$$

Henceforth, let us consider A and Φ to within $o(\xi_\alpha^2)$. Then, since $v_\alpha = v + \dot{\xi}_\alpha$, $v \equiv \dot{x}$, we have

$$L_I = (\Sigma_\alpha e_\alpha)v \cdot A + \Sigma_\alpha e_\alpha \dot{\xi}_\alpha \cdot A + \Sigma_\alpha e_\alpha v \cdot (\xi_\alpha \cdot \nabla)A + \Sigma_\alpha e_\alpha \dot{\xi}_\alpha \cdot (\xi_\alpha \cdot \nabla)A$$

$$+ \tfrac{1}{2}\Sigma_\alpha e_\alpha v \cdot [\xi_\alpha \cdot (\xi_\alpha \cdot \nabla)\nabla]A - (\Sigma_\alpha e_\alpha)\Phi - \Sigma_\alpha e_\alpha \xi_\alpha \cdot \nabla\Phi - \tfrac{1}{2}\Sigma_\alpha e_\alpha \xi_\alpha \cdot (\xi_\alpha \cdot \nabla)\nabla\Phi,$$

the potentials A, Φ and their derivatives being evaluated at the center of mass x. The interpretation of some terms in L_I is obvious. For example, in view of (2.2) we have

$$\Sigma_\alpha e_\alpha v \cdot (\xi_\alpha \cdot \nabla)A = v \cdot (P \cdot \nabla)A\,dv.$$

The meaning of the terms involving $\dot{\xi}_\alpha$ is not clear and some comments are in order. Consider, for instance, the fourth term; an integration by parts gives

$$\Sigma_\alpha e_\alpha \dot{\xi}_\alpha \cdot (\xi_\alpha \cdot \nabla)A = \frac{d}{dt}\Sigma_\alpha e_\alpha \xi_\alpha \cdot (\xi_\alpha \cdot \nabla)A - \Sigma_\alpha e_\alpha \xi_\alpha \cdot (\dot{\xi}_\alpha \cdot \nabla)A$$

$$- \Sigma_\alpha e_\alpha \xi_\alpha \cdot \left(\xi_\alpha \cdot \frac{d}{dt}\nabla\right)A$$

which implies that

$$\Sigma_\alpha e_\alpha \dot{\xi}_\alpha \cdot (\xi_\alpha \cdot \nabla)A = \tfrac{1}{2}\left\{\frac{d}{dt}[\Sigma_\alpha e_\alpha \xi_\alpha \cdot (\xi_\alpha \cdot \nabla)A] + \Sigma_\alpha e_\alpha (\xi_\alpha \otimes \dot{\xi}_\alpha\right.$$

$$\left. - \dot{\xi}_\alpha \otimes \xi_\alpha) \cdot \nabla A - \Sigma_\alpha e_\alpha (\xi_\alpha \otimes \xi_\alpha) \cdot \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) \nabla A\right\}.$$

Now, because

$$(\xi_\alpha \otimes \dot{\xi}_\alpha - \dot{\xi}_\alpha \otimes \xi_\alpha) \cdot \nabla A = (\xi_\alpha \times \dot{\xi}_\alpha) \cdot \nabla \times A,$$

in view of (2.3) and (2.4) we obtain

$$\Sigma_\alpha e_\alpha \dot{\xi}_\alpha \cdot (\xi_\alpha \cdot \nabla)A = \tfrac{1}{2}\frac{d}{dt}[\Sigma_\alpha e_\alpha \xi_\alpha \cdot (\xi_\alpha \cdot \nabla)A]$$

$$+ M \cdot (\nabla \times A)dv - Q \cdot \left(\frac{\partial}{\partial t} + v \cdot \nabla\right) \nabla A dv.$$

