RELATIVISTIC FERMION-ANTIFERMION EQUATION AND MASS SPECTRUM OF MESONS

By A. A. Bogush

Institute of Physics, BSSR Academy of Sciences, Minsk*

and A. V. Ostapenko

A. N. Sevchenko Institute of Applied Physical Problems, Minsk*

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In this work both the radial and angular parts of the relativistic fermion-antifermion equation with the Coulomb-like scalar potential are investigated. The formula is obtained determining a discrete energy mass spectrum of the composite quark-antiquark systems. The performed analysis of the angular part of the equations leads to the two series of solutions corresponding to the P-parity values $P = (-1)^J$ and $P = (-1)^{J+1}$, respectively. By using weak coupling approximation the satisfactory agreement between particle mass values calculated on this basis and experimental data for several families of the real mesons is obtained.

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1. *Introduction*

In [1–2] and later in [3–4], a procedure has been developed of reducing the Bethe-Salpeter equation for composite fermion-antifermion systems to two simple systems of equations for the radial and angular part of the wave function. In subsequent papers [5–7], Królikowski and Rzewuski considered a tensor representation of such a relativistic equation with a potential allowing for spin effects (Breit equation). There, the reduction of the equation with a complete potential, having both vector and scalar parts, was made and a classification of the equation solutions by the quantum numbers J, l, s was obtained. However, as it was also shown there, an analytical solution of the equation for the complete potential could not be found because of terms like $(dV/dr)/(E-V)$ (in [5], for example, only the approximate solutions of the Breit equation with the Coulomb potential in the case of parapositronium was obtained).

* Address: Institute of Physics, BSSR Academy of Sciences, Leninskii prospect 70, Minsk 220602, USSR.

** Address: A. N. Sevchenko Institute of Applied Physical Problems, Kurchatova 7, Minsk 220106, USSR.

Solving the relativistic fermion-antifermion equation with a potential which contains only a Lorenz scalar is a simpler problem and, in some cases, allows one to get useful information about the composite systems.

In the paper [3], an analytical solution of the equation in the case of the states with complete momentum $J = 0$ and the scalar potential corresponding to the square-well potential was obtained. Later, in [9], the analytical solution of the equation for the arbitrary J was obtained, where the same potential was used. A similar problem was solved in [10, 11] for the Coulomb-like scalar potential.

In the present work, the complete solution of the radial equation system with the Coulomb-like scalar potential is found for general case of fermions differing in mass; a formula for the angular part of the equation is investigated to classify the solutions by the sequences of states corresponding to the parities $P = (-1)^J$ or $P = (-1)^{J+1}$ of the bound system. No Breit interaction is considered in such approach, and so there is no separation of the states with different l and s for a given J in the classification proposed (contrary to [5–7]). Finally, a comparison between theoretical estimates and available experimental data for meson masses of different families is made. The results obtained are in qualitative agreement with experiment, even for the simplest case of the Coulomb-like scalar potential, which is an evidence of the consistency of the approach developed in [3–7] and [9–11].

2. *Separation of variables and solution of the radial part of the equation*

The relativistic fermion-antifermion equation with instantaneous interaction for the spherically symmetric scalar potential $V(r)$ may be written as (see [3–5]):

$$H_0(\boldsymbol{r})\Psi(\boldsymbol{r}) = 0, \tag{1}$$

where

$$H_0(\boldsymbol{r}) = E - i(\boldsymbol{\alpha}^{(2)} - \boldsymbol{\alpha}^{(1)})\frac{\partial}{\partial \boldsymbol{r}} - \beta^{(1)}m_1 - \beta^{(2)}m_2 - V(r)\,(\beta^{(1)} + \beta^{(2)}), \tag{2}$$

$$\boldsymbol{\alpha}^{(1)} = \boldsymbol{\alpha}\otimes I, \quad \boldsymbol{\alpha}^{(2)} = I\otimes\boldsymbol{\alpha}, \quad \beta^{(1)} = \beta\otimes I, \quad \beta^{(2)} = I\otimes\beta$$

and $\Psi(\boldsymbol{r}) = \{\Psi_q(\boldsymbol{r})\}$ $(q = 1, 2, \ldots, 16)$ is the 16-component wave function of the composite system under consideration. Here $\boldsymbol{\alpha}$, β are the known Dirac matrices, I is a unit 4×4 matrix, m_i $(i = 1, 2)$ are masses of bound particles.

The equation (1) with the Breit potential

$$V(\boldsymbol{r}) = V(r) - \tfrac{1}{2}\,[\boldsymbol{\alpha}^{(1)}\cdot\boldsymbol{\alpha}^{(2)} + (\boldsymbol{r}\cdot\boldsymbol{\alpha}^{(1)})\,(\boldsymbol{r}\cdot\boldsymbol{\alpha}^{(2)})/r^2]V'(r)$$

was considered in the works [5–7]. It was shown there that the presence of the vector part of the potential, as had been noted, did not permit to obtain the analytical solution of the equation. In the case of the scalar potential ($V'(r) = 0$) considered here, the situation becomes simpler and the radial part of Eq. (1) has an analytical solution.

