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SOLITON SCATTERING IN NUCLEAR MATTER IN ONE DIMENSION

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We have considered a model Hamiltonian system in one dimension with higher order nonlinearity to simulate the excitations of nuclear "drops" or "solitons" in the background of the usual vacuum. The theory is usually referred to as $\psi^4-\psi^6$ theory and has been already used in many different physical contexts. Essentially we have considered the interaction of two such solitary excitations and their subsequent evolution regarding amplitude and phase.

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1. Introduction

After more than ten years of explosive growth the interest for nonlinear models sustaining soliton solution is still increasing. Researchers find more and more physical situations where the concepts of solitary wave can be applied profitably. One of the most popular models is the nonlinear Schrödinger equation [1] which really describes a many-body system with a repulsive or attractive delta-like potential [2]. In the attractive case, the stable ground state is very simple — it is the state where ψ is identically zero. The excitations over this vacuum are either plane waves or droplets — that is solitons. In the case of repulsive potential the ground state is the state of the condensate of infinite number of bosons. One way to achieve both types of excitations in the same U(1) invariant model is to increase the nonlinearity. Such a model is the usual NLS equation with a $\psi|\psi|^4$ term added, representing a ψ^6 type [3] interaction. Such a model has been discussed by Friedberg et al. [4] for the soliton model of hadrons. The model is also useful for studying heavy ion collisions. Here we have not deduced this model because such calculations have been done quite recently[1]. We have analysed by a variant of reductive perturbation technique the collision of two such "droplet"-like (soliton) excitations and studied their subsequent dynamics in the $\psi^4-\psi^6$ theory.

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[1] H. D. Hefter — Preprint, Karl-Marx Universität, DDR.

2. *Formulation*

The nonlinear Schrödinger equation with a higher-order nonlinear term added, reads:

$$i\psi_t + \psi_{\kappa\kappa} + \alpha|\psi|^2\psi + \beta|\psi|^4\psi = 0, \tag{1}$$

or

$$i\psi_t + \psi_{\kappa\kappa} + V(\psi) = 0,$$

with

$$V(\psi) = \alpha|\psi|^2\psi + \beta|\psi|^4\psi.$$

To analyse the scattering of two droplets we choose the multiple scaled variables

$$\kappa_n = \varepsilon^n \kappa \qquad \text{and} \qquad t^n = \varepsilon^n t,$$

ε being a parameter having the smallness of the bandwidth of the wave travelling in the position x direction. The wave travelling in the negative x direction is assumed to be of the order of ε^2. Let us consider the following expansion of ψ [5] in ε;

$$\psi = \sum \varepsilon^n \psi_n(\kappa_0, \kappa_1, \kappa_2, \kappa_3, \ldots, t_0, t_1, t_2, t_3, \ldots). \tag{2}$$

Substituting in (1)

$$\frac{\partial}{\partial t} = \sum \varepsilon^n \frac{\partial}{\partial t_n} \quad \text{and} \quad \frac{\partial}{\partial \kappa} = \sum \varepsilon^n \frac{\partial}{\partial \kappa_n},$$

and equating the same powers of ε we get

$$\begin{aligned}
0(\varepsilon) &: \mathscr{L}\psi_1 = 0,\\
0(\varepsilon^2) &: \mathscr{L}\psi_2 = -\psi_{1_t} - 2\psi_{1_{\kappa_0\kappa_1}},\\
0(\varepsilon^3) &: \mathscr{L}\psi_3 = -\alpha[|\psi_1|^2\psi_1] - i[\psi_{2_{t_1}} + \psi_{1_{t_2}}] - 2[\psi_{2_{\kappa_0\kappa_1}} + \psi_{1_{\kappa_1\kappa_1}} + 2\psi_{1_{\kappa_0\kappa_2}}],\\
0(\varepsilon^4) &: \mathscr{L}\psi_4 = -\alpha[2|\psi_1|^2\psi_2 + \psi_1^2\psi_2^*]\\
&\quad - i[\psi_{3_{t_1}} + \psi_{2_{t_2}} + \psi_{1_{t_3}}] - 2[2\psi_{3_{\kappa_0\kappa_1}} + \psi_{2_{\kappa_1\kappa_1}}\\
&\quad + 2\psi_{2_{\kappa_0\kappa_2}} + 2\psi_{1_{\kappa_0\kappa_3}} + 2\psi_{1_{\kappa_2\kappa_1}}],
\end{aligned} \tag{3}$$

