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CAUSTICS OF SPACE-TIME FOLIATIONS IN GENERAL RELATIVITY

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Gravitation singularities are examined as singularities of space-time foliations. These singularities represent topology transitions and caustics.

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1. *Introduction*

The description of gravitation singularities as singularities of space-time foliations related to gravitational fields was suggested in [1, 2]. In this paper we aim at studying structure of these singularities. Our approach is based on the following propositions and their corollary.

Proposition 1. For any gravitational field g on an orientable manifold X^4 there exists a g-compatible pair of a nonvanishing 1-form ω and a Riemannian metric g^R on X^4 such that

$$g_{\mu\nu} = g^R_{\mu\nu} - 2\omega_\mu\omega_\nu/|\omega|^2, \tag{1}$$

where $|\omega|^2 = g^{R\mu\nu}\omega_\mu\omega_\nu = -g^{\mu\nu}\omega_\mu\omega_\nu$. Inversely, let ω be a nonvanishing 1-form on a manifold X^4. For any Riemannian metric g^R on X^4 there exists a pseudo-Riemannian metric g such that Eq. (1) holds. The form $\omega/|\omega|$ coincides with a tetrad form $h^0 = h^0_\mu dx^\mu$ of a gravitational field g.

Proposition 2. There is one-to-one correspondence between nonvanishing 1-forms ω and smooth orientable distributions F of 3-dimensional subspaces of tangent spaces to X^4 which are defined by the equation $\omega(F) = 0$.

Let us call space-time distribution a distribution F whose generating form ω is a tetrad form h^0 of some gravitational field g on X^4.

Corollary. Any gravitational field g generates space-time distributions on a manifold X^4. Inversely, any 3-dimensional orientable distribution on X^4 is a space-time distribution relative to some gravitational field.

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A space-time distribution induces the $(3+1)$ decomposition of the tangent bundle $T(X) = F \oplus T^0(X)$ in the 3-dimensional spatial (relative to g) subbundle F and its timelike orthocomplement $T^0(X)$ (relative to g^R and g). Thereby, a space-time distribution defines a space-time structure on a manifold, which is compatible with a given gravitational field g. The associated Riemannian metric g^R defines a g-compatible locally Euclidean topology on a space-time which is equivalent to the manifold topology on X^4.

A space-time structure is thought to be causal if a space-time distribution is integrable, and its generating form ω is exact, i.e. $\omega = df$. Let us remind that a distribution of codimension 1 is called integrable if its generating form obeys the equation $\omega \wedge d\omega = 0$. In this case, a manifold foliates in hypersurfaces such that fibers of the distribution are tangent to slices of this foliation.

This notion of causality coincides with the stable causality by Hawking [3]. Slices of a causal foliation represent level surfaces of a generating function f. No curve transversal to slices of a causal foliation intersects any slice more than once. If ∇f is bounded on X any transversal curve intersects every slice once and only once. In this case, foliation slices are diffeomorphic to each other, and a space-time represents the product $X = V \times R$ of a slice V and a line of reals R. Such a space-time is globally hyperbolic [3].

Taking into account this causality condition and the correspondence between space--time distributions and gravitational fields, we set up the following criterion of gravitation singularities [1, 2].

Singularity criterion. A gravitational field on a manifold X^4 is free from singularities if it admits a g-compatible pair of a complete Riemannian metric and a causal space-time foliation on X^4.

We distinguish three types of singularities with this criterion.

First, one may face singular gravitational fields characterized by singularities of a Riemannian metric g^R on X^4. This means that a manifold topology of X^4 is non-compatible with the metric topology defined by a gravitational field via a Riemannian metric g^R. Therefore, one ought to remove such singular points from a space-time. If the remainder is noncomplete as a Riemannian space, one can correct this by conformal transformation of metrics g^R and g [5]. Conformal transformations keep a space-time distribution. Thereby, the first-type singularities are reduced to conformal singularities characterized by finite values of conformal curvature at singularity points.

Secondly, there are gravitational fields admitting a complete Riemannian metric and a space-time distribution, but no causal foliation. Such a field, being regular itself, yields causal singularities of a space-time structure on a manifold X^4. Notice the global character of these singularities. Any gravitational field admits locally a causal space-time foliation.

