ON THE THEORY OF FIELDS IN FINSLER SPACES — III

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Some structural features caused by the intrinsic behavior of the internal variable (y) are reconsidered in more detail with respect to our newly introduced Finsler metric $g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x)+h_{\lambda\kappa}(x, y)$ (*Acta Phys. Pol.* **B15**, 757 (1984)), where $\gamma_{\lambda\kappa}$ denotes the Riemann metric in Einstein's sense and $h_{\lambda\kappa}$ the Finslerian metric induced by the internal field spanned by vectors $\{y\}$. In particular, the mapping process of the internal (y)-field on the external (x)-field is treated systematically.

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1. *Introduction*

In the theory of gravitational field in Finsler spaces [1, 2], the vector y is attached, as the internal variable, to each point x at some more microscopic stage than in Einstein's sense [3] and the line-element (x, y), instead of the point x, is chosen as the independent variable. Therefore, it may be said that the Finslerian field is regarded as "microscopic" and "nonlocal", while the Riemannian field in Einstein's sense is regarded as "macroscopic" and "local" (cf. [2]).

In this paper, as in the previous one [2], Greek indices $\kappa, \lambda, \ldots$ (= 1, 2, 3, 4) are used for the external quantities, while Latin indices $i, j, \ldots$ (= 1, 2, 3, 4) are used for the internal quantities, in order to distinguish the physical functions explicitly.

The internal vector $y(=y^i)$ belongs to the internal field called the (y)-field spanned by $\{y\}$ and shows its own intrinsic behavior. The (y)-field has, in general, a four-dimensional Riemann structure (R_4) with the Riemann metric $h_{ij}(y)$, although it may be reduced to Minkowskian, if necessary. This is supported by the fact that the tangent space spanned by $\{y\}$ at each point of Finsler space is Riemannian (cf. [4]).

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Therefore, there appear two fields around x: One is the external (x)-field spanned by points $\{x\}$ and the other is the internal (y)-field. The former is nothing else than the gravitational field in Einstein's sense, which has R_4-structure governed by the Riemann metric $\gamma_{\lambda\kappa}(x)$. The total space, i.e., the "unified" field between the (x)- and (y)-fields presents an aspect of eight-dimensional Riemann space (R_8), whose spatial structure is represented by the metric G_{AB} and the connection

$$DV^A = dV^A + \Delta_B{}^A{}_C V^B dX^C, \tag{1.1}$$

where X^A ($= (x^\kappa, y^i)$; $A = 1, 2, 3, \ldots, 8$) denotes the "unified" coordinate and $\Delta_B{}^A{}_C$ the Christoffel three-index symbol formed with G_{AB}.

In our case, at the first starting point, it may be assumed that the metric G_{AB} has the direct-product form such as

$$G_{AB} = \begin{pmatrix} \gamma_{\lambda\kappa}(x) & 0 \\ 0 & h_{ij}(y) \end{pmatrix}, \quad G^{AB} = \begin{pmatrix} \gamma^{\kappa\lambda}(x) & 0 \\ 0 & h^{ij}(y) \end{pmatrix}, \tag{1.2}$$

and then the non-zero components of $\Delta_B{}^A{}_C$ are only $\Delta_\lambda{}^\kappa{}_\mu = \left\{ {\kappa \atop \lambda\ \mu} \right\}$ and $\Delta_j{}^i{}_k = \left\{ {i \atop j\ k} \right\}$ (both of them are Christoffel three-index symbols derived from $\gamma_{\lambda\kappa}$ and h_{ij} respectively). Therefore, our problem is to obtain the four-dimensional Finsler metric $g_{\lambda\kappa}(x, y)$ by unifying $\gamma_{\lambda\kappa}(x)$ and $h_{ij}(y)$, that is to say, to extract $g_{\lambda\kappa}(x, y)$ from G_{AB} properly by means of the decomposition process such as $g_{\lambda\kappa} = A_\lambda^A A_\kappa^B G_{AB}$, where A is the decomposition factor (cf. [5] and see Section 3). This kind of decomposition has close analogy with the process adopted in the generalized Kaluza-Klein theory of gravitation (cf. [6]), where the state of (1.2) is regarded as the "vacuum" state without "fluctuation".

