ON SO(ν, ν) PURE SPINORS

By S. Giler, P. Kosiński, J. Rembieliński

Institute of Physics, University of Łódź*

and P. Maślanka

Institute of Mathematics, University of Łódź**

(Received November 13, 1985; final version received April 17, 1986)

Using the group-theoretical methods and the geometrical picture of pure spinors due to E. Cartan we give the explicit construction of the manifold of such spinors for the group SO(ν, ν). We apply this construction to solve the Dirac equation for pure spinors in the momentum space.

PACS numbers: 02.20.+b, 11.30.Na

1. Introduction

The nonlinear realisations of groups arised in physics more than a decade ago in connection with current algebra and low energy hadron physics [1–4] (see also [6]). This method is known under the name of the method of effective Lagrangians. Some models, like for example the nonlinear sigma model, are interesting on its own right [5].

It was noticed recently [7–9] that the natural basis for nonlinear realisations in the case of fermion fields is provided by the Cartan theory of pure spinors [10–11]. In this paper we discuss some topics concerning the mathematical structure of pure spinors. We choose the case of the pseudoorthogonal group SO(ν, ν) as an example because it is the simplest case. The general case of SO(p, q) will be treated in the subsequent paper [13]. In our work we emphasize on the geometrical aspects of the problem. Similar results can be obtained by algebraic methods [12].

2. Preliminaries

Let us remind some basic notions concerning the orthogonal groups and Clifford algebras.

* Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-236 Łódź, Poland.
** Address: Instytut Matematyki, Uniwersytet Łódzki, Banacha 22, 90-238 Łódź, Poland.

2.1. The group SO(ν, ν)

Let M be a 2ν-dimensional vector space over R with the pseudoeuclidean scalar product determined by the metric tensor $\hat{g}$

$$\hat{g} = (\hat{g}_{\alpha\beta}) = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

here I is the $\nu\times\nu$ unit matrix. The scalar product in M is invariant under action of the pseudoorthogonal group SO(ν, ν), namely

$$\hat{O}^T\hat{g}\hat{O} = \hat{g} \tag{1}$$

for all $O \in$ SO(ν, ν). According to the Witt theorem [11] M is the direct sum of two ν-dimensional maximal totally singular subspaces, say N and P. Note that this decomposition is not unique. One possible choice is given by the following orthogonal transformation of coordinates

$$x^\alpha = \hat{R}^\alpha{}_\beta \hat{x}^\beta, \tag{2}$$

where

$$\hat{R} = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -I \\ I & I \end{pmatrix}.$$

The metric tensor takes the form

$$\hat{g} = \hat{R}\hat{g}\hat{R}^{\mathrm{T}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{3}$$

The subspaces N and P are spanned by the coordinates x_N^k and x_P^k respectively, where

$$x_N^k \equiv x^k = \frac{1}{\sqrt{2}}(\hat{x}^k - \hat{x}^{k+\nu}), \tag{4}$$

$$x_P^k \equiv x^{k+\nu} = \frac{1}{\sqrt{2}}(\hat{x}^k + \hat{x}^{k+\nu})$$

for $k = 1, \ldots, \nu$. Note that

$$x_N^k = x_{Pk}, \quad x_P^k = x_{Nk}. \tag{5}$$

Let us consider the Lie algebra of SO(ν, ν). From Eq. (1) it follows immediately that the general element of this algebra takes the form

$$\underbrace{\hat{L} = \begin{pmatrix} \hat{A}_+ & \hat{S} \\ \hat{S}^T & \hat{A}_- \end{pmatrix}}_{\text{in the cartesian basis}} \quad \text{or} \quad \underbrace{L = \begin{pmatrix} R & A \\ B & -R^{\mathrm{T}} \end{pmatrix}}_{\text{in the Witt basis}}. \tag{6}$$

Here $\hat{A}_\pm$, $\hat{S}$, A, R and B are real $\nu\times\nu$ matrices and $\hat{A}^T_\pm = -\hat{A}$, $A^T = -A$, $B^T = -B$. We adopt the convention without imaginary unit in the definition of infinitesimal rotations.