Similar procedures apply to the remaining terms. Then, by observing that additive (total) time derivatives are ineffective in a variational approach, we arrive at the expression

$$L_1 = \left[qv \cdot A - q\Phi - P \cdot \nabla\Phi - P \cdot \frac{\partial A}{\partial t} + (P \times v) \cdot (\nabla \times A) - Q \cdot (\nabla\nabla\Phi)\right.$$

$$\left. - Q \cdot \frac{\partial}{dt} \nabla A + (Q \otimes v - v \otimes Q) \cdot (\nabla\nabla A) + M \cdot (\nabla \times A)\right] dv. \tag{3.6}$$

A more compact form is gained by introducing suitable new fields. First observe that, upon letting

$$\mathscr{E} = E + v \times B$$

and recalling the relations (3.3), we can write

$$L_1 = [qv \cdot A - q\Phi + P \cdot \mathscr{E} + Q \cdot \nabla E + (Q \otimes v - v \otimes Q) \cdot (\nabla\nabla A) + M \cdot (\nabla \times A)]dv.$$

Now, it can easily be shown that

$$(Q \otimes v - v \otimes Q) \cdot (\nabla\nabla A) = Q \cdot [\nabla(v \times B)] - [B \times (Q\nabla)] \cdot v;$$

in component form

$$[B \times (Q\nabla)] \cdot v = \varepsilon_{ijk} B_i Q_{jp} v_{k'p}.$$

Finally, letting

$$\mathscr{M} = M - (Q\nabla) \times v \tag{3.7}$$

we arrive at the Lagrangian

$$L_I = [q\boldsymbol{v}\cdot\boldsymbol{A} - q\Phi + \boldsymbol{P}\cdot\mathscr{E} + \boldsymbol{Q}\cdot(\nabla\mathscr{E}) + \mathscr{M}\cdot\boldsymbol{B}]dv$$

to which corresponds the Lagrangian density

$$\mathscr{L}_I = q\boldsymbol{v}\cdot\boldsymbol{A} - q\Phi + \boldsymbol{P}\cdot\mathscr{E} + \boldsymbol{Q}\cdot(\nabla\mathscr{E}) + \mathscr{M}\cdot\boldsymbol{B}. \tag{3.8}$$

As to the Lagrangian density corresponding to (3.4) we observe that the kinetic energy contribution of the internal variables ξ_α is usually disregarded in connection both with the electromagnetic properties and with the mechanical behaviour of the solid. More specifically, the inertial effects of the internal variables are negligible as far as acoustic waves are concerned. Accordingly, in order to get a fully macroscopic theory we disregard $\Sigma_\alpha m_\alpha \dot{\xi}_\alpha^2$ in (3.4). Then, letting ϱ be the mass density and V the potential energy density, viz.

$$\varrho dv = \Sigma_\alpha m_\alpha, \qquad V dv = \Sigma_\alpha V_\alpha,$$

we can write

$$\mathscr{L}_M = \tfrac{1}{2}\varrho v^2 - V. \tag{3.9}$$

Coherently with the macroscopic viewpoint, in next Section we let V depend on $\boldsymbol{F}$ and q, $\boldsymbol{P}$, $\boldsymbol{Q}$, $\mathscr{M}$. Indeed, for the sake of formal simplicity we set

$$V = V_1(q, \boldsymbol{F}) + V_2(\boldsymbol{F}, \boldsymbol{P}, \boldsymbol{Q}, \mathscr{M}). \tag{3.10}$$

4. Maxwell's equations, equations of motion, and constitutive equations

The total Lagrangian density $\mathscr{L}$ of the system (solid and electromagnetic field) is the sum of the Lagrangian densities $\mathscr{L}_F$, $\mathscr{L}_M$, $\mathscr{L}_I$. As we are dealing with contributions to the Lagrangian density both in the L-description and in the E-description, we need having recourse to the invariance axiom for the action of a system in different descriptions. This axiom amounts to saying that the Lagrangian density in the E-description, $\mathscr{L}^s$, and the corresponding one in the L-description, $\mathscr{L}^m$, are related by

$$\mathscr{L}^m = J\mathscr{L}^s. \tag{4.1}$$

Furthermore, as proved in a recent paper [3], in spite of the dependence on $\boldsymbol{x}(\boldsymbol{X}, t)$ introduced by J, there is no additional contribution to the Euler-Lagrange equations in passing from the E-description to the L-description. Henceforth this feature will be taken into account so as to avoid unnecessary calculations.