To separate the variables in Eq. (1), as is known from [3, 4], it is necessary to go over to the spherical system of coordinates and make a unitary transformation:

$$U = \exp(-iA\theta)\exp(-iB\varphi), \tag{3}$$

$$A = \frac{i}{2}(\alpha_3^{(1)}\alpha_2^{(1)} + \alpha_3^{(2)}\alpha_2^{(2)}), \quad B = \frac{i}{2}(\alpha_1^{(1)}\alpha_2^{(1)} + \alpha_1^{(2)}\alpha_2^{(2)}). \tag{3a}$$

With the transformation of (3), Eq. (2) transforms to

$$(UH_0(r)U^{-1})U\Psi(r) = 0 \tag{4}$$

and its solution can be written as:

$$U\Psi(r) = Z_J^m(\cos\theta, \varphi)\varrho^J(r). \tag{5}$$

As a result of such transformation to a spherical system of coordinates, the following expressions, essential in further considerations, are obtained [3, 4]:

$$U\left(\alpha^{(i)}\frac{\partial}{\partial r}\right)U^{-1} = \alpha_3^{(i)}\left(\frac{\partial}{\partial r} + \frac{1}{r}\right) + \frac{1}{2r}(\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)})\alpha_3^{(j)} + \frac{Q(i,j)}{r}, \tag{6}$$

$$Q(i,j) = \alpha_2^{(j)}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} - \tfrac{1}{2}\alpha_1^{(j)}\alpha_2^{(j)}\operatorname{ctg}\theta\right) + \alpha_1^{(i)}\left(\frac{\partial}{\partial\theta} + \tfrac{1}{2}\operatorname{ctg}\theta\right), \tag{7}$$

$$Z_J^m(\cos\theta, \varphi) = \left[\mu_+ + \frac{\mu_-}{\sqrt{J(J+1)}}\left(\frac{\partial}{\partial\theta} + \alpha_1^{(1)}\alpha_2^{(1)}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)e^{im\varphi}P_J^m(\cos\theta)\right], \tag{8}$$

$$(i = 1, 2; \quad j = 1, 2; \quad i \neq j; \quad \mu_\pm = (1 \pm \alpha_1^{(1)}\alpha_2^{(1)}\alpha_1^{(2)}\alpha_2^{(2)})/2,$$

which have the properties:

$$[\alpha_3^{(1,2)}, Z_J^m] = 0, \quad [\alpha_1^{(1)}\alpha_1^{(2)} + \alpha_2^{(1)}\alpha_2^{(2)}, Z_J^m] = 0, \tag{9}$$

$$Q(1,2)Z_J^m(\cos\theta, \varphi) = Z_J^m(\cos\theta, \varphi)\alpha_1^{(1)}\sqrt{J(J+1)},$$

$$Q(2,1)Z_J^m(\cos\theta, \varphi) = Z_J^m(\cos\theta, \varphi)\alpha_1^{(2)}\sqrt{J(J+1)}. \tag{10}$$

Relations (6)–(8) provide the separation of the radial part of the original equation. The obtained system of 16 equations for the radial function can, in turn, be subdivided into two independent systems, of eight equations each, which correspond to two eigenvalues $\mathfrak{M} = \pm 1$ of the normal divisor $\mathfrak{M}$ [3]. Further on, a system of eight equations for $\mathfrak{M} = +1$ will be considered. In this case, as has been shown in [8], the system of radial equations consisting of four algebraic and four first-order differential equations can be significantly simplified by introducing, instead of the functions $\varrho_q(r)$ ($r = 1, 2, \ldots 8$), new functions:

$$\Phi_1 = \chi_1 + \chi_6, \quad \Phi_2 = \chi_1 - \chi_6, \quad \Phi_3 = \chi_3 + \chi_8, \quad \Phi_4 = \chi_3 - \chi_8, \tag{11}$$

where $\chi_q(r) = r\varrho_q(r)$.

Now, the system of differential radial equations takes a simple form

$$\left[\frac{d^2}{dr^2} - \frac{J(J+1)}{r^2} + \omega^2\right] \Phi(r) = 0, \tag{12}$$

$$\Phi_2 = \frac{1}{a}\left(\frac{d}{dr} - \frac{\beta_+}{r} + \frac{i}{2r}\right)\Phi, \tag{13a}$$

$$\Phi_4 = \frac{1}{a}\left(\frac{d}{dr} - \frac{\beta_-}{r} - \frac{i}{2r}\right)\Phi, \tag{13b}$$

where

$$\Phi = \Phi_1 = \Phi_3, \quad \beta_\pm = \tfrac{1}{2} \pm \frac{(m_1 - m_2)\sqrt{J(J+1)}}{E}, \quad a = \frac{E^2 - (m_1 - m_2)^2}{2iE},$$

$$\omega^2 = \frac{1}{4E^2}[E^2 - (m_1 + m_2 + 2V(r)^2][E^2 - (m_1 - m_2)^2]. \tag{14}$$