2.2. The Clifford algebra

In the sequel we consider the real Clifford algebra $C^{(\nu,\nu)}$ with the anticommutation rules for generating elements

$$\{\hat{\gamma}_\alpha, \hat{\gamma}_\beta\} = 2\hat{g}_{\alpha\beta}I.$$

The vector space spanned by the generators of $C^{(\nu,\nu)}$ is naturally isomorphic to M by identification $(\hat{x}^\alpha) \equiv \hat{x} = \hat{x}^\alpha\hat{\gamma}_\alpha$. In the Witt basis we obtain

$$\{\gamma^i_N, \gamma^k_N\} = 0, \quad \{\gamma^i_P, \gamma^k_P\} = 0, \quad \{\gamma^i_N, \gamma^k_P\} = 2\delta^{ik}\hat{I} \tag{7}$$

and $x = x_{NK}\gamma^k_N + x_{Pk}\gamma^k_P$, $\gamma^k_N = \gamma_{Pk}$, $\gamma^k_P = \gamma_{Nk}$ (see Eqs. (4), (5)).

2.3. The Chevalley construction [11]

In the case of the Clifford algebra under consideration one can give the elegant construction of its representation resembling the construction of the adjoint representation for Lie algebras. This is possible because of the existence of the Witt decomposition of M. Let us note that the vector belonging to N or P can be written in the form $x_N = x^k_N\gamma_{Nk}$ or $x_P = x^k_P\gamma_{Pk}$, respectively. The elements γ_{Nk} and γ_{Pk} generate two 2^ν-dimensional Grassman algebras C^N and C^P over N and P respectively. Let f_P be the element of C^P of maximal order i.e.

$$f_P = \gamma_{P1}\gamma_{P2}\cdots\gamma_{P\nu}. \tag{8}$$

Now, f_P, up to a multiplicative factor, does not depend on the particular choice of basis in P. Furthermore, let us consider the left ideal $C^{(\nu,\nu)}f_P = C_N f_P$. This ideal span the space of the representation ϱ of the Clifford algebra. We define this representation by the formula

$$u \in C^{(\nu,\nu)}: \quad \varrho(u)C^N f_P = uC^N f_P \subset C^N f_P. \tag{9}$$

Note that the generators γ_{Nk} and γ_{Pk} act under ϱ as Grassman multiplication and differentiation respectively. The representation ϱ determines the representations of the Clifford group as well as the groups Pin, Spin and SO(ν, ν) [11].

2.4. Pure spinors

Let Z be an arbitrary maximal totally singular subspace and let $M = Z \oplus Z'$ be the corresponding Witt decomposition. We put

$$f_Z = \gamma_{Z^1}\gamma_{Z^2}\cdots\gamma_{Z^\nu}.$$

As previously f_Z is determined by the choice of Z up to a multiplicative factor. Now $f_Z C^{(\nu,\nu)} = f_Z C^{Z'}$ is the maximal right ideal in $C^{(\nu,\nu)}$. It follows then that the intersection $C^N f_P \bigcap f_Z C^{Z'}$ is the one-dimensional subspace of $C^{(\nu,\nu)}$ [11]. Consequently we can write

$$C^N f_P \bigcap f_Z C^{Z'} = \{s_Z f_P\} \quad \text{where} \quad s_Z \in \text{Spin}\,(\nu, \nu).$$

We call this one-dimensional subspace of the space of pure spinors associated with the maximal totally singular subspace Z. Remark that for the particular case $Z = P$ this subspace has the form βf_P with $\beta \in \boldsymbol{R}$. Let us note that the pure spinor ψ associated with the subspace Z is determined up to a multiplicative factor by the equations.

$$z^{\alpha}\gamma_{\alpha}\psi = 0 \tag{10}$$

for all $z^{\alpha}\gamma_{\alpha} \in Z$.

