AGAINST THE EXISTENCE OF PHYSICAL INTERPRETATION OF THE REGULARIZATION PROCEDURE

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The regularization procedure in QFT is defined. The problem of the existence of "physical" regularization is discussed.

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1. *Introduction*

In order to deal with ultraviolet divergences in perturbation calculations of QFT, we use some auxiliary, well-known procedure, called regularization. Thanks to it we avoid writing poorly defined expressions, namely, the products of free Green functions connected with some (in practice — with majority) of Feynman diagrams with closed loops. Next, the $\hat{R}$-operation (a set of subtractions) has to be performed. Regularization, plus subtractions, is an unfailing method of calculation [1] leading to well defined finite results. In this way, we obtain an algorithm. However, it apparently has no physical meaning.

The physicists do not agree in estimating the reasons for the ultraviolet divergences. For some of them the fault is the perturbation approach, whereas the others believe in perturbation calculation as a realistic picture of interaction. In the latter case some "physical" corrections of the theory are needed. In this context, for example, the non-local trials in QFT may be regarded. All these efforts are nothing but looking for "physical regularization", namely, a regularization giving a finite and correct result without taking the regularization off[1]. It seems difficult to exclude the possibility of such a regularization at all, but we are going to give some arguments against the existence of "physical regularization".

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[1] By "taking-off the regularization" we mean taking the distributional limit (for regularized perturbation calculation) canceling the regularization after the $\hat{R}$-operation has been performed.

2. *Regularization*

We define the term "regularization" as used in this paper. Let us limit ourselves to scalar theories.

Making use of the standard Feynman formula

$$\Delta_F(p) = \frac{1}{p^2-m^2+i\varepsilon} = \frac{1}{i}\int_0^\infty e^{i\alpha(p^2-m^2+i\varepsilon)}d\alpha \tag{1}$$

and integrating (for given Feynman diagram) over internal momenta, we have [1]:

$$\iint_0^\infty d\alpha_1 \dots d\alpha_L D^{-2}(\alpha)\exp\left(i\frac{A(k,\alpha)}{D(\alpha)}\right)\exp\left(-i\sum_{l=1}^{L}\alpha_l(m_l^2-i\varepsilon)\right), \tag{2}$$

where $D(\alpha)$ and $A(k, \alpha)$ are standard expressions connected with trees and two-trees of the diagram respectively, k is the set of external momenta.

In the cases when the index ω_Γ of the diagram ($\omega_\Gamma = 2L-4(n-1)$; L — number of lines of the diagram, n — number of vertices) is non-negative or the index of any one-particle irreducible subdiagram is non-negative, the integral (2) is not well defined because in this cases the function $D^2(\alpha)$ vanishes in the point $\alpha = 0$ (or, respectively, in the point $\alpha_i, \dots, \alpha_j = 0$, where $i, \dots, j$ are numbers of the lines of the divergent subdiagram), giving a nonintegrable singularity.

In order to remove this singularity we introduce regularization, namely, instead of (2) we start with

$$I_{\text{reg}}(k,\gamma) = \iint d\alpha\chi_\gamma(\alpha)D^{-2}(\alpha)\exp\left(i\frac{A(k,\alpha)}{D(\alpha)}\right)\exp\left(-i\sum\alpha_l(m_l^2-i\varepsilon)\right). \tag{3}$$

The function $\chi_\gamma(\alpha)$ is zero in the same points where $D^{-2}(\alpha)$ is singular and, moreover, it tends (distributionally) to 1 when $\gamma \to \gamma_0$ (the set of parameters γ is responsible for the shape of the function $\chi(\alpha)$).

The standard methods of regularization, namely, the analytical [2], the dimensional [3, 4] and the Pauli-Villars regularization methods [5, 6] are of the type (3). By the term "regularization" we understand here each procedure illustrated by (3).

3. *The $\hat{R}$-operation*

We limit ourselves to the primitively divergent diagrams Γ ($\omega_\Gamma \geqslant 0$, no divergent subdiagrams). In accordance to the BPHZ-theorem [1, 7, 8], if the set of subtractions is performed on (3), the "potential source" of singularity is removed from (3) (when (3) is differentiated with respect to k, some products of α-s appear in the numerator of the integrand and cancel the singularity of $D^{-2}(\alpha)$).

If the integration $d\alpha$ is performed in (3), we come to the expression which can be divided into three parts (for primitively divergent diagrams there is only one parameter γ):

$$I_{\text{reg}}(\boldsymbol{k}, \gamma) = Sing(\boldsymbol{k}, \gamma) + I(\boldsymbol{k}) + T(\boldsymbol{k}, \gamma), \tag{4}$$

where $Sing(\boldsymbol{k}, \gamma)$ is singular for $\gamma \to \gamma_0$ and disappears under the $\hat{R}$-operation. (The $\hat{R}$-operation, the integration $d\alpha$ and the limit $\gamma \to \gamma_0$ are not trivially interchangeable. This problem has been discussed in [9]);

$T(\boldsymbol{k}, \gamma)$ is the Taylor expansion (with respect to γ) of the regular component of $I_{\text{reg}}(\boldsymbol{k}, \gamma)$ around $\gamma = \gamma_0$, without the "zero" term ($I(\boldsymbol{k})$ is the "zero" term). Of course $T(\boldsymbol{k}, \gamma)$ vanishes for $\gamma = \gamma_0$;

$I(\boldsymbol{k})$ is the final result of calculation, because, after the $\hat{R}$-operation is realized, $Sing(\boldsymbol{k}, \gamma)$ disappears and after the limit $\gamma \to \gamma_0$ has been performed (taking-off the regularization), $T(\boldsymbol{k}, \gamma)$ also disappears.