The Lagrangian density $\mathscr{L}$ depends on the unknown functions $\boldsymbol{x}$, $\boldsymbol{A}$, Φ, q, $\boldsymbol{P}$, $\boldsymbol{Q}$, $\mathscr{M}$; $\mathscr{E}$ and $\boldsymbol{B}$ are related to $\boldsymbol{A}$ and Φ through (3.3). Observe now that the Lagrangian density $\mathscr{L}^s$ is given by

$$\mathscr{L}^s = \mathscr{L}^s_F + \mathscr{L}^s_I + \mathscr{L}^s_M, \tag{4.2}$$

$\mathscr{L}^{\mathrm{s}}_{\mathrm{M}}$ being independent of $\boldsymbol{A}$ and Φ. Then a direct calculation shows that, in component form, the Euler-Lagrange equations corresponding to the variations of $\boldsymbol{A}$ and Φ are

$$0 = \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta A_a} = (-Q_{ab}v_c + v_a Q_{bc})_{,bc} - Q_{ab'bt} - \Big[\frac{1}{\mu_0}(A_{b'a} - A_{a'b}) + P_b v_a - P_a v_b + \varepsilon_{bac}\mathscr{M}_c\Big]_{,b} - (\varepsilon_0 \Phi_{,a} + \varepsilon_0 A_{a't} - P_a)_{,t} + q v_a, \tag{4.3}$$

$$0 = \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta \Phi} = -Q_{ab'ab} + (P_a - \varepsilon_0 \Phi_{,a} - \varepsilon_0 A_{a't})_{,a} - q, \tag{4.4}$$

where $_{,a} = \partial/\partial x_a$ and $_{,t} = \partial/\partial t$. Letting

$$\boldsymbol{R} = \boldsymbol{P} - \nabla \cdot \boldsymbol{Q}$$

and

$$\gamma = q - \nabla \cdot \boldsymbol{R}$$

we can write (4.4) in the form

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \gamma; \tag{4.5}$$

the quantity γ can then be viewed as the effective charge density. As to Eq. (4.3) we observe that, after some rearrangement, it can be given the form

$$-\frac{1}{\mu_0}\nabla \times \boldsymbol{B} + \varepsilon_0 \boldsymbol{E}_{,t} + \nabla \times \boldsymbol{M} + \overset{*}{\boldsymbol{P}} + \gamma \boldsymbol{v} - (\nabla \cdot \boldsymbol{Q})^* - \nabla \cdot (\boldsymbol{L}\boldsymbol{Q} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}) = \boldsymbol{0}.$$

$\boldsymbol{L}$ denoting the (spatial) velocity gradient and the divergence being taken on the last (second) index; for any vector $\boldsymbol{w}$, the derivative $\overset{*}{\boldsymbol{w}}$ is defined as

$$\overset{*}{\boldsymbol{w}} = \boldsymbol{w}_{,t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{w} + \boldsymbol{w}\nabla \cdot \boldsymbol{v} - \boldsymbol{L}\boldsymbol{w}.$$

Now, it is an easy matter to see that

$$\nabla \cdot (\boldsymbol{L}\boldsymbol{Q} - \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{L}^{\mathrm{T}}) = \nabla \times (\mathscr{M} - \boldsymbol{M}).$$

Then a direct substitution and the definition of $\boldsymbol{R}$ allow us to write (4.3) as

$$\frac{1}{\mu_0}\nabla \times \boldsymbol{B} - \varepsilon_0 \boldsymbol{E}_{,t} = \gamma \boldsymbol{v} + \overset{*}{\boldsymbol{R}} + \nabla \times \mathscr{M}. \tag{4.6}$$

Consistent with (4.6) we regard $\zeta = \overset{*}{\boldsymbol{R}} + \nabla \times \mathscr{M}$ as the effective current density.