3. *The construction of pure spinors*

From the discussion given above it is obvious that the pure spinors form a nonlinear representation of the group $SO(v, v)$. Having this in mind we can proceed in the standard fashion and construct the manifold of all pure spinors as follows. We choose an arbitrary but fixed pure spinor and determine its stability subgroup $G_0 \subset SO(v, v)$. Now the group $SO(v, v)$ acts transitively on the coset manifold $SO(v, v)/G_0$. It follows then that we can identify the manifold of all pure spinors with the above coset space. Every pure spinor can be obtained from the standard one by applying a suitable transformation from $SO(v, v)$.

3.1. The stability group

Let Π be the homomorphism of the group Spin (v, v) into $SO(v, v)$. By s we denote the element of Spin (v, v) such that $\Pi(s) = O(s) \in SO(v, v)$. Putting $s_Z f_P = f_Z w$ we obtain

$$\varrho(s)s_Z f_P = s s_Z f_P = s f_Z w = (s f_Z s^{-1}) s w$$

but

$$s f_Z s^{-1} = \sum_{k=1}^{v} (s\gamma_{Zk} s^{-1}) \quad \text{and} \quad s\gamma_{\alpha} s^{-1} = O_{\alpha}^{\ \beta}(s^{-1})\gamma_{\beta}.$$

Consequently, from the above formulas we conclude that $s f_Z s^{-1} = \beta(s) f_Z$, $\beta(s) \in \boldsymbol{R}$, if and only if the subspace Z is invariant under the action $O(s)$. Then

$$s s_Z f_P = f_Z \beta(s) s w \in f_Z C^{Z'} \cap C^N f_P.$$

It follows from the above considerations that *the stability subgroup of the one-dimensional subspace of pure spinors associated with Z consists of those elements of $SO(v, v)$ which leave Z invariant.*

Let us choose the spinor f_P related to the subspace P as a standard one, i.e. we put $Z = P$. It is easily seen from the above that the stability group of P should leave invariant the vectors of the form $\begin{pmatrix} (X^k_N) \\ 0 \end{pmatrix}$. However, we need to know the stability group G_0 of the f_P alone rather than the subspace P, namely the elements $s \in$ Spin (v, v) such that

$$s f_P = f_P. \tag{11}$$

It can be easily shown [13] with the use of the main involution of the Clifford algebra [11] that from Eq. (11) it follows *that $\Pi(s)$ when restricted to the subspace P should have the determinant equal to one.*

Now, the condition that the global transformations leave the subspace P unchanged reads

$$(I-\Pi_P)L\Pi_P = 0, \qquad \Pi_P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where L is the general element of the Lie algebra of $\mathrm{SO}(\nu, \nu)$ as given by Eq. (6). The condition on the determinant mentioned above can be written in the form

$$\det{}_P(\Pi_P e^L \Pi_P) = 1. \tag{13}$$

It follows from Eqs. (6, 12, 13) that the general element of the Lie algebra of the stability group $G_0 \subset \mathrm{SO}(\nu, \nu)$ of f_P is

$$L_0 = \begin{pmatrix} R_0 & A \\ 0 & -R_0^{\mathrm{T}} \end{pmatrix}, \tag{14}$$

where $A^{\mathrm{T}} = -A$, $\operatorname{Tr} R_0 = 0$. From Eq. (14) we see that Lie algebra of G_0 contains two subalgebras

$$\mathscr{R}_0 = \left\{ \begin{pmatrix} R_0 & 0 \\ 0 & -R_0^{\mathrm{T}} \end{pmatrix} \right\} \quad \text{and} \quad \mathscr{A} = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right\}.$$

$\mathscr{R}_0$ is isomorphic to the Lie algebra of $\mathrm{SL}(\nu, \boldsymbol{R})$ and $\mathscr{A}$ is the $\binom{\nu}{2}$-dimensional abelian algebra. Moreover, it follows from the commutation rule between $\mathscr{R}_0$ and $\mathscr{A}$ that $\mathscr{A}$ is an ideal. The general element of G_0 can be written as

$$e^{\mathscr{A}} e^{\mathscr{R}_0} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} e^{R_0} & 0 \\ 0 & e^{-R_0^{\mathrm{T}}} \end{pmatrix} \equiv \{A, e^{R_0}\}_0 \tag{15}$$

with the composition law

$$\{A, e^{R_0}\}_0 \{A', e^{R_0'}\}_0 = \{A + e^{R_0} A' e^{R_0^{\mathrm{T}}}, e^{R_0} e^{R_0'}\}_0.$$