Concerning this kind of Finsler metric $g_{\lambda\kappa}$ (x, y), we have already introduced it in Section 3 of [2] in the form

$$g_{\lambda\kappa}(x, y) = \gamma_{\lambda\kappa}(x) + h_{\lambda\kappa}(x, y), \tag{1.3}$$

where $h_{\lambda\kappa}$ is induced by the following mapping process:

$$y^\kappa = e_i^\kappa(x) y^i,$$
$$h_{\lambda\kappa}(x, y) = e_\lambda^i(x) e_\kappa^j(x) h_{ij}(y), \tag{1.4}$$

by which the internal (y)-field is embedded in the external (x)-field. But the physical and geometrical meanings of (1.3) and (1.4) cannot yet be given fully. So, in this paper, we shall reconsider the mapping process (1.4) and the resulting metric $g_{\lambda\kappa}$ (x, y) (1.3) from the standpoint of the above-mentioned decomposition process of G_{AB}, where the components $g_{\lambda\kappa}$, $g_{\lambda i}$, etc. are caused by the "interactions" between the (x)- and (y)-fields in the sense of generalized Kaluza-Klein theory (cf. [6]).

For that purpose, we must, first of all, determine the decomposition factors such as A (see Section 3) by setting the base and dual base suitably in the "unified" field, which must be done by taking account of the intrinsic behavior of y^i in the internal (y)-field.

2. *Intrinsic behavior of y*

As has already been mentioned in the previous papers [1, 2, 7], the intrinsic behavior of y^i in the (y)-field can be represented by such a rotational property as, as Asanov's K-group [8],

$$\bar{y}^i = K^i_j(x, y)y^j(\equiv y^i + dy^i), \tag{2.1}$$

where the rotation matrix K^i_j is assumed to be a function of (x, y) from a general standpoint. (Some special cases will be considered in Section 4). Of course, (2.1) is different from the coordinate transformation in the (y)-field.

For our purpose, the intrinsic behavior given by (2.1) must be "geometrized" as the parallelism ($\delta y^i = 0$). This is actually done as follows:

$$\delta y^i \equiv dy^i + K_j{}^i{}_\mu y^j dx^\mu + L_j{}^i{}_k y^j dy^k \equiv P^i_\mu dx^\mu + Q^i_k dy^k (= 0), \tag{2.2}$$

where $K_j{}^i{}_\mu = -\dfrac{\partial K^i_j}{\partial x^\mu}$, $L_j{}^i{}_k = -\dfrac{\partial K^i_j}{\partial y^k}$ and $P^i_\mu = K_j{}^i{}_\mu y^j$, $Q^i_k = \delta^i_k + L_j{}^i{}_k y^j$. (The homogeneity conditions of K^i_j and $L_j{}^i{}_k$ with respect to y are not assumed here from a general standpoint, cf. [4].) (2.2) corresponds to the so-called base connection of y from the standpoint of the theory of higher order spaces (cf. [9]). (Finsler space is the higher order space of order 1.)

This intrinsic behavior δy^i (2.2) must be reflected in the whole spatial structure of the "unified" field. This reflection can be done by decomposing (1.1) as follows:

$$\begin{aligned} DV^\kappa &= dV^\kappa + \Gamma_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + \Gamma_\lambda{}^\kappa{}_i V^\lambda dy^i \\ &= dV^\kappa + F_\lambda{}^\kappa{}_\mu V^\lambda dx^\mu + \Theta_\lambda{}^\kappa{}_k V^\lambda \delta y^k, \\ DV^i &= dV^i + \Gamma_j{}^i{}_\mu V^j dx^\mu + \Gamma_j{}^i{}_k V^j dy^k \\ &= dV^i + F_j{}^i{}_\mu V^j dx^\mu + \Theta_j{}^i{}_k V^j \delta y^k, \end{aligned} \tag{2.3}$$

where $V^\kappa = A^\kappa_A V^A$, $V^i = B^i_A V^A$, and

$$\begin{aligned} F_\lambda{}^\kappa{}_\mu &= \Gamma_\lambda{}^\kappa{}_\mu - N^i_\mu \Gamma_\lambda{}^\kappa{}_i, \quad \Theta_\lambda{}^\kappa{}_k = Q^{-1}{}^i_k \Gamma_\lambda{}^\kappa{}_i, \\ F_j{}^i{}_\mu &= \Gamma_j{}^i{}_\mu - N^k_\mu \Gamma_j{}^i{}_k, \quad \Theta_j{}^i{}_k = Q^{-1}{}^l_k \Gamma_j{}^i{}_l, \\ N^i_\mu &\equiv P^k_\mu Q^{-1}{}^i_k, \end{aligned} \tag{2.4}$$