The corresponding system of algebraic equations now reduces to equations:

$$\begin{aligned} 2\chi_2 &= (\alpha_1^- - \alpha_2^-)\Phi - (\alpha_2^- + \alpha_1^-)\Phi_4, \\ 2\chi_5 &= -(\alpha_1^+ - \alpha_2^+)\Phi + (\alpha_1^+ - \alpha_2^+)\Phi_4, \\ 2\chi_4 &= -(\alpha_1^+ + \alpha_2^+)\Phi - (\alpha_1^- - \alpha_2^-)\Phi_2, \\ 2\chi_7 &= (\alpha_1^- + \alpha_2^-)\Phi - (\alpha_1^- - \alpha_2^-)\Phi_2, \end{aligned} \tag{15}$$

where

$$\alpha_i^\pm = (m_i + V + i\sqrt{J(J+1)}/r), \quad (i = 1, 2). \tag{15a}$$

Taking the scalar potential as:

$$V(r) = -\frac{\alpha_s}{r} \tag{16}$$

Eq. (12) can be rewritten in the form:

$$\left[\frac{d^2}{dr^2} - \frac{s^2}{r^2} + \frac{2d^2}{r} - \varepsilon^2\right]\Phi(r) = 0, \tag{17}$$

where

$$\begin{aligned} s^2 &= J(J+1) + \alpha^2 v^2, \\ d^2 &= \tfrac{1}{2} v^2 \alpha_s (m_1 + m_2), \\ v^2 &= [E^2 - (m_1 - m_2)^2]/E^2, \end{aligned} \tag{18}$$

$$\varepsilon^2 = \frac{1}{4E^2}[E^2 - (m_1 - m_2)^2][(m_1 + m_2)^2 - E^2]. \tag{19}$$

It can be seen from the properties of second-order differential equations (such as Eq. (12)) and from expression (19) for ε^2 that the discrete energy spectrum is in the range:

$$|m_1 - m_2| \leqslant E \leqslant m_1 + m_2. \tag{20}$$

The general solution $\Phi(r)$ of Eq. (17) is expressed via the confluent hypergeometric function F as:

$$\Phi = Cr^{\lambda}e^{-\varepsilon r}F(\lambda - d^2/\varepsilon, 2\lambda, 2\varepsilon r), \tag{21}$$

where

$$\lambda = \tfrac{1}{2}(1+\sqrt{1+4J(J+1)+4\alpha^2\nu^2}). \tag{22}$$

Substituting (21) into (13a) and (13b) and using the properties of confluent hypergeometrical functions, it can also be written:

$$\Phi_2 = \frac{2i}{E}e^{-\varepsilon r}C\left[-r^{\lambda}\varepsilon F + r^{\lambda-1}\left(\lambda - \frac{d^2}{\varepsilon}\right)F' + r^{\lambda-1}\left(\frac{d^2}{\varepsilon} - \tfrac{1}{2} + i\right)F\right], \tag{23a}$$

$$\Phi_4 = \frac{2i}{E}e^{-\varepsilon r}C\left[-r^{\lambda}\varepsilon F + r^{\lambda-1}\left(\lambda - \frac{d^2}{\varepsilon}\right)F' + r^{\lambda-1}\left(\frac{d^2}{\varepsilon} - \tfrac{1}{2} - i\right)F\right], \tag{23b}$$

where the following definitions have been introduced:

$$F = F\left(\lambda - \frac{d^2}{\varepsilon}, 2\lambda, 2\varepsilon r\right), \quad F' = F\left(\lambda - \frac{d^2}{\varepsilon} + 1, 2\lambda, 2\varepsilon r\right). \tag{24}$$

The discrete spectrum of solutions (21) of Eq. (17) can be found from the condition of finiteness of the confluent hypergeometric function at $r \to \infty$. For this purpose it is necessary to assume:

$$1/\Gamma(\lambda - d^2/\varepsilon) = 0. \tag{25}$$

It follows from the properties of the Γ-function that

$$\lambda - d^2/\varepsilon = -n_r, \quad n_r = 0, 1, 2, \ldots \tag{26}$$

(n_r is a radial quantum number). Now, taking into account that the function F' (Eq. (24)) enters into Eq. (23) only as a product $(\lambda - d^2/\varepsilon)F'$, the condition of finiteness of the second term in Eq. (23) at $r \to \infty$ yields

$$(\lambda - d^2/\varepsilon)/\Gamma(\lambda - d^2/\varepsilon + 1) = 0. \tag{27}$$

Since $\Gamma(z+1) = z\Gamma(z)$, it can easily be seen that condition (27) is actually reduced to Eq. (25). Thus, to find all the solutions of the system of radial equations (12)–(13), which correspond to the discrete energy spectrum of a q-$\bar{\text{q}}$ system, the same formula (26) can be used.