Summaiizing we see that G_0 is isomorphic to the semidirect product of the group $\mathrm{SL}(\nu, \boldsymbol{R})$ and the $\binom{\nu}{2}$-dimensional abelian group N i.e.

$$G_0 \simeq \mathrm{SL}(\nu, \boldsymbol{R}) \otimes N.$$

3.2. The coset space W

Let us write the general element of the Lie algebra of $\mathrm{SO}(\nu, \nu)$ in the following form

$$\begin{pmatrix} R & A \\ B & -R^{\mathrm{T}} \end{pmatrix} = \underbrace{\begin{pmatrix} R_0 & 0 \\ 0 & -R_0^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}}_{\text{the stability subalgebra}} + \underbrace{\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} + \begin{pmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{pmatrix}}_{\text{the complement}},$$

where $\nu\alpha = \operatorname{Tr} R$, $R_0 = R - \left(\frac{1}{\nu} \operatorname{Tr} R\right) I$. *Note that the Lie algebra elements complementing the stability subalgebra do form the subalgebra by themselves.* Its structure is very simple.

It consists of the one-dimensional algebra

$$\left\{\begin{pmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{pmatrix}\right\} \quad \text{and} \quad \binom{\nu}{2} \text{ dimensional abelian ideal } \left\{\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right\}.$$

The elements of the space W (which form the group) we parametrize as follows

$$W = \exp\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix} \exp\begin{pmatrix} \alpha I & 0 \\ 0 & -\alpha I \end{pmatrix} = \begin{pmatrix} I & 0 \\ B & I \end{pmatrix}\begin{pmatrix} e^{\alpha} I & 0 \\ 0 & e^{-\alpha} I \end{pmatrix} \equiv \{\alpha, B\}. \tag{16}$$

The composition law in W reads

$$\{\alpha, B\}\{\alpha', B'\} = \{\alpha+\alpha', B+e^{-2\alpha}B'\}.$$

Consequently W is the semidirect product $W \simeq R \otimes N'$ with $\binom{\nu}{2}$-dimensional abelian N'. The group W is nilpotent so the exponential mapping gives the global map for the group manifold. In our case the group manifold is diffeomorphic to $R^{\binom{\nu}{2}+1}$. Denoting by $b_{ik} = -b_{ki}$ the matrix elements of B we can parametrize the coset manifold $W = \mathrm{SO}(\nu, \nu)/G_0$ (and consequently the manifold of pure spinors) by the set of real numbers α and b_{ik}.

4. *Transformation properties of pure spinors*

To determine the nonlinear action of SO(ν, ν) on the manifold of pure spinors we proceed in standard way. Let $O \in \mathrm{SO}(\nu, \nu)$ be any element and $W \in W$; the action of O is given by the relation $O: W \to W'$, where $W' \in W$ is uniquely determined from

$$OW = W' g_0 \tag{17}$$

with $g_0 \in G_0$. According to Eq. (16) we will denote $W = \{\alpha, B\}$. Then the following transformation rules are obtained from (17):
Stability group G_0:

$$\mathrm{SL}(\nu, R) \ni \{0, \exp R_0\}_0: \begin{cases} \alpha' = \alpha \\ B' = e^{-R_0^{\mathrm{T}}} B e^{-R_0}, \end{cases}$$

$$N \ni \{A, I\}_0: \begin{cases} \alpha' = \alpha + \dfrac{1}{\nu} \ln\det(I+AB), \\ B' = B(I+AB)^{-1}. \end{cases}$$

Note the constraint $A \neq -B^{-1}$ which follows from the fact that SO(ν, ν) cannot be covered by exponential map.
The coset subgroup W:

$$W \ni \{\beta, C\}: \begin{cases} \alpha' = \alpha+\beta \\ B' = e^{-2\beta} B + C. \end{cases}$$

We see that the action of SO(ν, ν) on the manifold W is essentially nonlinear.