The third type of singularities includes gravitational fields admitting no regular space--time distributions. These singularities can be described as distribution singularities. One may indicate them by infinite values of the scalar exterior curvature

$$K = -\partial_\mu h_0^\mu - \tfrac{1}{2} h_0^\mu \partial_\mu \ln |g^R|$$

of fibers of a space-time distribution [6]. The scalar K obeys Raychaudhuri's equation which governs the evolution of K in a space-time with a given Ricci curvature. In this particular case of a causal foliation F generated by an exact tetrad form $h^0 = df$ (the dual vector field h_0 is geodesic, and F defines a synchronous coordinate frame) Raychaudhuri's equation for K reads:

$$\frac{dK}{ds} = -R_{00}-2\sigma^2+\tfrac{1}{3}K^2, \qquad \sigma_{ab} = K_{ab}-\tfrac{1}{3}\gamma_{ab}K.$$

Singular solutions of this equation are well known and, e.g., are applied in singularity theorems by Hawking and Penrose [3]. These singularities represent focal and conjugate points, i.e. critical points of the exponential map defined by a geodesic field.

However, in a general case the dual field h_0 is not geodesic, Raychaudhuri's equation includes additional terms preventing singularity formation, and exponential map fails to be applicable to describing distribution singularities.

Our approach is based on the following speculations. Since a regular gravitational field admits locally a causal foliation, singularities of a space-time distributions can be described locally as singularities of causal foliations. Therefore, let a Riemannian metric g^R be complete everywhere on X^4 and a space-time distribution represent a causal foliation in a domain U_r where it is regular. Let us consider again the particular case of a geodesic dual field h_0. It is geodesic relative to both g and g^R inside the domain U_r. Being complete relative to g^R, geodesics of h_0 i.e. time lines are prolonged outside U_r. However, the corresponding extension of the foliation F meets with singularities at intersections of time lines. One can overcome these points by lifting a space-time foliation F onto a total space of the tangent bundle to X^4, by extension of this lifted foliation along geodesics in the total space, and by projection of the extended foliation onto the base X^4. Then, singularities of F can be described as singularities of this projection.

We apply this method in a general case, and we show that foliation singularities represent locally critical points of a generating function and caustics, i.e. branch points of f.

In Sections 2–4 the mathematics we need is briefly reminded. Throughout these sections X denotes a n-dimensional manifold.

2. *Geodesic spray*

Let $L(X)$ denote a principal reper bundle associated with $T(X)$. Let $\{x^\mu, X^\mu\}$ be the bundle coordinates on a total space tl $L(X)$, which are associated with coordinates $\{x^\mu\}$ on a manifold X, i.e. $\{X^\mu_\nu\}$ are matrix elements of the transformation of the basis reper $\{\partial/\partial x^\mu\}$ to a given one. A connection form ω on the bundle $L(X)$ reads:

$$\omega^\mu_\nu = (X^{-1})^\mu_\varepsilon(dX^\varepsilon_\nu+\Gamma^\varepsilon_{\alpha\beta}X^\alpha_\nu dx^\beta),$$

where $\Gamma^\varepsilon_{\alpha\beta}$ denote coefficients of a local connection form Γ on the base X. Lift of geodesics onto tl $T(X)$ is based on the following proposition [5].

Proposition 3. Projection of an integral curve of any standard horizontal vector field on tl $L(X)$ onto X is geodesic in X. Inversely, any geodesic in X can be built in this way.

A field τ on tl $L(X)$ is called horizontal and standard if $\omega(\tau) = 0$ and $\theta^\mu(\tau) = \xi^\mu$ where $\theta^\mu = (X^{-1})^\mu_\nu dx^\nu$ is the canonical form, and ξ^μ is a constant vector. By these conditions a standard horizontal field reads:

$$\tau = (X^{-1})^\mu_\alpha \xi^\alpha \left(\frac{\partial}{\partial x^\mu} - \Gamma^\nu_{\mu\beta} X^\beta_\varepsilon \frac{\partial}{\partial X^\nu_\varepsilon} \right).$$