Note that at $m_1 = m_2 = \mu$

$$E = 2\mu \sqrt{1-4\alpha_s^2/(1+2n_r^2+\sqrt{1+4J(J+1)+4\alpha_s^2})} \tag{28}$$

follows from (26).

3. *Classification of states by P-parity*

Consider now the properties of the original Eqs. (1) and (4) and their solutions with respect to spatial inversion (P-parity) transformations.

Acting on Eq. (1) with the spatial inversion operator, in accordance with the general rules [8], the following expression can be obtained for the complete parity operator:

$$P = \eta\beta^{(1)}\beta^{(2)}I_r(\boldsymbol{r} \to -\boldsymbol{r}) = P_0 I_r, \tag{29}$$

where $P_0 = \eta\beta^{(1)}\beta^{(2)}$, and η is a quantum number determining the internal parity of the system. For the q-$\bar{\text{q}}$ system $\eta = -1$ [12].

It should be noted, however, that under unitary transformation by the operator $U(\theta, \varphi)$, Eq. (1) is changed to Eq. (4). Naturally, the P-parity operator Eq. (29) should also undergo such transformation. Therefore, the behaviour of Eq. (4) due to spatial reflections requires further investigations.

Let us consider the behaviour of the operator $P_0 = \eta\beta^{(1)}\beta^{(2)}$ due to the transformation $U(\theta, \varphi)$. By expanding in series the exponentials: $\exp(-iA\theta)$ and $\exp(-iB\varphi)$ entering into Eq. (3), and using the properties of the matrices A and B (see Eq. (3a)), one can easily obtain:

$$\exp(-iA\theta) = 1-iA\sin\theta-A^2(1-\cos\theta),$$

$$\exp(-iB\varphi) = 1-iB\sin\varphi-B^2(1-\cos\varphi). \tag{30}$$

It can be seen that the operator $U(\theta, \varphi)$ commutes with the matrices $\beta^{(1)}$, $\beta^{(2)}$

$$[U, \beta^{(1,2)}] = 0 \tag{31}$$

and, hence

$$[U, P_0] = 0, \quad P_0' = UP_0U^{-1} = P_0 = \eta\beta^{(1)}\beta^{(2)}, \tag{32}$$

i.e. the operator P_0 under unitary transformation considered does not change its form.

Returning now to the original Eq. (1), it is easy to see that the action of $U(\theta, \varphi)$ transformation (see (3)) on the operator consists only in modification of the second term in the operator:

$$i(\boldsymbol{\alpha}^{(2)}-\boldsymbol{\alpha}^{(1)})\frac{\partial}{\partial \boldsymbol{r}} \to U\left[i(\boldsymbol{\alpha}^{(2)}-\boldsymbol{\alpha}^{(1)})\frac{\partial}{\partial \boldsymbol{r}}\right]U^{-1}. \tag{33}$$

In turn, it is with this term that the sign changes under the action of the operator $P_0 = \eta\beta^{(1)}\beta^{(2)}$. Therefore, by virtue of (31), (32) the operator P_0 will have the same action on the operator $H(\boldsymbol{r})$ in the transformed Eq. (4). Indeed, taking into account relations (6)–(10), it is easy to see that under the action of the operator $P_0' = P_0$ on the transformed operator $H(\boldsymbol{r}) = UH_0(\boldsymbol{r})U^{-1}$ also the sign of the above mentioned term of (33) is changed in Eq. (4).

Thus, the operator $P_0' = P_0$ is also a reflection operator for Eq. (4) after unitary transformation of (3), (30) as well. Now it is seen that the action of the total spatial reflection operator $P' = P_0' I_r(\boldsymbol{r} \to -\boldsymbol{r}) = P_0 I_r$ on the operator $H(\boldsymbol{r})$ of the transformed Eq. (4) leads to their commutation, i.e. $[P', H(\boldsymbol{r})] = 0$.

Now, let us examine the action of the spatial reflection operator P' on the wave function $U\Psi(\boldsymbol{r})$. Write the column matrix $U\Psi(\boldsymbol{r})$ of (5) in the explicit form. Taking into account the possibility of breaking sixteen radial equations into two independent systems, we obtain for the case $\mathfrak{M} = +1$ according to (5)–(8) the function with the following eight components:

$$U\Psi(\boldsymbol{r}) = \{(\widetilde{U\Psi(\boldsymbol{r})})_q\} = \{K_-(J, \theta, \varphi)\varrho_1(r), \varrho_2(r), K_-(J, \theta, \varphi)\varrho_3(r),$$

$$\varrho_4(r), \varrho_5(r), K_+(J, \theta, \varphi)\varrho_6(r),$$

$$\varrho_7(r), K_+(J, \theta, \varphi)\varrho_8(r)\} \exp(im\varphi) P_J^m(\cos\theta),$$

$$K_\pm(J, \theta, \varphi) = \frac{1}{\sqrt{J(J+1)}}\left(\frac{\partial}{\partial r} \pm \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right). \tag{34}$$