and $\Gamma_\lambda{}^\kappa{}_\mu$, $\Gamma_\lambda{}^\kappa{}_i$, etc. are obtained by, e.g.,

$$\begin{aligned} \Gamma_\lambda{}^\kappa{}_\mu &= A^\kappa_A A^B_\lambda A^C_\mu \Delta_B{}^A{}_C + A^\kappa_A \left(A^B_\mu \frac{\partial}{\partial X^B}\right) A^A_\lambda, \\ \Gamma_\lambda{}^\kappa{}_i &= A^\kappa_A A^B_\lambda B^C_i \Delta_B{}^A{}_C + A^\kappa_A \left(B^B_i \frac{\partial}{\partial X^B}\right) A^A_\lambda. \end{aligned} \tag{2.5}$$

In (2.4) and (2.5), A, B denote the so-called decomposition factors [5], which will be determined below (see (3.1)). The quantity N^i_μ in (2.4) plays the role of nonlinear connection [4], which prescribes the "interaction" between the (x)- and (y)-fields (see Section 3).

From (2.2) and (2.3), the base and dual base can be set for the "unified" field as follows:

$$\left(\frac{\delta}{\delta x^\lambda} \equiv A^B_\lambda \frac{\partial}{\partial X^B} = \frac{\partial}{\partial x^\lambda} - N^i_\lambda \frac{\partial}{\partial y^i},\right.$$

$$\left.\frac{\delta}{\delta y^k} \equiv B^A_k \frac{\partial}{\partial X^A} = Q^{-1i}{}_k \frac{\partial}{\partial y^i}\right),$$

$$(dx^\kappa \equiv A^\kappa_A dX^A, \quad \delta y^k \equiv B^k_A dX^A = P^k_\mu dx^\mu + Q^k_l dy^l). \tag{2.6}$$

It is also understood from (2.4) and (2.6) that the intrinsic gauge fields $K_j{}^i{}_\mu$ and $L_j{}^i{}_k$ are absorbed into the "unified" gauge fields $F_\lambda{}^\kappa{}_\mu$ and $F_j{}^i{}_\mu$ at the level of horizontal covariant derivative with respect to x (cf. [10]), that is, the partial differentials $\left(\frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial y^i}\right)$ and the connection coefficients $(\Gamma_\lambda{}^\kappa{}_\mu, \Gamma_\lambda{}^\kappa{}_i)$ or $(\Gamma_j{}^i{}_\mu, \Gamma_j{}^i{}_k)$ are both "unified" by means of the nonlinear connection N at the level of covariant derivative by x.

3. *Finslerian structure — I*

In our case, the internal field is assumed to have R_4 with the Riemann metric $h_{ij}(y)$, which is attached to each point x of the external field with the Riemann metric $\gamma_{\lambda\kappa}(x)$. Therefore, as mentioned already, the "unified" field of the (x)- and (y)-fields has some R_8-structure with the "unified" Riemann metric G_{AB}, in which the frame conditioned by (2.6) is established as the result of taking account of the intrinsic behavior of y (i.e., δy (2.2)).

Under these conditions, it is necessary for our purpose to extract some kind of Finslerian structure (F_4) from the R_8-structure, and then obtain a Finsler metric ($g_{\lambda\kappa}(x, y)$) from G_{AB}. That is to say, it is necessary to decompose G_{AB} into such components as ($g_{\lambda\kappa}$, $g_{\lambda i} = g_{i\lambda}$, g_{ij}) adapted to the base and dual base given by (2.6). For that purpose, we shall in the following consider the decomposition process of R_8-structure by analogy with the generalized Kaluza-Klein theory [6].