Equations (4.5), (4.6) constitute the sought pair of Maxwell's equations in matter. Our purpose now is to derive the equation of motion and then, in turn, the force density acting on the electromagnetic solid. Mathematically the equation of motion is the Euler-

-Lagrange equation corresponding to the variation of the function $x(X, t)$. In order to obtain this equation it is convenient to perform the necessary calculations within the L-description and to have recourse to the fields $\hat{q}$, $\hat{P}$, $\hat{Q}$, $\hat{M}$ which are respectively equal to J times q, P, Q, M. The fields $\hat{q}$, $\hat{P}$, $\hat{Q}$, $\hat{M}$ arise because of the need of specific quantities in the material volume dV. For example,

$$P dv = \hat{P} dV;$$

the relation $dv = JdV$ provides the stated connection $\hat{P} = JP$. It is worth emphasizing, however, that $\hat{P}$, $\hat{Q}$, $\hat{M}$ are not the usual material representations of P, Q, M (cf. Ref. [6], § 12.4). What is important for our procedure is that $\hat{q}$, $\hat{P}$, $\hat{Q}$, $\hat{M}$ are to be viewed as functions of the material coordinate X and time t. Specifically, the dependence of $\hat{P}$ (and $\hat{Q}$, $\hat{M}$) on t stems from the definition

$$\hat{P} dV = \Sigma_\alpha q_\alpha \xi_\alpha$$

as a consequence of the dependence of the ξ_α's on t. As to $\hat{q}$, instead, the charge conservation, q_α = constant, results in the condition $\hat{q} = \hat{q}(X)$.

By the same token we can write the Lagrangian $\mathscr{L}^{\mathrm{m}}_{\mathrm{M}}$ as

$$\mathscr{L}^{\mathrm{m}}_{\mathrm{M}} = \tfrac{1}{2} \hat{\varrho} v^2 - \hat{V} \tag{4.7}$$

with $\hat{\varrho}(X) = J\varrho$, $\hat{V} = JV$.

In view of (4.1) we have

$$\mathscr{L}^{\mathrm{m}} = J\mathscr{L}^{\mathrm{s}}_{\mathrm{F}} + J\mathscr{L}^{\mathrm{s}}_{\mathrm{I}} + \mathscr{L}^{\mathrm{m}}_{\mathrm{M}}.$$

Because $\mathscr{L}^{\mathrm{s}}_{\mathrm{F}}$ is independent of x, as shown in the Appendix the Euler-Lagrange equation $\delta\mathscr{L}^{\mathrm{m}}/\delta x = \mathbf{0}$ reduces to

$$\frac{\delta}{\delta x} (\tilde{\mathscr{L}}^{\mathrm{m}}_{\mathrm{I}} + \tilde{\mathscr{L}}^{\mathrm{m}}_{\mathrm{M}}) = \mathbf{0}, \tag{4.8}$$

where $\tilde{\mathscr{L}}^{\mathrm{m}}(x) = \mathscr{L}^{\mathrm{m}}(A(x), \Phi(x), x)$. Now, on account of (4.7) we see immediately that

$$\frac{\delta \tilde{\mathscr{L}}^{\mathrm{m}}_{\mathrm{M}}}{\delta x} = -\hat{\varrho}\ddot{x} + \frac{\partial}{\partial X} \cdot \frac{\partial \hat{V}}{\partial F}.$$

On the other hand, owing to (3.6) we can write

$$\mathscr{L}^{\mathrm{m}}_{\mathrm{I}} = \hat{q} v \cdot A - \hat{q}\Phi - \hat{P} \cdot \nabla\Phi - \hat{P} \cdot A_{,t} + (\hat{P} \times v) \cdot (\nabla \times A) - \hat{Q} \cdot (\nabla\nabla\Phi) - \hat{Q} \cdot (\nabla A_{,t})$$

$$+ (\hat{Q} \otimes v - v \otimes \hat{Q}) \cdot (\nabla\nabla A) + \hat{M} \cdot (\nabla \times A).$$