Integral curves of this field are governed by equations:

$$\frac{dx^\mu}{ds} = X^\mu_\nu \xi^\nu, \qquad \frac{dX^\mu_\nu}{ds} = -\Gamma^\mu_{\alpha\beta} X^\alpha_\nu X^\beta_\varepsilon \xi^\varepsilon, \tag{2}$$

where s denotes a geodesic parameter. Trivial manipulations bring Eq. (2) into the familiar form

$$\ddot{x}^\mu = -\Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad (\dot{x}^\mu(0) = \xi^\mu)$$

of the geodesic equation on X. Denoting $P^\mu = X^\mu_\nu \xi^\nu$, we rewrite Eq. (2) in the form

$$\dot{x}^\mu = P^\mu, \qquad \dot{P}^\mu = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \qquad (P^\mu(0) = \xi^\mu) \tag{3}$$

of geodesic equations on a total space tl $T(X)$ provided with coordinates $\{x^\mu, P^\mu\}$. A geodesic field on tl $T(X)$ reads [7]:

$$\tau = P^\mu \frac{\partial}{\partial x^\mu} - \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta \frac{\partial}{\partial P^\mu}.$$

Its projection onto X is a geodesic field on X.

Proposition 4. A line connection is complete if and only if each standard horizontal vector field τ on tl $L(X)$ (or on tl $T(X)$) is complete [5].

If a field τ on tl $T(X)$ is complete, Eq. (3) has solutions

$$x(s, x(0), P(0)), \qquad P(s, x(0), P(0))$$

defining the exponential map $R \times \text{tl}\, T(X) \to \text{tl}\, T(X)$:

$$\text{Exp}_s : \{x^\mu, P^\mu\} \to \{x^\mu(s, x, P), P^\mu(s, x, P)\}.$$

By Eq. (3) the map Exp possesses the property

$$x(ks, x, P) = x(s, x, kP), \qquad P(ks, x, P) = P(s, x, kP) \tag{4}$$

and a field τ on tl $T(X)$ is called geodesic spray.

Let K be a submanifold of X, and $N(K)$ be a n-dimensional submanifold of tl $T(X)$ which consists of points (x, P) of tl $T(X)$, such that $x \in K$ and P is orthonormal to $T_x(K)$.

For instance, if K reduces to a point x, $N(x) = T_x(X)$. The exponential map Exp_1 yields the map

$$\exp_{s=1} = \pi \circ \mathrm{Exp}_{s=1} : \{x^\mu, P^\mu\} \to \{x^\mu(1, x, P)\}$$

of $N(K)$ onto X. This map is defined everywhere on N but, in general, it fails to be $1:1$ on $N(K)$. Singular points of

$$\exp_1 : N(K) \to X$$

are called focal points (conjugate points if $N(K) = T_x(X)$) [8].

In gravitation theory a geometrical locus of focal or conjugate points is named caustic by analogy with geometrical optics [9–11].

3. *Caustics*

From the mathematical view-point, caustics represent singularities of maps of a rather special kind: Lagrange maps [12, 13]. Any caustic can be brought locally (in the germ form) into the standard form we show here.

Let a space R^{2n} be endowed with coordinates $\{x^\mu, P_\mu\}$. Consider a 1-form $\alpha = P_\mu dx^\mu$ on R^{2n} and a submanifold N of R^{2n} such that $d\alpha(N) = 0$, i.e. α restricted onto N becomes an exact form $\alpha(N) = dz(N)$. Such a manifold of maximal dimension n is called Lagrange manifold. A Lagrange manifold can be built by means of a generating function $S(x^i, P_j)$ of n variables $\{x^i, P_j; i \in I, j \in J\}$ (where (I, J) is some partition of the set $1, \ldots, n$), and this is defined by relations:

$$x^j = -\frac{\partial S}{\partial P_j}, \qquad P_i = \frac{\partial S}{\partial x^i}.$$

Let $\pi : \{x^\mu, P_\mu\} \to \{x^\mu\}$ be projection of R^{2n} onto R^n. This projection, restricted to a Lagrange manifold N

$$\pi_N : \{x^i, P_j\} \to \left\{x^i, x^j = -\frac{\partial S}{\partial P_j}\right\}$$

is called Lagrange map. Caustic is defined to be a set of critical points of Lagrange map, i.e. the points where the matrix $\partial^2 S/\partial P_j \partial P_{j'}$ becomes singular.