Now, we can first determine the decomposition factors from (2.6) as follows:

$$A^\kappa_B = (\delta^\kappa_\lambda, 0), \quad A^B_\lambda = (\delta^\kappa_\lambda, -N^i_\lambda),$$
$$B^i_A = (P^i_\lambda, Q^i_l), \quad B^A_i = (0, Q^{-1j}{}_i). \tag{3.1}$$

Therefore, we can obtain the following components:

$$g_{\lambda\kappa} = \gamma_{\lambda\kappa}(x) + N^i_\lambda N^j_\kappa h_{ij}(y),$$
$$g_{\lambda i} = -N^k_\lambda Q^{-1l}{}_i h_{kl}(y),$$
$$g_{ij} = Q^{-1k}{}_i Q^{-1l}{}_j h_{kl}(y). \tag{3.2}$$

$$g^{\kappa\lambda} = \gamma^{\kappa\lambda}(x),$$

$$g^{\kappa i} = P^i_\lambda \gamma^{\kappa\lambda}(x),$$

$$g^{ij} = P^i_\kappa P^j_\lambda \gamma^{\kappa\lambda}(x) + Q^i_k Q^j_l h^{kl}(y).$$

In (3.2), $g_{\lambda\kappa}(x, y)$ has quite a similar form to (1.3) with $e(x)$ being replaced by $N(x, y)$, but other additional components such as $g_{\lambda i}$, $g^{\kappa i}$, etc. do appear. And those quantities $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{N}$ represent the "interactions" in our sense caused by the intrinsic behavior of y.

Next, if we assume that as in the ordinary theory of vector bundles (cf. [11]), only such components as $g_{\lambda\kappa}$, $g^{\kappa\lambda}$ and g_{ij}, g^{ij} appear at the stage of metric (i.e., $g_{\lambda i} = 0$ and $g^{\kappa i} = 0$) and then the rotation matrix $K^i_j(x, y)$ in (2.1) is positively homogeneous of degree 0 in y (i.e., $Q^i_k = \delta^i_k$, since $L_j{}^i{}_k y^j = 0$, see (2.2)), then we can further reduce (3.2) to

$$g_{\lambda\kappa} = \gamma_{\lambda\kappa}(x) + N^i_\lambda N^j_\kappa h_{ij}(y),$$

$$g_{ij} = h_{ij}(y).$$

$$g^{\kappa\lambda} = \gamma^{\kappa\lambda}(x),$$

$$g^{ij} = h^{ij}(y), \tag{3.3}$$

where $N^i_\lambda = P^i_\lambda$ (see (2.4)).

At this final stage, if only $g_{\lambda\kappa}$ and $g^{\kappa\lambda}$ are taken into account by "compactifying" h_{ij} and h^{ij} in the "unified" field, then the resulting field obtained by this decomposition process appears to have F_4-structure based on the Finsler metric $g_{\lambda\kappa}(x, y)$ of (3.3). This process resembles the "reduction" process of dimension number from 8 to 4. The nonlinear connection $N(x, y)$ plays the role of mapping operator of the (y)-field on the (x)-field, (which will be called the N-mapping), as $e(x)$ does in (1.4), and causes the same effects as the "fluctuation" does in "vacuum" state from the viewpoint of generalized Kaluza-Klein theory (cf. [6]). Thus, the mapping process given by (1.4) and the Finsler metric $g_{\lambda\kappa}(x, y)$ given by (1.3) can now be supported by the above-mentioned decomposition process. Of course, the F_4-structure based on $g_{\lambda\kappa}(x, y)$ of (3.3) is found to be more suitable.

4. Finslerian structure — II

Now, taking account of the above results obtained in Section 3, we shall hereafter consider that our Finslerian gravitational field has the Finsler metric $g_{\lambda\kappa}(x, y)$ given by (3.3). Then, the intrinsic behavior of y given by (2.2) is brought to the "unified" Finsler field as follows:

$$\delta y^\kappa \equiv N^\kappa_i \delta y^i = dy^\kappa + K_\lambda{}^\kappa{}_\mu y^\lambda dx^\mu + L_\lambda{}^\kappa{}_\mu y^\lambda dy^\mu$$

$$\equiv P^\kappa_\mu dx^\mu + Q^\kappa_\mu dy^\mu \,(= 0), \tag{4.1}$$

where

$$K_{\lambda}{}^{\kappa}{}_{\mu} = N_i^{\kappa} N_{\lambda}^{j} K_j{}^i{}_{\mu} - \frac{\partial N_i^{\kappa}}{\partial x^{\mu}} N_{\lambda}^{i},$$

$$L_{\lambda}{}^{\kappa}{}_{\mu} = N_i^{\kappa} N_{\lambda}^{j} N_{\mu}^{k} L_j{}^i{}_k - \frac{\partial N_i^{\kappa}}{\partial y^{\mu}} N_{\lambda}^{i}, \tag{4.2}$$