The amplitude $I(\boldsymbol{k})$ is, of course, determined up to some polynomial $W(\boldsymbol{k})$. The order of this polynomial depends on the number of subtractions in the operator $\hat{R}$. The order of $W(\boldsymbol{k})$ is the maximal one, for which $\hat{R}W(\boldsymbol{k})$ still vanishes. The coefficients of $W(\boldsymbol{k})$ depend on the choice of regularization method (choice of $\chi_\gamma(\alpha)$).

It seems reasonable to treat $I(\boldsymbol{k})$ as a standard result. Every calculation based on the Feynman diagrams has to be in agreement with this result.

All this "trouble" with regularization and subtractions cannot be avoided when the Feynman-diagram technique is used. The best we can achieve, is the absence of the singular term $Sing(\boldsymbol{k}, \gamma)$ in (4) *before* the $\hat{R}$-operation is performed. This is realized, e.g., on the ground of Pauli-Villars regularization method [1, 10]. Roughly speaking: the singular component of the nonregularized integrand of (2) is then "orthogonal" to $\chi(\alpha)$. However, even in this case, it is difficult to speak about any physical interpretation of the regularization procedure, because the limit is indispensable as well, in order to cancel $T(\boldsymbol{k}, \gamma)$.

From this point of view, the following question seems to be natural: is it possible to find some regularization for which the non-singular term $T(\boldsymbol{k}, \gamma)$ would be of the same structure (with respect to the variables $\boldsymbol{k}$) as the term $Sing(\boldsymbol{k}, \gamma)$? In this case one would not need to take-off the regularization at all, because — even for $\gamma \neq \gamma_0$ — the term $T(\boldsymbol{k}, \gamma)$ would be a polynomial with respect to $\boldsymbol{k}$. The *whole* expression (4) taken for $\gamma \neq \gamma_0$ would be in agreement with our standard result. The regularization would be "physical".

Let us stress here: we are not going to look for "physical" regularization — we are going to show the non-existence of it.

4. *The non-existence of the "physical" regularization*

Let us start with the standard formula for the regularized amplitude (primitively divergent diagram → one parameter γ only)

$$I_{\text{reg}}(\boldsymbol{k}, \gamma) = \iint\limits_0^\infty d\alpha \chi_\gamma(\alpha) D^{-2}(\alpha) \exp\left(i\frac{A(\boldsymbol{k}, \alpha)}{D(\alpha)}\right) \exp\left(-i\sum \alpha_l(m_l^2 - i\varepsilon)\right). \tag{5}$$

Let us introduce a symbol $\partial_k^{\frac{\omega}{2}+1}$:

$$\partial_k^{\frac{\omega}{2}+1} \equiv \frac{\partial^{t_1}}{\partial k_1^{t_1}} \cdots \frac{\partial^{t_s}}{\partial k_s^{t_s}},$$

where $\sum_1^s t_i = \frac{\omega}{2}+1$, ω — index of the diagram, $k_1 \dots k_s$ is any set of external invariants k. For a "physical" regularization one would have:

$$\partial_k^{\frac{\omega}{2}+1} I_{reg}(k, \gamma) = \partial_k^{\frac{\omega}{2}+1} I(k) \quad \text{for} \quad \gamma \neq \gamma_0 \tag{6}$$

($\partial_k^{\frac{\omega}{2}+1}$ acts under the integral!)

$$\iint_0^\infty d\alpha \chi_\gamma(\alpha) \frac{1}{D^2}\left[\partial_k^{\frac{\omega}{2}+1} \exp\left(i\frac{A}{D}\right)\right] \exp\left(-i\sum \alpha_l(m_l^2 - i\varepsilon)\right)$$

$$= \iint_0^\infty d\alpha \frac{1}{D^2}\left[\partial_k^{\frac{\omega}{2}+1} \exp\left(i\frac{A}{D}\right)\right] \exp\left(-i\sum \alpha_l(m_l^2 - i\varepsilon)\right) \quad \text{for} \quad \gamma \neq \gamma_0; \tag{7}$$

$$\iint_0^\infty d\alpha(1-\chi_\gamma(\alpha)) \frac{1}{D^2}\left[\partial_k^{\frac{\omega}{2}+1} \exp\left(i\frac{A}{D}\right)\right] \exp\left(-i\sum \alpha_l(m_l^2 - i\varepsilon)\right) = 0 \quad \text{for } \gamma \neq \gamma_0. \tag{8}$$

Thanks to the presence of the operator $\partial_k^{\frac{\omega}{2}+1}$, the integral on the right-hand side of (7) is convergent (the lower limit of integration is of interest) and — what follows — the integral (8) is convergent as well. The upper limit of integration is "safe" until masses m_l are different from zero. In order to avoid difficulties with infrared divergences in our discussion, we understand ε as being kept different from zero. As far as ε is a positive constant, the integral (8) defines a function of L real variables m_l^2 (L — number of internal lines of the Feynman diagram).

We can write (8) in the form:

$$\iint_0^\infty d\alpha \varphi_k(\alpha) \exp(i\alpha M) \prod_{l=1}^L \exp(-\varepsilon\alpha_l) = F(M). \tag{9}$$

The integral (9) is well defined for every $M \in R^L$ because of the presence of the ε-term. So, we have here a Fourier transform of $\Phi(\alpha)\prod_l \exp(-\varepsilon\alpha_l)$, where $\Phi(\alpha) = 0$ outside the area of integration in (9), and $\Phi(\alpha) = \varphi_k(\alpha)$ inside this area. The function $F(M)$ is the right-hand side of (8), and hence, it is zero. It results that $\varphi(\alpha)$ has to be at least a distributional zero as well and, according to (8), $\chi_\gamma(\alpha)$ has to be (at least) a distributional 1.

No "physical" regularization exists.

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