The analysis of the behaviour of (34) due to the spatial reflection transformation presents a severe problem. This is due to the fact that under the action of transformation of (29), in (34) the structure of terms with derivatives $\partial/\partial r$ changes so that it is impossible to see how the sign changes in individual components of $(U\Psi)_q$ $(q = 1, 2, \ldots, 8)$ of the function $U\Psi$ in (34). However, these difficulties can be avoided if one goes over to such basis in the space of the function $U\Psi = \{(U\Psi)_q\}$ of (34) where the X_q components of the new function $X = \{X_q\}$ $(q = 1, 2, \ldots, 8)$ are associated with the original components by the following relations:

$$X_1 = (U\Psi)_1 + (U\Psi)_6, \quad X_2 = (U\Psi)_1 - (U\Psi)_6,$$

$$X_3 = (U\Psi)_3 + (U\Psi)_8, \quad X_4 = (U\Psi)_3 - (U\Psi)_8, \tag{35}$$

$$X_5 = (U\Psi)_2, \quad X_6 = (U\Psi)_4, \quad X_7 = (U\Psi)_5, \quad X_8 = (U\Psi)_7.$$

According to this and taking into account (5) and (34) we obtain

$$X_1 = X_3 = \frac{\exp(im\varphi)}{\sqrt{J(J+1)}}\frac{\partial}{\partial\theta} P_J^m(\cos\theta)\frac{\Phi(r)}{r} = \Omega_{L,k}^{Jm}\frac{\Phi(r)}{r}, \tag{36}$$

$$X_2 = -\frac{2im\exp(im\varphi)}{\sqrt{J(J+1)}} P_J^m(\cos\theta)\frac{\Phi_2(r)}{r} = \Omega_{L,2}^{Jm}\frac{\Phi_3(r)}{r}, \tag{37}$$

$$X_4 = -\frac{2im\exp(im\varphi)}{\sqrt{J(J+1)}}P_J^m(\cos\theta)\frac{\Phi_4(r)}{r} = \Omega_{L,4}^{Jm}\frac{\Phi_4(r)}{r}, \tag{38}$$

$$X_5 = X_8 = \exp(im\varphi)P_J^m(\cos\theta)\alpha^-\frac{\Phi(r)}{r} = \Omega_{L,k_1}^{Jm}\frac{\Phi(r)}{r}, \tag{39}$$

$$X_6 = X_7 = -\exp(im\varphi)P_J^m(\cos\theta)\alpha^+\frac{\Phi(r)}{r} = \Omega_{L,k_2}^{Jm}\frac{\Phi(r)}{r},$$

$$(k = 1, 3; \quad k_1 = 5, 6; \quad k_2 = 6, 7), \tag{40}$$

where (see (15))

$$\alpha^{\pm} = \alpha_1^{\pm} = \alpha_2^{\pm} = (\mu \pm i\sqrt{J(J+1)}/r)/E$$

and $\Omega_{L,q}^{Jm}$ $(q = 1, 2, ..., 8)$ denotes the angular part of the X_q components of the function X.

Performing now spatial reflection, which corresponds to the substitution:

$$\theta \to \theta' = \pi - \theta, \qquad \varphi \to \varphi' = \varphi + \pi$$

in the angular variables used, it can easily be seen that in the functions X_1 and X_3 of (36), the factor $(-1)^{J+1}$, and the remaining functions $X_{2,4,5,6,7,8}$ of (37)–(40), the factor $(-1)^J$ arise.

This is evidently due to the fact that different values l and s are possible for any given value J, i.e. at a fixed value J of the functions $X = \{X_q\}$, states with different sets of quantum numbers l and s can be described. The function X describes in general a certain mixture of states. This feature was noticed earlier in [5].

The problem is now reduced to isolating such functions from the obtained set of $X = \{X_q\}$ $(q = 1, 2, ..., 8)$ functions of (36)–(40) for which one and the same factor would appear for all the components under spatial inversion according to the law of P-parity conservation. It is functions of the states of the system with a given P-parity are to be constructed.

To this effect, let us first introduce, according to (13), standard spectroscopic notations for the states of q-$\bar{\text{q}}$ systems:

a) P-parity for q-$\bar{\text{q}}$ systems is equal to $P = (-1)^J$.

Then the following set of states is isolated:

$$J^P = 0^+, 1^-, 2^+, \ldots \tag{41}$$

or, with other notations, this will be:

$$(J \pm 1)_J = {}^3\mathrm{P}_0,\ {}^3\mathrm{S}_1 + {}^3\mathrm{D}_1,\ {}^3\mathrm{P}_2 + {}^3F_2, \ldots \tag{41a}$$

b) P-parity for q-$\bar{\text{q}}$ systems is equal to $P = (-1)^{J+1}$.