So the function $\tilde{\mathscr{L}}^{\mathrm{m}}_{\mathrm{I}}$ depends on x through A and Φ while the dependence on $\dot{x} = v$ is expressed explicitly; hence

$$\frac{\partial \tilde{\mathscr{L}}^{\mathrm{m}}_{\mathrm{I}}}{\partial x} = \hat{q}(\nabla A)v - \hat{q}\nabla\Phi - (\nabla A_{,t})\hat{P} + [\nabla(\nabla \times A)](\hat{P} \times v) - (\nabla\nabla\nabla\Phi)\hat{Q}$$

$$-(\nabla\nabla A_{,t})\hat{Q}+(\nabla\nabla\nabla A)\,(\hat{Q}\otimes v-v\otimes\hat{Q})+[\nabla(\nabla\times A)]\hat{M},$$

$$\frac{\partial\tilde{\mathscr{L}}_{\mathrm{I}}^{\mathrm{m}}}{\partial\dot{x}}=\hat{q}A+(\hat{P}\cdot\nabla)A-(\nabla A)\hat{P}+[\hat{Q}\cdot(\nabla\nabla)]A-(\nabla\nabla A)\hat{Q}.$$

Then, taking into account that $\dot{\hat{q}}=0$, after some rearrangement we arrive at

$$\frac{\partial\tilde{\mathscr{L}}_{\mathrm{I}}^{\mathrm{m}}}{\delta x}=\hat{q}(E+v\times B)+(\hat{P}\cdot\nabla)E+v\times(\hat{P}\cdot\nabla)B+\dot{\hat{P}}\times B+[\hat{Q}\cdot(\nabla\nabla)]E$$

$$+v\times[\hat{Q}\cdot(\nabla\nabla)]B+(\dot{\hat{Q}}\nabla)\times B+(\nabla B)\hat{M}.$$

In conclusion, upon dividing by J we can write the equation (4.8) in terms of the fields $\mathscr{E}$, $\mathscr{M}$, R, γ as

$$\varrho\ddot{x}=\gamma\mathscr{E}+(\overset{*}{R}+\nabla\times\mathscr{M})\times B+\frac{1}{J}\frac{\partial}{\partial x}\cdot\frac{\partial\hat{V}}{\partial F}$$

$$+\nabla\cdot\{\mathscr{E}\otimes R+[(Q\nabla)\otimes\mathscr{E}]^{\mathrm{T}}+\mathscr{B}\overset{*}{Q}-(B\cdot\mathscr{M})\mathbf{1}+\mathscr{M}\otimes B\}\tag{4.9}$$

where $\mathscr{B}$ is the skew-symmetric tensor associated with the vector B. The right hand side of (4.9) constitutes the expression for the force density acting on the electromagnetic solid. In a natural way we may regard such a force as consisting of three different contributions, namely the body force

$$b=\gamma\mathscr{E}+(\overset{*}{R}+\nabla\times\mathscr{M})\times B,\tag{4.10}$$

the Piola-Kirchhoff stress

$$S=\frac{\partial\hat{V}}{\partial F},$$

and the electromagnetic stress

$$T^{\mathrm{em}}=\mathscr{E}\otimes R+[(Q\nabla)\otimes\mathscr{E}]^{\mathrm{T}}+\mathscr{B}\overset{*}{Q}-(B\cdot\mathscr{M})\mathbf{1}+\mathscr{M}\otimes B.\tag{4.11}$$