The caustic picture on manifolds is locally equivalent to this one. The form $\alpha = P_\mu dx^\mu$ on a total space tl $T^*(X)$ (endowed with the cotangent bundle coordinates $\{P_\mu, x^\mu\}$) defines n-dimensional Lagrange submanifolds of tl $T^*(X)$. A Riemannian metric g^R on X defines the diffeomorphism of $T^*(X)$ to $T(X)$, and images of Lagrange submanifolds of tl $T^*(X)$ under this diffeomorphism are Lagrange submanifolds of tl $T(X)$ defined by the form $\alpha = g_{\mu\nu} P^\mu dx^\nu$ on tl $T(X)$. Singular points of projection of Lagrange submanifolds of tl $T^*(X)$ (or tl $T(X)$) onto the base X compose caustics.

Caustics of focal and conjugate points of Riemannian and timelike pseudo-Riemannian geodesics are also caustics by Arnol'd. Indeed, the sympletic form $d\alpha = d(g_{\mu\nu}P^\mu) \wedge dx^\nu$

equals zero on a submanifold $N(K) \subset \mathrm{tl}\, T(X)$ defined in the previous Section. Since this form is invariant under the exponential map Exp_1 [10], $d\alpha$ equals zero on the submanifold $\mathrm{Exp}_1\, N(K)$ which consequently is a Lagrange manifold. Since Exp_1 is a diffeomorphism, critical points of the exponential map

$$\exp_1 : N(K) \to \mathrm{Exp}_1\, N(K) \overset{\pi}{\to} X$$

coincide with the critical points of the map $\pi_{N(K)}$ of a Lagrange manifold $\mathrm{Exp}_1\, N(K)$ onto the base X, i.e. they compose caustic by Arnol'd.

4. *Haefliger structures*

A smooth map $\varphi : Y \to X$ of a manifold Y into a manifold X endowed with a foliation F is called transversal to F, if $T_x(X) = T_x(Y) \oplus \mathrm{Im}\,(d\varphi)_x$ at each point $x \in X$. If the map φ is transversal to F, the pre-images of slices of F compose the induced foliation φ^*F on a manifold Y, and codim $\varphi^*F =$ codim F. When the map $\varphi : Y \to X$ fails to be transversal to the foliation F on X, the induced construction φ^*F, called Haefliger structure, makes a certain geometric sense, too [14]. This may be interpreted as a singular foliation.

We restrict our consideration to the case when a Haefliger structure on a manifold X is generated by a real function f possessing critical points, i.e. points where $df = 0$. Such a Haefliger structure can be constructed as induced by the imbedding $X \ni x \to (x, f(x)) \in X \times R$ when the product $X \times R$ is endowed with the foliation $\{R \ni z = \mathrm{const}\}$. There is a classification of critical points of real functions [12]. We pay special attention to Morse functions possessing only non-degenerate critical points, i.e. points where the matrix $\partial f / \partial x^\mu \partial x^\nu$ is nonsingular. Remind the following theorems [4].

Morse's lemma Let x_0 be a non-degenerate point of a differentiable function $f : X \to R$. Then, there is a local coordinate system in a neighborhood u of x_0 such that

$$f(x) = f(x_0) - \sum_{i=1}^{k} (x^i)^2 + \sum_{j=k+1}^{n} (x^j)^2.$$

The number k is called index of the function f at the point x_0.

Theorem 1. Let f be a Morse function on a manifold X. Let M_- and M_+ denote slices before and after transition through a critical point of index k. Then, there are a k-dimensional cell e^k and a $(n-k)$-dimensional cell e^{n-k} such that $e^k \cap e^{n-k} = x_0$, $M_- \cap e^k = \partial e^k$, $M_+ \cap e^{n-k} = \partial e^{n-k}$ and $M_- - \partial e^k$ is diffeomorphic to $M_+ - \partial e^{n-k}$ (∂ denotes a boundary).

This theorem describes changes of topology of level surfaces under transition through a non-degenerate critical points [15].

5. *Foliation singularities*

Proposition 5. For any foliation of level surfaces F on a manifold X there is a foliation F' on a Lagrange submanifold of $\mathrm{tl}\, T^*(X)$ such that F is an image of F' under Lagrange map. A foliation F' is induced on a Lagrange submanifold by a foliation $\{R \ni z = \mathrm{const}\}$ on $\mathrm{tl}\, T^*(X) \times R$.