where

$$\mathscr{L} = i\frac{\partial}{\partial t_0} + \frac{\partial^2}{\partial \kappa_0^2}.$$

The solution of the lowest-order equation in (3) is given by:

$$\begin{aligned}
\psi_1 &= A_1(\kappa_1, \kappa_2, \ldots, \kappa_5, t_1, t_2, \ldots, t_5)e^{i\phi_+}\\
&\quad + B_1(\kappa_2, \kappa_3, \ldots, \kappa_5, t_2 t_3, \ldots, t_5)e^{i\phi_-},
\end{aligned} \tag{4}$$

which represents two waves of amplitude A_1, B_1 and phase ϕ_+ and ϕ_- given as

$$\phi_+ = k_+\kappa_0-\omega t_0, \quad \phi_- = k_-\kappa_0-\omega t_0 \tag{5}$$

along with

$$k_+ = -k = \sqrt{\omega}.$$

In equation (4) we have assumed that the two-band widths of the spectra of A_1 and B_1 are of the order of ε and ε^2 respectively, so that while A_1 is assumed to depend on $(\kappa_1, \kappa_2, \ldots, \kappa_5, t_1, t_2, \ldots, t_5)$ the amplitude B_1 is considered to depend on $(\kappa_2, \ldots, \kappa_5, t_2, \ldots, t_5)$. Plugging ψ_1 from (4) in the second equation of (3) we get;

$$\mathscr{L}\psi_2 = \left(-i\frac{\partial A_1}{\partial t_1} - 2k_+ i\frac{\partial A_1}{\partial \kappa_1}\right)e^{i\phi_+}. \tag{6}$$

Imposition of nonsecularity condition leads to

$$\frac{\partial A_1}{\partial t_1} + c_g^+\frac{\partial A_1}{\partial \kappa_1} = 0, \tag{7}$$

where $c_g^+ = \dfrac{d\omega}{dk_+} = 2k_+$ is the group velocity of the wave in the positive x direction.

The second-order solution, which is the homogeneous solution of (6) is given by

$$\psi_2 = A_2(\kappa_1, \ldots, \kappa_5, t_1, \ldots, t_5)e^{i\phi_+} + B_2(\kappa_1, \ldots, \kappa_5, t_1, \ldots, t_5)e^{i\phi_-}. \tag{8}$$

Substituting ψ_1 from (4) and ψ_2 from (8) the next-order equation in (3) yields;

$$\mathscr{L}\psi_3 = \left\{-i\left(\frac{\partial A_2}{\partial t_1}+\frac{\partial A_1}{\partial t_2}\right) - 2ik_+\left(\frac{\partial A_1}{\partial \kappa_2}+\frac{\partial A_2}{\partial \kappa_1}\right) - \frac{\partial^2 A_1}{\partial \kappa_1^2} - \alpha(|A_1|^2+2|B_1|^2)A_1\right\}e^{i\phi_+} + \ldots \tag{9}$$

So that third-order nonsecularity condition reads:

$$-i\left(\frac{\partial A_1}{\partial t_2}+\frac{\partial A_2}{\partial t_1}\right) - 2ik_+\left(\frac{\partial A_1}{\partial \kappa_2}+\frac{\partial A_2}{\partial \kappa_1}\right) - \frac{\partial^2 A_1}{\partial \kappa_1^2} = \alpha(|A_1|^2+2|B_1|^2)A_1. \tag{10}$$