It is an advantage of the variational approach that the constitutive equations can be derived along with the balance equations. Specifically, first observe that, when q, P, Q, and $\mathscr{M}$ are the quantities to be varied, owing to (3.2), (3.8), and (3.9) the pertinent Lagrangian is

$$\mathscr{L}^{\mathrm{m}}=Jq(v\cdot A-\Phi)+JP\cdot\mathscr{E}+JQ\cdot(\nabla\mathscr{E})+J\mathscr{M}\cdot B-JV_1(F,q)$$

$$-JV_2(F,P,Q,\mathscr{M})+\ldots,$$

the dots standing for the contribution $\frac{1}{2}\hat{\varrho}v^2+J\mathscr{L}_{\mathrm{F}}^{\mathrm{s}}$ which is independent of q, P, Q, $\mathscr{M}$. Then the Euler-Lagrange equations corresponding to variations of q, P, Q $\mathscr{M}$ are

$$v\cdot A-\Phi-\frac{\partial V_1}{\partial q}=0,\tag{4.12a}$$

$$\mathscr{E} - \frac{\partial V_2}{\partial \boldsymbol{P}} = \boldsymbol{0}, \tag{4.12b}$$

$$\nabla \mathscr{E} - \frac{\partial V_2}{\partial \boldsymbol{Q}} = \boldsymbol{0}, \tag{4.12c}$$

$$\boldsymbol{B} - \frac{\partial V_2}{\partial \mathscr{M}} = \boldsymbol{0}. \tag{4.12d}$$

Is is apparent from (4.12b, c) that the derivatives $\partial V_2/\partial \boldsymbol{P}$, $\partial V_2/\partial \boldsymbol{Q}$ satisfy the condition

$$\nabla \frac{\partial V_2}{\partial \boldsymbol{P}} = \frac{\partial V_2}{\partial \boldsymbol{Q}}.$$

Once the functions V_1, V_2 are chosen, the relation (4.12a) provides q in terms of $\boldsymbol{A}$, Φ while (4.12b, c, d) are in fact constitutive relations in that they give $\boldsymbol{P}$, $\boldsymbol{Q}$ $\mathscr{M}$ in terms of $\mathscr{E}$ and $\boldsymbol{B}$ (or vice versa).

Within the theory of elastic dielectrics the dependence of the internal energy on the polarization gradient is motivated on several grounds. In the present case this is accomplished by letting V_2 depend also on $\partial \boldsymbol{P}/\partial \boldsymbol{X}$. Hence in view of the identity

$$\left(J \frac{\partial V_2}{\partial \boldsymbol{P}_{,K}}\right)_{,K} = J \left(\frac{\partial V_2}{\partial \boldsymbol{P}_{,K}} F_{kK}\right)_{,h}$$

Eq. (4.12) becomes

$$\mathscr{E} - \frac{\partial V_2}{\partial \boldsymbol{P}} + \nabla \cdot \left[\left(\frac{\partial V_2}{\partial \nabla \boldsymbol{P}}\right)^{\mathrm{T}} \boldsymbol{C}\right] = 0,$$

where $\boldsymbol{C} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$.

As shown by Maugin [7], electromagnetic internal variables prove very useful in setting up models for electromagnetic solids such as elastic ferroelectrics and ferromagnets. If such dependences would be in order we could generalize our scheme by letting the response functions (e.g. V_2) depend on the time derivatives of $\boldsymbol{P}$, $\boldsymbol{Q}$, $\mathscr{M}$. In so doing, however, we would model non-dissipative (or gyroscopic) effects, which is expected in view of the variational character of the approach. A simple way of accounting also for dissipative effects is to consider the constitutive equations for $\mathscr{E}$ and $\boldsymbol{B}$ in terms of $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\mathscr{M}$, arising from (4.12), and to supplement them with dissipative terms like $\boldsymbol{N}\dot{\boldsymbol{P}}$ in the expression for $\mathscr{E}$ (see, e.g., Ref. [8]). The characterization of the phenomenological quantities so introduced should then be found through a thermodynamic analysis. This point is not developed here.