Then, the following set of states is isolated:

$$J^P = 0^-, 1^+, 2^-, \ldots \tag{42}$$

or

$$^{(1-3)}J_J = {}^1\mathrm{S}_0,\ {}^1\mathrm{P}_1 + {}^3\mathrm{P}_1,\ {}^1\mathrm{D}_2 + {}^3\mathrm{D}_2,\ \ldots \tag{42a}$$

Now, the wave function $Z^{Jm}_{L,q}$ can be split into the orbital and the spin parts by using the Clebsh-Gordon coefficients $C^{Jm}_{Lmsm_s}$. The formula for splitting is:

$$Z^{Jm}_{L,q} = \sum C^{Jm}_{Lmsm_s} Y^{Jm}_q(\cos\theta, \varphi)\chi_{m_s}, \tag{43}$$

where $Y^{Jm}_q(\cos\theta, \varphi)$ are spherical vectors, χ_{m_s} are Dirac spinors.

Making use of the possibility of expanding (43) as well as of the law of P-parity conservation, we can now isolate two sets of wave functions by the following rules:

Case 1. P-parity of q-$\bar{\mathrm{q}}$ systems is equal to $P = (-1)^J$.

a) To the functions $X^J_{L,m,k}$ $(k = 1, 3)$ there corresponds $L = J$. This means that (see Eq. (30)):

$$X^J_k = \Omega^{J(s=1, m_s=0)}_{J,m,k} \frac{\Phi(r)}{r}, \tag{44}$$

$$X_k = \Omega^{J(s=0)}_{J,m,k} \frac{\Phi(r)}{r}. \tag{45}$$

b) To the functions $X^J_{L,m,k'}$ $(k' = 2, 4, 5, 6, 7, 8)$ there corresponds $L = J \pm 1$ so that (see Eq. (37)–(40)):

$$X^J_{k'} = \Omega^{J(s=1)}_{J+1,m,k'} \frac{\Phi_q(r)}{r}, \tag{46}$$

$$X^J_{k'} = \Omega^{J(s=1)}_{J-1,m,k'} \frac{\Phi_q(r)}{r}. \tag{47}$$

Case 2. P-parity of q-$\bar{\mathrm{q}}$ systems is equal to $P = (-1)^{J+1}$.

a) To the functions $X^J_{L,m,k}$ there corresponds $L = J \pm 1$. This means that

$$X^J_k = \Omega^{J(s=1)}_{J+1,m,k} \frac{\Phi(r)}{r}, \tag{48}$$

$$X^J_k = \Omega^{J(s=1)}_{J-1,m,k} \frac{\Phi(r)}{r}. \tag{49}$$

b) To the functions $X^J_{L,m,k'}$ there corresponds $L = J$, so that

$$X^J_{k'} = \Omega^{J(s=1, m_s=0)}_{J,m,k'} \frac{\Phi_q(r)}{r}, \tag{50}$$

$$X^J_{k'} = \Omega^{J(s=0)}_{J,m,k'} \frac{\Phi_q(r)}{r} \tag{51}$$

where $\Phi_q = \Phi_{2,4}$ for $k' = 2, 4$ and $\Phi_q = \Phi$ for $k' = 5, 6, 7, 8$. In these formulae $s = 1, 0$ are spin quantum numbers determining the eigenvalues $s\,(s+1)$ of the operator of the spin momentum square $\hat{s}^2$ ($m_s = 1, 0, -1$ at $s = 1$; $m_s = 0$ at $s = 0$).

Thus, to the two sets of states of Eq. (40) and (41) for the P-parities $(-1)^J$ and $(-1)^{J+1}$, there correspond two different sets of wave functions, (44)–(47) and (48)–(51), respectively. *Note 1.* In the case $L = 0$, considerable simplifications take place. Indeed, by using the equation

$$(U\hat{S}^2U^{-1})U\Psi = s(s+1)U\Psi \tag{52}$$

which is true when $L = 0$, it is easy to establish the following:

a) $s = 0$. All the functions X_k ($k = 1, 3, 4, 5, 6, 7, 8$) (36), (38)–(40) but X_2 vanish, i.e. in the case of antiparallel spins of composite particles there is only one function X_2 which describes the q-$\bar{\text{q}}$ system and its P-parity is always equal to -1 (see (51)).

b) $s = 1$. In this case $X_5 = X_6 = X_7 = X_8 = 0$ and only functions X_1, X_3 and X_4 are other than zero which is in agreement with the presence of three possible spin states for the q-$\bar{\text{q}}$ system at $s = 1$. However, the P-parity can be different now (see Eq. (44), (46), (48)–(50)).

The above classification of states by P-parities equal to $(-1)^J$ or $(-1)^{J+1}$ can be clearly interpreted if the approximation of the so-called strong or weak coupling is considered. It turns out that in such approximation, components X_q are isolated which correspond to certain P-parity without distinction in all possible quantum numbers, and the states of q-$\bar{\text{q}}$ systems can be classified using only allowable sets J^P.