Let a foliation F be generated by a real function f on a manifold X. Define the imbedding

$$\gamma:\{x^\mu\} \to \{x^\mu, P_\mu = \partial f/\partial x^\mu\}$$

of X into the space tl $T^*(X)$. Its image $\gamma(X)$ is obviously a Lagrange submanifold of tl $T^*(X)$. Let F' be the induced foliation $\pi^*_{\gamma(X)}F$ on $\gamma(X)$, where $\pi_{\gamma(X)}$ is the Lagrange map $\pi_{\gamma(X)}:\gamma(X) \to X$. The foliation F' on $\gamma(X)$ can also be induced by the imbedding

$$\varphi:\gamma(X) \ni (x, P) \to (x, P, z = f(x)) \in \text{tl}\, T^*(X) \times R,$$

where the space tl $T^*(X) \times R$ is endowed with the foliation $\{R \in z = \text{const}\}$. Since γ and $\pi_{\gamma(X)}$ are diffeomorphisms between X and $\gamma(X)$ ($\pi_{\gamma(X)}\gamma = \text{Id}\, X$), the foliation F on X can be represented as the image of the foliation F' on $\gamma(X)$ under the Lagrange map $\pi_{\gamma(X)}$.

Now, let $N \subset$ tl $T^*(X)$ be a Lagrange submanifold generated locally by a function $S(x^i, P_j)$. Let us denote by F' a Haefliger structure induced on N by the foliation $\{R \ni z = \text{const}\}$ on tl $T^*(X) \times R$. This is formed locally by level sets F_z, $z \in R$ of the real function

$$f(x^i, P_j) = S - P_j \frac{\partial S}{\partial P_j}$$

defined on the Lagrange submanifold N. The image $\pi_N(F')$ of F' under the Lagrange map $\pi_N: N \to X$ represents a certain construction on a manifold X. By Proposition 5 this construction represents a foliation on the image $\pi_N(U)$ of a domain U of N where the Haefliger structure F' is a foliation and the Lagrange map π_U possesses no critical points. The foliation structure on X is destroyed at critical points of the function f and at caustic points of the Lagrange map π_N.

Note that caustic points can also be described as branch points of the multiple-valued functions $f'(x) = f(\pi_N^{-1}(x))$ on a manifold X.

In Sections 6 and 7 we shall discuss some properties of foliation singularities whose type is stable under small deformations of a generating function.

6. *Changes of spatial topology*

Proposition 6. Critical points of general position of real functions on a manifold are non-degenerate points described by Morse's lemma [4].

According to this proposition, changes of topology of spatial slices in a space-time are performed, as a rule, by transition through non-degenerated critical points (Morse's singularities) [15]. The following theorem proves this fact [16].

Definition. Let M_- and M_+ be differentiable 3-manifolds. A cobordism between M_- and M_+ is a connected differentiable 4-manifold whose boundary is the disjoint union of M_- and M_+.

Theorem 2. There is a cobordism between any two 3-dimensional compact manifolds, and any such cobordism admits a Morse function taking different values at different critical points.

By this theorem a compact manifold M_+ can be obtained from M_- by a finite sequence of topology transitions through non-degenerate critical points. This statement can be extended to include noncompact 3-manifolds if one assumes that a topology transition is localized inside a compact domain.

The structure of Morse's singularities is described by Theorem 1. The topology transition $M_- \to M_+$ through a Morse's singularity of index k consists of the following operations. A submanifold $M_- - \partial e^k$ of M_- is mapped onto a submanifold $M_+ - \partial e^{4-k}$ of M_+. A set $\partial e^k \subset M_-$ is retracted into the critical point, and then this point is inflated onto the set $\partial e^{4-k} \subset M_+$.

In a gravitation theory, describing of topological transitions implies also describing of evolution of a spatial metric γ_- on M_- to a spatial metric γ_+ on M_+. To describe such metric evolution we apply the formalism of superspaces by Wheeler and De Witt [17]. We shall construct a connected superspace of spatial geometries on nonhomeomorphic slices M_- and M_+ before and after topology transition.