5. Comparison with other theories

Before ending this paper it is worth presenting a brief comparison with the analogous results which already appeared in the literature. Explicitly, the comparison involves Maxwell's equations and the force acting on the electromagnetic solid. It is immediately apparent

that the expressions (4.5), (4.6) for Maxwell's equations coincide with the analogous ones given by Dixon and Eringen [9], Lax and Nelson [4], and Maugin and Eringen [10]. The same is true for the statistical formulation and the formulation of Lorentz and Minkowski; the formal coincidence is obtained once suitable identifications are made [2]. As a consequence the equations (4.5), (4.6) do not coincide with Maxwell's equations in the Chu formulation [2] because of the extra term $-\nabla\times(\boldsymbol{M}\times\boldsymbol{v})$ in Faraday's equation.

The same conclusion holds for the expression of the force. In this regard, however, we have to observe that the magnetic moment vector $\boldsymbol{M}$ ($\boldsymbol{M}_e$ in Ref. [2]) is related to the magnetization

$$\boldsymbol{M}_G dv = \Sigma_\alpha e_\alpha \boldsymbol{\xi}_\alpha \times (\boldsymbol{v}+\dot{\boldsymbol{\xi}}_\alpha)$$

by

$$\boldsymbol{M} = \boldsymbol{M}_G + \boldsymbol{v}\times\boldsymbol{P}.$$

In addition we remark that, in the present theory, the contributions due to the electric quadrupole moment $\boldsymbol{Q}$ are evaluated both for Maxwell's equations and for the expression of the force; this constitutes an improvement on the theories in Ref. [2].

Finally, a comparison with the paper by Lax and Nelson [4] is in order especially because their approach is variational in character. From a mathematical standpoint the peculiar feature of the approach of Lax and Nelson is that the unknown functions are the position vectors $\boldsymbol{x}_\alpha$ of the single charged particles. That is why Lax and Nelson first derive the equations of motion as Euler-Lagrange equations and then proceed by averaging over the forces acting on the charged particles. In the present approach, instead, the interaction Lagrangian is written in terms of the electromagnetic field and of the position $\boldsymbol{x}$ of the center of mass (of the groups of charged particles). Physically this means that the interaction Lagrangian has the form of a multipole expansion, which is associated with a twofold advantage: it makes the generalization to higher-order expansions immediate and renders the derivation of the sought equations purely deductive. In a sense our approach is similar to and generalizes that developed by Lax and Nelson in connection with elastic pyroelectrics [5].

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APPENDIX

With each material system we can associate a spatial Lagrangian (density) $\mathscr{L}^{\mathrm{s}}$ or a material Lagrangian $\mathscr{L}^{\mathrm{m}}$ depending on whether we describe the system through the E-description or the L-description. Specifically, consider a (solid) body interacting with a set of fields ψ_σ, $\sigma = 1, \ldots, n$. The Lagrangians $\mathscr{L}^{\mathrm{s}}$, $\mathscr{L}^{\mathrm{m}}$ are given by the expressions

$$\mathscr{L}^{\mathrm{s}} = \mathscr{L}^{\mathrm{s}}(\psi_\sigma, X), \qquad \mathscr{L}^{\mathrm{m}} = \mathscr{L}^{\mathrm{m}}(\psi_\sigma(\boldsymbol{x}), \boldsymbol{x}),$$

which mean that the unknown functions are, respectively, $\psi_\sigma(x, t)$, $X(x, t)$ and $\psi_\sigma(x(X, t), t)$, $x(X, t)$ [3]; to save writing the dependence on the derivatives of the unknown functions is unspecified and understood.