It follows from the previously obtained condition of energy spectrum discreteness (Eq. (20)) that the strong coupling between the quarks will occur when

$$E \ll 2\mu. \tag{53}$$

Going over to a corresponding limit in Eqs. (36)–(40) we find that the functions X_q break into "big" and "small" components. "Big" components in this case are $X_{2,4,5,6,7,8}$, i.e. the strong coupling approximation isolates a series of states (42) with $P = (-1)^{J+1}$.

Under the condition of weak coupling:

$$E \to 2\mu \tag{54}$$

"big" components are the functions X_1 and X_3, i.e. such limiting transition isolates a series of states of Eq. (44) corresponding to $P = (-1)^J$.

Note 2. Note that if $L = 0$, $s = 0$ (see Note 1, case a) the $^1S_0$ state of the q-$\bar{\text{q}}$ system does not exist in the limit of weak coupling. This state is described here by one function X_2^0 which vanishes in the case being considered. At the same time, the state $^3S_1$ ($s = 1$) of the system exists at $L = 0$ in the limit of both strong and weak coupling (the nonzero components are X_1, X_3, X_4).

It will be shown below that good agreement with experiment is observed only for a series corresponding to the weak coupling for all the families of mesons. Since in the involved model a Coulomb-like scalar potential is chosen, i.e. the one which is essential for the meson-forming colored quarks only at small distances between them, one can say that the approach being developed agrees with the well-known notions of asymptotic freedom of quarks, though it is not based on the quantum chromodynamic ideology.

4. *Comparison with experiment*

An essential feature of the proposed approach is that the consideration is carried out with a single Coulomb-like scalar potential. Only two parameters are changed depending on the quark aroma, they are: the effective constant of interaction α_s and the effective mass of quark μ.

Based on the model involved, we shall consider families of mesons consisting of light u, d and s quarks as systems of the light quarkonium type and those consisting of heavy quarks c and b as systems of the heavy quarkonium type.

a) Light quarkonium

Attempts to classify low-lying mesons consisting of light quarks u, d and s and their antiquarks were made by many authors on the basis of both non-relativistic and relativistic models. The most satisfactory values for masses of a number of mesons were obtained by introducing a large number of phenomenological parameters (see, for example, [14]). The simple model considered here does not require such procedure.

Table I summarizes the results of numerical calculations by formula (28) for the mass spectrum of mesons consisting of u, d quarks ($\mu_{u,d} = \mu_u = \mu_d$). They are well fitted with the series corresponding to the weak coupling of quarks (41). To estimate the unknown parameters α_s and $\mu_{u,d}$, the states $E^{J=1}_{n_r=0}$ and $E^{J=1}_{n_r=1}$ were identified with the vector mesons $\varrho(776)$ and (1110) (taken from [15]) respectively. $\alpha_{u,d} = 6.37$, $\mu_{u,d} = 909$ (MeV) were obtained. The well-known problem of calculating a reasonable value for the pion mass is naturally avoided in our model. The $^1S_0$ state which is usually compared with the π-meson

TABLE I

Light quarkonium states (for non-strange quarks)

J	n_r	E_{theor} (MeV)	E_{exp} (MeV)	J^P_{theor}	J^P_{exp}
1	0	776	$\rho(776)$	1^-	1^-
0	1	1072	$\delta^+(981)$	0^+	0^+
1	1	1110	(1110) [15]	1^-	1^-
0	2	1268	$\varepsilon(1300)$	0^+	0^+
1	2	1290	$\rho'(1250)$	1^-	1^-
1	3	1405	(1384) [15]	1^-	1^-
1	4	1485	—	1^-	—
1	5	1531	$\rho''(1600)$	1^-	1^-
2	1	1174	f(1273)	2^+	2^+
2	2	1330	$A_2(1317)$	2^+	2^+

TABLE II

Light quarkonium states (for strange quarks)

J	n_r	E_{theor} (MeV)	E_{exp} (MeV)	J^P_{theor}	J^P_{exp}
0	0	992	S*(980)	0^+	0^+
0	1	1452	ε(1300)	0^+	0^+
1	0	1020	Φ(1020)	1^-	1^-
1	1	1498	(1498) [16]	1^-	1^-
1	2	1764	X(1690)	1^-	?
1	3	1939	—	1^-	—
1	4	2064	Φ′(2136)	1^-	?
2	0	1175	f(1273)	2^+	2^+
2	1	1577	f′(1516)	2^+	2^+

does not appear in the series of states (41) allowable for the weak coupling approximation, i.e. it is excluded from consideration (note 2).

In general, it may be concluded that the satisfactory agreement has been obtained for both, the masses and the quantum numbers.

Similar results are obtained for a family of mesons treated as $S\bar{S}$ systems. These results are given in Table II. The states (1020) and (1498) (see [16]) for $J = 1$ ($n_r = 0, 1$) were used as the initial masses.