Let M be a 3-dimensional manifold. Denote by $B(M)$ a linear space of symmetric 2-covariant tensor fields on M, endowed with the topology of uniform convergence in all derivatives. Denote by $R(M)$ a subspace of $B(M)$ of Riemannian metrics on M, endowed with relative topology. This space $R(M)$ represents an open cone in $B(M)$, and it inherits the structure of a Frechet manifold. The superspace $S(M)$ of geometries on M is defined to be the quotient $R(M)/\text{Diff}\,(M)$ of $R(M)$ modulo diffeomorphisms of M; each geometry is an orbit of the action of Diff (M) in $R(M)$. The superspace $S(M)$ possesses a good topological structure, being connected and metrizable, but it fails to be a manifold in general.

In the case under discussion there are two nondiffeomorphic manifolds M_- and M_+, and our goal is to glue spaces $R(M_-)$, $R(M_+)$ or superspaces $S(M_-)$, $S(M_+)$. One can use the diffeomorphism between $M_- - \partial e^k$ and $M_+ - \partial e^{4-k}$.

Let us consider a subspace T_- of $B(M_-)$ which consists of tensor fields γ_- representing Riemannian metrics on $M_- - \partial e^k$, and whose all components vanish on $\partial e^k \subset M_-$. Any such field γ_- can be obtained as a limit point of metrics on M_-. Therefore, the subspace T_- belongs to the boundary of an open cone $R(M_-)$ in $B(M_-)$, but T_- is not closed in $B(M_-)$. Let T_+ be the analogous subspace of $B(M_+)$. Since $M_- - \partial e^k$ is diffeomorphic to $M_+ - \partial e^{4-k}$ and all γ_-, γ_+ vanish, respectively, on ∂e^k, ∂e^{4-k}, the completion $\overline{M}_-$ of $M_- - \partial e^k$ relative to any γ_- is homeomorphic to the completion $\overline{M}_+$ of $M_+ - \partial e^{4-k}$ relative to any γ_+. Tensor fields γ_- (γ_+) can be represented as fields on $\overline{M}_-$ $(\overline{M}_+)$ which vanish on $\overline{M}_- - (M_- - \partial e^k)$ $(\overline{M}_+ - (M_+ - \partial e^{4-k}))$, and, thereby, T_- is homeomorphic to T_+.

Now, one can glue spaces $R(M_-) \bigcup T_-$ and $R(M_+) \bigcup T_+$ together at points of T_- and T_+ by their identification. The resulting set $\overline{R} = R(M_-) \bigcup R(M_+) \bigcup (T_+ = T_-)$ is provided with the clutching topology such that a neighborhood of a point $\gamma \in \overline{R}$ obtained by identifications of points $\gamma_- \in T_-$ and $\gamma_+ \in T_+$ is defined to be a set whose intersections with $R(M_-) \bigcup T_- \subset \overline{R}$ and $R(M_+) \bigcup T_+ \subset \overline{R}$ are neighborhoods of γ_- in $R(M_-) \bigcup T_-$ and γ_+ in $R(M_+) \bigcup T_+$, respectively. The topology space $\overline{R}$ is connected, and the evolution of a spatial metric under the topology transition can be expressed by a trajectory in $\overline{R}$ which joins points of $R(M_-)$ and $R(M_+)$.

To construct a superspace of topology transition let us consider the group G_- of diffeomorphisms of M_-, keeping the set $\partial e^k \subset M_-$, and the group G_+ of diffeomorphisms of M_+, keeping the set $\partial e^{4-k} \subset M_+$. Representations of G_- on T_- and G_+ on T_+ are equivialent. Therefore, the quotient S of R modulo action of G_- and G_+ on $\bar{R}$ is defined, and this space can be treated as the generalization of the superspace construction to the case of topology transition. Points $T_\pm/G_\pm$ are not isolated in S, and there is a trajectory in S which joins 3-geometries before and after topology transition.

Notice that a superspace S fails to be glueing of superspaces $S(M_-)$ and $S(M_+)$ because they are quotients modulo Diff(M_-) and Diff(M_+), but not G_- and G_+.