The other wave is governed by the equation

$$-i\left(\frac{\partial B_1}{\partial t_2}+\frac{\partial B_2}{\partial t_1}\right) - 2ik_-\left(\frac{\partial B_1}{\partial \kappa_2}+\frac{\partial B_2}{\partial \kappa_1}\right) = \alpha(|B_1|^2+2|A_1|^2)B_1. \tag{11}$$

Similarly, the fourth-order nonsecularity condition reads:

$$-i\left(\frac{\partial A_1}{\partial t_3}+\frac{\partial A_2}{\partial t_2}+\frac{\partial A_3}{\partial t_1}\right) - 2ik_+\left(\frac{\partial A_1}{\partial \kappa_3}+\frac{\partial A_2}{\partial \kappa_2}+\frac{\partial A_3}{\partial \kappa_1}\right)$$

$$-\left(\frac{\partial^2 A_2}{\partial \kappa_1^2}+2\frac{\partial^2 A_1}{\partial \kappa_1 \partial \kappa_2}\right)=\alpha[2A_2(|A_1|^2+|B_1|^2)$$
$$+A_1(A_2^* A_1+2B_2B_1^*+2B_2^*B_1)] \tag{12}$$

and

$$-i\left(\frac{\partial B_1}{\partial t_3}+\frac{\partial B_2}{\partial t_2}+\frac{\partial B_3}{\partial t_1}\right)-2ik_-\left(\frac{\partial B_1}{\partial \kappa_3}+\frac{\partial B_2}{\partial \kappa_2}+\frac{\partial B_3}{\partial \kappa_1}\right)$$
$$-\left(\frac{\partial^2 B_2}{\partial \kappa_1^2}\right)=\alpha[2B_2(|B_1|^2+|A_1|^2+B_1(B_2^*B_1+2A_2A_1^*+2A_2^*A_1)]. \tag{13}$$

The actual amplitudes A, B may now be represented as

$$A=\varepsilon A_1+\varepsilon^2 A_2+\varepsilon^3 A_3+\varepsilon^4 A_4+\dots$$
$$B=\varepsilon B_1+\varepsilon^2 B_2+\varepsilon^3 B_3+\varepsilon^4 B_4+\dots \tag{14}$$

Now, by using the expansions noted in (2a) we can deduce two separate equations for A and B which are written as follows:

$$-i\left(\frac{\partial A}{\partial t}+2k_+\frac{\partial A}{\partial \kappa}\right)-\frac{\partial^2 A}{\partial \kappa^2}=\alpha(|A_1|^2+2|B_1|^2)A$$
$$+\beta(|B|^4+3|A|^4+6|B|^2|A|^2)A, \tag{15}$$

$$-i\left(\frac{\partial B}{\partial t}+2k_-\frac{\partial B}{\partial \kappa}\right)-\frac{\partial^2 B}{\partial \kappa^2}=\alpha(|B|^2+2|A|^2)B$$
$$+\beta(|A|^4+3|B|^4+6|B|^2|A|^2)B. \tag{16}$$

It should be noted that though (15) and (16) are of the same form, yet each term in them has different order of magnitude. So in general, we have the following equations:

$$i\left(\frac{\partial A}{\partial t}+c_g\frac{\partial A}{\partial \kappa}\right)+\frac{\partial^2 A}{\partial \kappa^2}=[q|A|^2+r|B|^2+s|A|^4+t|B|^4+u|B|^2|A|^2]A$$

and

$$i\left(\frac{\partial B}{\partial t}-c_g\frac{\partial B}{\partial \kappa}\right)+\frac{\partial^2 B}{\partial \kappa^2}=[q|B|^2+r|A|^2+s|B|^4+t|A|^4+u|A|^2|B|^2]B. \tag{17}$$