On the basis of the invariance of the action under the transformation mapping the E-description and the L-description into each other, we have $\mathscr{L}^{\mathrm{m}} = J\mathscr{L}^{\mathrm{s}}$. As proved in Ref. [3], the variational derivatives of $\mathscr{L}^{\mathrm{s}}$ and $\mathscr{L}^{\mathrm{m}}$ with respect to ψ_σ and x are related by

$$\frac{\delta \mathscr{L}^{\mathrm{m}}}{\delta \psi_\sigma} = J \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta \psi_\sigma}, \quad \sigma = 1, \ldots, n,$$

$$\frac{\delta \mathscr{L}^{\mathrm{m}}}{\delta x} + \frac{\delta \mathscr{L}^{\mathrm{m}}}{\delta \psi_\sigma} \frac{\partial \psi_\sigma}{\partial x} = -JF^{-1} \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta X}. \tag{A.1}$$

Some care must be exercised in writing explicitly the variational derivatives $\delta\mathscr{L}^{\mathrm{m}}/\delta\psi_\sigma$ and $\delta\mathscr{L}^{\mathrm{s}}/\delta\psi_\sigma$. Specifically, $\mathscr{L}^{\mathrm{s}}$ is meant to depend on ψ_σ through the values ψ_σ themselves and the derivatives $\partial\psi_\sigma/\partial t$, $\partial\psi_\sigma/\partial x$, ... Analogously, the dependence of $\mathscr{L}^{\mathrm{m}}$ on ψ_σ occurs explicitly through the values of ψ_σ and the derivatives $\dot{\psi}_\sigma$, $\partial\psi_\sigma/\partial X$, ...

A different viewpoint may be adopted for deriving the connection between the variational derivatives of $\mathscr{L}^{\mathrm{s}}$ and $\mathscr{L}^{\mathrm{m}}$ with respect to the position variables x, X. Since the variables ψ_σ are held fixed, we may consider the function $\tilde{\mathscr{L}}^{\mathrm{m}}$ defined as $\tilde{\mathscr{L}}^{\mathrm{m}}(x) = \mathscr{L}^{\mathrm{m}}(\psi_\sigma(x), x)$. Hence (A.1) becomes

$$\frac{\delta \tilde{\mathscr{L}}^{\mathrm{m}}}{\delta x} = -JF^{-1} \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta X}. \tag{A.2}$$

A significant simplification occurs when the Lagrangian $\mathscr{L}^{\mathrm{s}}$ takes the form

$$\mathscr{L}^{\mathrm{s}} = \mathscr{L}^{\mathrm{s}}_1(\psi_\sigma) + \mathscr{L}^{\mathrm{s}}_2(\psi_\sigma, X),$$

with a nonvanishing $\mathscr{L}^{\mathrm{s}}_1$; such is the case for the Lagrangian (4.2), $\mathscr{L}^{\mathrm{s}}_1$ being the field Lagrangian. Then (A.1) and (A.2) imply that

$$\frac{\delta \mathscr{L}^{\mathrm{m}}_1}{\delta x} + \frac{\delta \mathscr{L}^{\mathrm{m}}_1}{\delta \psi_\sigma} \frac{\partial \psi_\sigma}{\partial x} = \mathbf{0}, \tag{A.3}$$

$$\frac{\delta \tilde{\mathscr{L}}^{\mathrm{m}}_1}{\delta x} = \mathbf{0}. \tag{A.4}$$

Whenever a variational formulation holds, namely

$$\frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta \psi_\sigma} = 0, \quad \frac{\delta \mathscr{L}^{\mathrm{s}}}{\delta X} = \mathbf{0},$$

we have

$$\frac{\delta \mathscr{L}^{\mathrm{m}}}{\delta \psi_\sigma} = 0, \quad \frac{\delta \mathscr{L}^{\mathrm{m}}}{\delta x} = \mathbf{0}.$$

Moreover, in view of (A.3) and (A.4) we obtain the equivalent conditions

$$\frac{\delta \mathscr{L}_2^{\mathrm{m}}}{\delta x} + \frac{\delta \mathscr{L}_2^{\mathrm{m}}}{\delta \psi_\sigma} \frac{\partial \psi_\sigma}{\partial x} = \mathbf{0}, \tag{A.5}$$

$$\frac{\delta \tilde{\mathscr{L}}_2^{\mathrm{m}}}{\delta x} = \mathbf{0}. \tag{A.6}$$

Often, when deriving the Euler-Lagrange equations, the form (A.6) turns out to be more convenient than (A.5).

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