7. Stable caustics

According to Arnol'd stable caustics on a 4-dimensional manifold possess the following generating functions:

$$\begin{aligned}
A_2&: S = p_0^3,\\
A_3&: S = \pm p_0^4 + x^1 p_0^2,\\
A_4&: S = p_0^5 + x^1 p_0^3 + x^2 p_0^2,\\
D_4&: S = p_0^3 \pm p_0 p_1^2 + x^2 p_0^2,\\
A_5&: S = p_0^4 x^1 \pm p_0^6 + x^2 p_0^3 + x^3 p_0^2,\\
D_5&: S = p_0 p_1^2 \pm p_0^4 + x^2 p_0^3 + x^3 p_0^2.
\end{aligned} \tag{7}$$

Any germ of Lagrange map in some neighborhood (in the Whitney topology) of a germ of Lagrange map of types (7) is equivalent to this germ, i.e. its generating function S can be brought into the canonical form (7) by fiber bundle diffeomorphism of $T^*(X)$ and summation with an orbitrary function φ of x^i, $i \in I$. Here, we show the character of foliation caustics of types A_2, A_3.

The A_2-caustic. The generating function $S = p_0^3 + \varphi(x^1, x^2, x^3)$. The Lagrange manifold N is given by

$$\left\{x^0 = -3p_0^2, \quad p_{1,2,3} = \frac{\partial \varphi}{\partial x^{1,2,3}}\right\}.$$

The Lagrange map reads:

$$x^0 = -3p_0^2.$$

The caustic set where $\partial^2 S/\partial p_0^2 = 0$ consists of points $p_0 = 0$, and its image on X under the Lagrange map consists of points $x^0 = 0$. The generating function of a foliation on the Lagrange manifold reads:

$$f' = S - p_0 \frac{\partial S}{\partial p_0} = -2p_0^3 + \varphi(x^1, x^2, x^3).$$

The generating function of the Lagrange image F of the foliation F' on X reads:

$$f = \pm \frac{2}{3\sqrt{3}}(x^0)^{3/2} + \varphi(x^1, x^2, x^3).$$

One sees f to be a two-valued function on X. Thus, the A_2-germ of foliation caustics is characterized by a component $p_0 = \dfrac{\partial f}{\partial x^2}$ of a generating form of a foliation being doubled at caustic points.

The A_3-caustic. The generating function reads:

$$S = -p_0^4 + x^1 p_0^2 + \varphi(x^1, x^2, x^3).$$

The Lagrange manifold N is given by

$$\left\{x^0 = 4p_0^3 - 2x^1 p_0, \quad p_{1,2,3} = \frac{\partial \varphi}{\partial x^{1,2,3}}\right\}.$$

The Lagrange map reads:

$$x^0 = 4p_0^3 - 2x^1 p_0. \tag{8}$$

The caustic set where $\partial^2 S/\partial p_0^2 = 0$ consists of points

$$x^1 = 6p_0^2,$$

and its Lagrange image on X consists of points

$$x^0 = \pm \frac{8}{6\sqrt{6}}(x^1)^{5/2}. \tag{9}$$

The generating function of a foliation F' on the Lagrange manifold N reads:

$$f' = 3p_0^4 - x^1 p_0^2 + \varphi(x^1, x^2, x^3).$$

The generating function of the Lagrange image of F' on X reads:

$$f = 3p_0^4(x^0, x^1) - x^1 p_0^2(x^0, x^1) + \varphi(x^1, x^2, x^3),$$

where the function $p_0(x^0, x^1)$ is determined by Eq. (8). At caustic points (9) this function (and consequently the generating function f) becomes a three-valued function. Thus, the A_3-germ of foliation caustics is characterized by the component $p_0 = \partial f/\partial x^0$ of a generating form of a foliation being tripled at caustic points.

So, our consideration leads to the following gradation of foliation singularities in terms of generating functions.

(*i*) A one-valued real function f possessing a non-vanishing differential df on X generates a foliation of its level surfaces on X.

(*ii*) A one-valued function possessing critical points where $df = 0$ generates a Haefliger structure (singular foliation) of level sets of f on X. Level sets of such function change topology under transition through a critical point of f.

(*iii*) A multiple-valued function f on X defines a structure when a foliation slices on a domain, where f is one-valued, begin to intersect each other at branch points of f, and level sets corresponding to different branches of f intersect each other at every point outside U. Branch points of f, where a foliation is destroyed, compose caustic.

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