In both cases, the experimental data available are indicative of the presence of weak coupling between the quarks (asymptotic freedom).

b) Heavy quarkonium

(*i*) Charmonium. For charmonium, the series with $P = (-1)^J$ is also principal. Identifying the J/Ψ-particle mass with $E^{J=1}_{n_r=0}$ and Ψ' with $E^{J=1}_{n_r=1}$, we have from (28)

$$\alpha_c = 2.286, \quad \mu_c = 2.188\ (\text{GeV}).$$

The results of calculations are given in Table III. In the case of charmonium, the state η_c (2.98) which is the ground singlet state can be evaluated if we go to another series (42) and take into account that the effective constant of interaction for the $^1S_0$ state, which is a bound state of the quark and the antiquark with antiparallel spins ($s = 0$), must be different, α'_c. To evaluate α'_c consider the level $\chi(3.51)$ as the ground state with $J^P = 1^+$ ($n_r = 0$). For the same masses μ_c of charmed quarks as before, formula (28) yields $\alpha'_c = 1.65$. It can be seen from Table III that the states 0^1S_0 and 1^3S_1 of charmonium, calculated by the parameters α'_c and μ_c, agree well with the masses of η_c and η'_c mesons.

(*ii*) Bottonium. Presented in Table IV are the calculated and experimental values for the masses of a series of Υ-particles. The values for the parameters α_b and μ_b have been obtained under the condition that $E^{J=1}_{n_r=0} = M_{\Upsilon(9.46)}$, $E^{J=1}_{n_r=1} = M_{\Upsilon(10.02)}$, and are:

$$\alpha_b = 1.012, \quad \mu_b = 5.262\ (\text{GeV}).$$

TABLE III

Charmonium states

J	n_r	E_{theor} (GeV)	E_{exp} (GeV)	J^P_{theor}	J^P_{exp}
0	0	2.94	$\eta_c(2.98)$	0^-	0^-
0	1	3.76	$\eta_c(3.59)$	0^-	0^-
0	0	2.6	—	0^+	—
0	1	3.51	$\chi(3.41)$	0^+	0^+
1	0	3.1	$\psi(3.1)$	1^-	1^-
1	1	3.68	$\psi(3.68)$	1^-	1^-
1	2	3.97	$\psi(3.77)$	1^-	1^-
1	3	4.07	$\psi(4.03)$	1^-	1^-
1	4	4.15	$\psi(4.16)$	1^-	1^-
1	5	4.21	$\psi(4.44)$	1^-	1^-
1	0	3.51	$\chi(3.51)$	1^+	1^+
1	1	3.93	—	1^+	—
2	0	3.59	$\chi(3.55)$	2^+	2^+

TABLE IV

Bottonium states

J	n_r	E_{theor} (GeV)	E_{exp} (GeV)	J^P_{theor}	J^P_{exp}
0	0	8.25	?	0^+	?
0	1	9.71	—	0^+	—
1	0	9.46	$\Upsilon(9.46)$	1^-	1^-
1	1	10.02	$\Upsilon(10.02)$	1^-	1^-
1	2	10.23	$\chi_b(10.25)$	1^-	J^{++}
1	3	10.33	$\Upsilon(10.35)$	1^-	1^-
1	4	10.39	$\Upsilon(10.57)$	1^-	1^-

In general, Table IV shows a good agreement between calculated and experimental data for the masses of Υ-particles.

Note the following distinguishing feature resulting from the above calculations: the effective coupling constant α_s for meson states decreases with the growth of the quark masses. A similar result was also obtained using other approaches to the problem under study.

Thus, the precise solution of Eq. (12) even in the simplest case of the Coulomb-like scalar potential is very appropriate for describing the masses of the meson families. It is the natural step to continue the study by considering the potential which describes the effect of the confinement of quarks. We have also a plan to use a tensor representation of the Breit equation and apply the multipole reduction technique of the radial equation proposed in [5–7].

It should be noted that the estimates obtained by us for masses of light mesons are in agreement with experiment, not worse than that obtained in [18] using the relativistic two-particle equation with the potential consisting of scalar and vector parts.

5. *Conclusion*

The relativistic fermion-antifermion Bethe-Salpeter-like equation is considered. To reduce it, the technique developed in [1–4] is applied. The use of the simple scalar (without the vector part) Coulomb-like potential (i.e. the potential which is essential when describing the hyperfine structure of quark-antiquark systems — see, for example, [17]) permits one to obtain the precise analytical solution of radial part of equation and to derive a simple (with two free parameters) formula for the discrete energy spectrum. On the basis of the analysis of the angular part of the equation and its solutions, two series of states are separated that correspond to the different parities: $P = (-1)^J$ or $P = (-1)^{J+1}$. The bound q-$\bar{q}$ system calculated in the same way within the meson classification pertinent to such separation agrees satisfactorily with the experimental data both, for light and heavy mesons. Thus, one can conclude that the results obtained in the work evidence the simple relativistic model of composite two-particle systems and, hence, this model can be used for the primary description of the basic features of mesons when considered as bound states of a quark and antiquark.

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