Now, we search for steady wave solution and define $\xi=\kappa-\lambda t$. Substituting $A=R_+e^{i\phi_+}$ and $B=R_-e^{i\phi_-}$ we get (here R_+, R_-, ϕ_+, ϕ_- all are functions of ξ)

$$\frac{\partial^2 R_+}{\partial \xi^2}-R_+\left(\frac{\partial \phi_+}{\partial \xi}\right)^2-R_+(-\lambda+c_g)\frac{\partial \phi_+}{\partial \xi}$$
$$=[qR_+^2+rR_-^2+sR_+^4+tR_-^4+uR_+^2R_-^2]R_+ \tag{18}$$

and

$$\frac{\partial R_+}{\partial \xi}\left[(-\lambda+c_g)+2\frac{\partial \phi_+}{\partial \xi}\right]+R_+\frac{\partial^2\phi_+}{\partial \xi^2}=0. \tag{19}$$

From (19) we deduce

$$-\tfrac{1}{2}(-\lambda+c_g)\frac{d}{d\xi}(R_+^2)+\frac{d}{d\xi}\left(R_+^2\frac{\partial\phi_+}{\partial\xi}\right)=0. \tag{20}$$

Integrating we have

$$\tfrac{1}{2}(-\lambda+c_g)R_+^2+R_+^2\frac{\partial\phi_+}{\partial\xi}=I_+. \tag{21}$$

I_+ is a constant of integration.

Eliminating $\frac{\partial\phi_+}{\partial\xi}$ between (18) and (21), we get

$$\frac{d^2R_+}{d\xi^2}-\frac{I_+^2}{R_+^3}+\frac{(-\lambda+c_g)R_+}{4}-\sigma(R_+,R_-)R_+=0, \tag{22}$$

with

$$\sigma=qR_+^2+rR_-^2+sR_+^4+tR_-^4+uR_+^2R_-^2.$$

Multiplying by $2\frac{dR_+}{d\xi}$ and integrating (22) leads to

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{dR_+}{d\xi}\right)^2-\mu_1R_+^6+\mu_2R_+^4+\mu_{3+}^2R_+^2+\frac{\delta_+^2}{R_+^2}\\&+\gamma_1\int R_-^2\frac{d}{d\xi}(R_+^2)d\xi+\gamma_2\int R_-^4\frac{d}{d\xi}(R_+^2)d\xi\\&+\gamma_3\int R_-^2\frac{d}{d\xi}(R_+^4)d\xi=C_1.\end{aligned} \tag{23}$$

Similarly we deduce:

$$\begin{aligned}&\tfrac{1}{2}\left(\frac{dR_-}{d\xi}\right)^2+\mu_1R_-^6+\mu_2R_-^4+\mu_{3-}^2R_-^2+\frac{\delta_-^2}{R_-^2}\\&+\gamma_1\int R_+^2\frac{d}{d\xi}(R_-^2)d\xi+\gamma_2\int R_+^4\frac{d}{d\xi}(R_-^2)d\xi\\&+\gamma_3\int R_+^2\frac{d}{d\xi}(R_-^4)d\xi=C_2,\end{aligned} \tag{24}$$

where

$$\underset{1}{\mu} = -\frac{s}{12}, \quad \underset{2}{\mu} = -\frac{q}{4}, \quad \underset{3\pm}{\mu} = \frac{1}{2\sqrt{2}}(\mp\lambda + c_g)^{1/2},$$

$$\delta_\pm^2 = \frac{I_\pm^2}{2}, \quad \gamma_1 = -\frac{r}{2}, \quad \gamma_2 = -\frac{t}{4}, \quad \gamma_3 = -\frac{u}{8}.$$

Adding these two equations we obtain

$$\tfrac{1}{2}\left[\left(\frac{dR_+}{d\xi}\right)^2 + \left(\frac{dR_-}{d\xi}\right)^2\right] + U(R_+, R_-) = E. \tag{25}$$