and $P_{\mu}^{\kappa} = K_{\lambda}{}^{\kappa}{}_{\mu} y^{\lambda}$, $Q_{\mu}^{\kappa} = \delta_{\mu}^{\kappa} + L_{\lambda}{}^{\kappa}{}_{\mu} y^{\lambda}$ (see (2.2)). (4.1) gives the intrinsic parallelism or connection of y in the Finslerian field, so that the metrical conditions $\delta h_{\lambda\kappa} = 0$ can be assumed for the connection δ under the absolute parallelism of N (i.e., $\delta N = 0$). (It is easily understood that the condition $\delta N = 0$ is compatible with the relation (4.2).) It should be remarked that $\delta g_{\lambda\kappa} \neq 0$, even if $\delta h_{\lambda\kappa} = 0$ (see below).

From the most general case given by (4.1) and (4.2), we can consider some special cases as follows: if K_j^i is a function of x alone, then $L_j{}^i{}_k = 0$ in (2.2) and (4.2); if K_j^i is a function of y alone, then $K_j{}^i{}_{\mu} = 0$ in (2.2) and (4.2); If the (y)-field is flat (i.e., Minkowskian), then K_j^i = constant and (2.2) reduces to $\delta y^i = dy^i = 0$, so that $K_{\lambda}{}^{\kappa}{}_{\mu} = -\frac{\partial N_i^{\kappa}}{\partial x^{\mu}} N_{\lambda}^{i}$ and $L_{\lambda}{}^{\kappa}{}_{\mu} = -\frac{\partial N_i^{\kappa}}{\partial y^{\mu}} N_{\lambda}^{i}$ in (4.2); etc.

Our Finslerian gravitational field has the Finsler metric $g_{\lambda\kappa}(x, y)$ given by (3.3), so that its spatial structure must be made consistent with the metrical conditions $Dg_{\lambda\kappa} = 0$. Concerning the connection D, which is first introduced by (2.3), it should be now summarized in the form

$$\begin{aligned} DV^{\kappa} &= dV^{\kappa} + \Gamma_{\lambda}{}^{\kappa}{}_{\mu} V^{\lambda} dx^{\mu} + C_{\lambda}{}^{\kappa}{}_{\mu} V^{\lambda} dy^{\mu} \\ &= dV^{\kappa} + F_{\lambda}{}^{\kappa}{}_{\mu} V^{\lambda} dx^{\mu} + \Theta_{\lambda}{}^{\kappa}{}_{\mu} V^{\lambda} \delta y^{\mu}, \end{aligned} \tag{4.3}$$

where $\Gamma_{\lambda}{}^{\kappa}{}_{\mu}$ and $C_{\lambda}{}^{\kappa}{}_{\mu}$ denote, as usual [4], the horizontal and vertical coefficients of connection of the "unified" field, and

$$F_{\lambda}{}^{\kappa}{}_{\mu} = \Gamma_{\lambda}{}^{\kappa}{}_{\mu} - N_{\mu}^{\nu} C_{\lambda}{}^{\kappa}{}_{\nu}, \quad \Theta_{\lambda}{}^{\kappa}{}_{\mu} = Q^{-1}{}^{\nu}_{\mu} C_{\lambda}{}^{\kappa}{}_{\nu},$$
$$N_{\mu}^{\nu} = Q^{-1}{}^{\nu}_{\lambda} P_{\mu}^{\lambda}. \tag{4.4}$$

The quantities in (4.4) correspond to those appearing in (2.4) and N_{μ}^{ν} is the nonlinear connection in this case. And for the total Finslerian structure represented by (4.3), the base $\left(\frac{\partial}{\partial x^{\lambda}} - N_{\lambda}^{\nu} \frac{\partial}{\partial y^{\nu}},\ Q^{-1}{}^{\nu}_{\lambda} \frac{\partial}{\partial y^{\nu}}\right)$ and the dual base $(dx^{\kappa}, \delta y^{\lambda})$ can be set, and the intrinsic behavior of y^i represented by $L_j{}^i{}_k$ or $L_{\lambda}{}^{\kappa}{}_{\mu}$ is summarized by the quantity $C_{\lambda}{}^{\kappa}{}_{\mu}$, which is finally absorbed into the quantity $F_{\lambda}{}^{\kappa}{}_{\mu}$ through N_{μ}^{ν}, because in our case, the intrinsic parallelism holds good, i.e., $\delta y^i = 0$ and then $\delta y^{\kappa} = 0$.