5. *Pure spinors in the spinor representation*

Let us construct the elements of the Lie algebra of SO(ν, ν) in the spinor representation. From the standard choice of generators $\sim [\gamma_\alpha, \gamma_\beta]$ we conclude that they are bilinear forms in γ's (traceless in the matrix realization). We can give the following general construction of the spinor generators. Let L be an element of Lie algebra of SO(ν, ν) (see Eq. (6)) and $\hat{L}$ the corresponding element in the spinor representation. Let us introduce the following notation

$$\gamma \equiv \begin{pmatrix} (\gamma_N^k) \\ (\gamma_P^k) \end{pmatrix}, \quad \tilde{\gamma} \equiv (\gamma_N^1, \dots, \gamma_N^\nu, \gamma_P^1, \dots, \gamma_P^\nu)g.$$

Then

$$\hat{L} = -\tfrac{1}{4}\tilde{\gamma}L\gamma. \tag{18}$$

The above formula can be checked by considering the commutation rule $[\hat{L}, \gamma] = L\gamma$ which is easily obtained by direct calculation with help of the relation $gL^{\mathrm{T}}g = -L$ valid for all elements of the Lie algebra of SO(ν, ν).

We can construct now the general pure spinor corresponding to the point $\{\alpha, B\}$ of the manifold W. If we put

$$\boldsymbol{B} = \left(\begin{array}{c|c} 0 & 0 \\ \hline B & 0 \end{array}\right), \quad \sigma = \left(\begin{array}{c|c} \alpha I & 0 \\ \hline 0 & -\alpha I \end{array}\right)$$

then we obtain

$$f(\alpha, B) = \exp(-\tfrac{1}{4}\tilde{\gamma}\boldsymbol{B}\gamma)\exp(-\tfrac{1}{4}\tilde{\gamma}\sigma\gamma)f_P$$

or expanding in the power series

$$f(\alpha, B) = e^{\frac{\alpha\nu}{2}} \prod_{\substack{(i,k) \\ i<k}} (1 - \tfrac{1}{2} b_{ik}\gamma_{Ni}\gamma_{Nk})f_P. \tag{19}$$

6. *The Dirac equation in momentum space*

Let us consider the Dirac equation for massless spinor field in the 2ν-dimensional space M:

$$\gamma^\alpha \partial_\alpha \psi(x) = 0. \tag{20}$$

In momentum space it reads

$$\gamma^\alpha p_\alpha \tilde{\psi}(p) = 0, \tag{21}$$

where $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$.

Let us note now that we can impose the condition that $\psi(x)$ or $\tilde{\psi}(p)$ are pure spinors. However, because the manifold of pure spinors is essentially nonlinear these two conditions are not equivalent. In the sequel we assume that $\tilde{\psi}(p)$ is the pure spinor. We remind that the pure spinor ψ_Z associated with a maximal totally singular subspace Z can be determined up to a multiplicative factor by Eq. (10). Now it follows from the Dirac equation (21) that $\tilde{\psi}(p)$ is supported on the surface $p^2 = 0$. Therefore the vector p is isotropic and belongs to some maximal totally singular subspace Z. The Dirac equation (21) can be viewed as the condition that $\tilde{\psi}(p)$ is a pure spinor associated with such maximal totally singular subspace Z which contain p. This statement form the basis of our construction. It consists simply of finding of all maximal totally singular subspaces which contain p and then the pure spinors associated with them. We proceed as follows. For any given isotropic vector p we choose in some standard way the vector $q(p) \in P$ and the boost $L_p \in SO(\nu, \nu)$ such that $p = L_p q(p)$. Then the spinor $\hat{L}_p^{-1} f_p$ is obviously the pure spinor associated with the maximal totally singular subspace $L_p P$; but $p \in L_p P$ and consequently $\hat{L}_p^{-1} f_P$ fulfil the Dirac equation. Let $G_p \subset SO(\nu, \nu)$ be the stability group of direction in M determined by the vector p. Then any maximal totally singular subspace containing p can be obtained from $L_p P$ by acting with the elements of G_p. Consequently any spinor fulfiling the Dirac equation (21) can be obtained by acting of $\hat{L}_p^{-1} f_P$ with the corresponding elements in spinor representation. We can simplify this construction by noting that $G_p L_p = L_p(L_p^{-1} G_p L_p)$ and that $L_p^{-1} G_p L_p$ belong to the stability group G_q of the direction $q(p) \in P$. Further, we can exclude from G_q those elements which belong to G_0 and consider only the elements of $G_q \cap W$.