The relation between ($\Gamma_{\lambda}{}^{\kappa}{}_{\mu}$, $C_{\lambda}{}^{\kappa}{}_{\mu}$) of (4.3) and ($K_{\lambda}{}^{\kappa}{}_{\mu}$, $L_{\lambda}{}^{\kappa}{}_{\mu}$) of (4.1) can be obtained in more detail as follows: As to the connection δ, it is assumed, from the beginning, to be metrical for $h_{\lambda\kappa}$, i.e., $\delta h_{\lambda\kappa} = 0$ under the premises of $\delta h_{ij} = 0$ and $\delta N = 0$. And as to the connection D, it is also assumed to be metrical for $g_{\lambda\kappa}$, i.e., $Dg_{\lambda\kappa} = 0$. Then, we can obtain the relation that $Dg_{\lambda\kappa} = 0$, $Dh_{\lambda\kappa} \neq 0$ and $\delta h_{\lambda\kappa} = 0$, $\delta g_{\lambda\kappa} \neq 0$. Therefore, we can reconsider, from the standpoint of Kawaguchi's theorem [12], that the connection D is a metrical connection for $g_{\lambda\kappa}$ (i.e., $Dg_{\lambda\kappa} = 0$) derived from the non-metrical one δ (i.e., $\delta g_{\lambda\kappa} \neq 0$). So, by use of Kawaguchi's theorem which supplies a method to make a non-metrical connection metrical, the relation between D and δ can be obtained in the form (with neglect of arbitrariness)

$$Dy^{\kappa} = \delta y^{\kappa} + \tfrac{1}{2} g^{\kappa\nu}(\delta g_{\nu\lambda})y^{\lambda}, \tag{4.5}$$

by which the relations between $(\Gamma_{\lambda}{}^{\kappa}{}_{\mu}, C_{\lambda}{}^{\kappa}{}_{\mu})$ and $(K_{\lambda}{}^{\kappa}{}_{\mu}, L_{\lambda}{}^{\kappa}{}_{\mu})$ can be obtained as follows:

$$\Gamma_{\lambda}{}^{\kappa}{}_{\mu} = K_{\lambda}{}^{\kappa}{}_{\mu} + \tfrac{1}{2} g^{\kappa\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial x^{\mu}} - K_{\nu}{}^{\tau}{}_{\mu} g_{\tau\lambda} - K_{\lambda}{}^{\tau}{}_{\mu} g_{\nu\tau}\right),$$

$$C_{\lambda}{}^{\kappa}{}_{\mu} = L_{\lambda}{}^{\kappa}{}_{\mu} + \tfrac{1}{2} g^{\kappa\nu}\left(\frac{\partial g_{\nu\lambda}}{\partial y^{\mu}} - L_{\nu}{}^{\tau}{}_{\mu} g_{\tau\lambda} - L_{\lambda}{}^{\tau}{}_{\mu} g_{\nu\tau}\right). \tag{4.6}$$

With the use of (4.6), we can further obtain the relation between $(F_{\lambda}{}^{\kappa}{}_{\mu}, \Theta_{\lambda}{}^{\kappa}{}_{\mu})$ of (4.3) and $(K_{\lambda}{}^{\kappa}{}_{\mu}, L_{\lambda}{}^{\kappa}{}_{\mu})$ of (4.1), but we shall omit them for simplicity's sake. (Of course, (4.5) and (4.6) can be applied to the case where the (y)-field is flat.) (These considerations have also been mentioned in Section 3 of [2].)

5. *Conclusions*

In the theory described in this paper, a "unification" between the external (x)-field and the internal (y)-field has been performed at the stage of metric by the N-mapping, by which the resulting "unified" field appears to have the F_4-structure with the metric $g_{\lambda\kappa}(x, y)$ of (3.3). This N-mapping has been justified, as mentioned in Section 3, from the decomposition process of the "unified" Riemann metric G_{AB}. And the intrinsic behavior δy^i of the internal vector y^i is given by (2.2) and (4.1), which is reflected in the total "unified" Finslerian field at the stage of connection as in (4.3). The relation between $Dg_{\lambda\kappa} = 0$ and $\delta h_{\lambda\kappa} = 0$ is obtained in the form of (4.5). Thus, we have clarified the Finslerian structure of the gravitational field caused by the internal vector y.

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