After calculations described above we obtain as result

$$\psi(p, Q, \alpha) = \exp(-\hat{L}_p) \exp(\hat{B}) \exp(\hat{\sigma}) f_P, \tag{22}$$

where $\hat{L}_p$ and $\hat{B}$ are given by Eq. (18) with

$$L_p = \left(\begin{array}{c|c} I & 0 \\ \hline \dfrac{p_P \otimes p_N}{p_N^2} & I \end{array} \right)$$

and

$$B = Q + \frac{p_N \otimes Q p_N - Q p_N \otimes p_N}{p_N^2},$$

where Q is any real antisymmetric matrix, $(p_N^k) \equiv p_N$, $(p_P^k) \equiv p_P$, $k = 1, \ldots, \nu$, $p_N^2 = \sum_{k=1}^{\nu} p_N^k p_N^k$. Explicitly

$$\psi(p, Q, \alpha) = \delta(p_N^2) \exp\left(\frac{\alpha \nu}{2}\right)$$

$$\times \exp\left\{ -\tfrac{1}{4} \gamma_N \left[\frac{-p_P \otimes p_N + p_N \otimes p_P + p_N \otimes Q p_P - Q p_N \otimes p_P}{p_N^2} + Q \right] \gamma_N \right\} f_P.$$

The expression obtained here solves the problem posed in [9].

7. *Pure spinors in configuration space*

We consider now the Dirac Equation (20) with the condition that $\psi(x)$ is pure spinor, i.e. $\psi(x) = \exp(\hat{B}(x))\exp(\hat{\sigma}(x))f_P$ (see Eq. (19)). After substitution this last expression to Eq. (20) we easily obtain set of differential equations covariant under nonlinear action of SO(ν, ν) on the coset space W parametrized by $\{\alpha, B\}$:

$$\frac{\nu}{2}\,\partial_{Pk}\alpha(x)+\partial_{Ni}b_{ik}(x)+\frac{\nu}{2}\,b_{ki}(x)\partial_{Ni}\alpha(x) = 0,$$

$$\partial_{P[k}b_{ij]}(x)+b_{[kr}(x)\partial_{Nr}b_{ij]}(x) = 0.$$

We are greatly indebted to Professor R. Rączka for inspiring discussions and permanent encouragement. We also would like to thank Professor P. Budinich for the discussion during his visit in Warsaw.

REFERENCES

[1] C. G. Callan, S. Coleman, J. Wess, B. Zumino, *Phys. Rev.* **177**, 2247 (1969).
[2] S. Coleman, J. Wess, B. Zumino, *Phys. Rev.* **177**, 2239 (1969).
[3] S. Gosiorowicz, D. A. Geffen, *Rev. Mod. Phys.* **41**, 333 (1969).
[4] A. Salam, J. Strathdee, *Phys. Rev.* **184**, 1750, 1960 (1969).
[5] M. Forger, *Lecture Notes in Mathematics* **1037**, 18 (1983).
[6] C. J. Isham, *Nuovo Cimento* **59A**, 356 (1969).
[7] P. Budinich, ISAS preprint 36/83/EP (1983).
[8] P. Budinich, ISAS preprint 40/83/EP (1983).
[9] P. Furlan, R. Rączka, *Phys. Lett.* **152B**, 75 (1985).
[10] E. Cartan, *Leçons sur la théorie des spineurs*, Herman, Paris 1938.
[11] C. C. Chevalley, *The Algebraic Theory of Spinors*, Columbia Press, New York 1955.
[12] R. Rączka, private communication.
[13] S. Giler, P. Kosiński, P. Maślanka, J. Rembieliński, 85/IM/UŁ/6 preprint (1985).