Equations (23) and (24) are two simultaneous equations describing the evolution of the amplitudes of the solitary excitations, and equation (25) may be interpreted as the energy equation for a particle with two degrees of freedom. The first term in square bracket is the analogue of kinetic energy while $U(R_+, R_-)$ represent the potential energy given as

$$U(R_+, R_-) = \underset{1}{\mu}(R_+^6 + R_-^6) + \underset{2}{\mu}(R_+^4 + R_-^4)$$

$$+(\underset{3+}{\mu}{}^2 R_+^2 + \underset{3-}{\mu}{}^2 R_-^2) + \left(\frac{\delta_+^2}{R_+^2} + \frac{\delta_-^2}{R_-^2}\right) + \gamma_1 R_+^2 R_-^2 + \gamma_2 (R_+^4 R_-^2 + R_-^4 R_+^2), \tag{26}$$

(taking $\gamma_2 = \gamma_3$).

3. *Solution for R_+ and R_-*

For obtaining explicit solutions we initially consider the case $\delta_+ = 0$, $\delta_- \neq 0$ whence from (23) and (24) we obtain:

$$\frac{d^2 R_+}{d\xi^2} + 6\underset{1}{\mu} R_+^5 + 4\underset{2}{\mu} R_+^3 + 2\underset{3+}{\mu}{}^2 R_+ + 2\gamma_1 R_-^2 R_+^2$$

$$+2\gamma_2 R_-^4 R_+ + 4\gamma_2 R_-^2 R_+^3 = 0, \tag{27}$$

$$\frac{d^2 R_-}{d\xi^2} + 6\underset{1}{\mu} R_-^5 + 4\underset{2}{\mu} R_-^3 + 2\underset{3-}{\mu}{}^2 R_- - \frac{2\delta_-^2}{R_-^3} + 2\gamma_1 R_+^2 R_-$$

$$+2\gamma_2 R_+^4 R_- + 4\gamma_2 R_+^2 R_-^3 = 0; \tag{28}$$

integration is not possible unless we neglect the interaction term between R_+ and R_- which introduces an error of the order of ε^5. Then, we can obtain R_+ by solving:

$$\frac{dR_+}{d\xi^2} + 6\underset{1}{\mu} R_+^5 + 4\underset{2}{\mu} R_+^3 + 2\underset{3+}{\mu}{}^2 R_+ = 0. \tag{29}$$

Setting $R_+^2 = g_+$ we get

$$\int d\xi = \frac{1}{2}\int_0^{g_+} \frac{dg_+}{\sqrt{V(g_+)}},$$

with $V(g_+)$ being a fourth power polynomial in g_+:

$$V(g_+) = cg_+ - 2\mu_{3+}^2 g_+^2 - 2\mu_2 g_+^3 - 2\mu_1 g_+^4. \tag{30}$$

The integral written above is usually done with the help of elliptic functions and we can write a special solution in the form [6]:

$$g_+ = \frac{g_2 g_4 (1-\mathrm{cn}^2(\xi_0, K))}{[g_4 - g_2\,\mathrm{cn}^2(\xi_0, K)]}, \tag{31}$$

where

$$\xi_0 = \sqrt{g_1(g_4-g_2)2\mu_1}\cdot\xi,$$

$$K = \sqrt{\frac{(g_1-g_4)g_2}{g_1(g_2-g_4)}}, \tag{32}$$

and g_i ($i = 1, \ldots, 4$) are the roots of the biquadratic relation $V(g_+) = 0$ and it is assumed that $g_3 = 0$. When $g_1 = g_2 \neq g_4$ the modulus of the elliptic function reduces to unity and we obtain

$$g_+ = \frac{g_2\left(\frac{\mu_2}{\mu_1} - 2g_2\right) - g_2\left(\frac{\mu_2}{\mu_1} - 2g_2\right)\mathrm{sech}^2\xi_0}{\left(\frac{\mu_2}{\mu_1} - 2g_2\right) - g_2\,\mathrm{sech}^2\xi_0}. \tag{33}$$

In the case $g_1 = g_4 \neq g_2$ the modulus K of the elliptic function tends to zero and then cn function becomes the sine function. Thus we get

$$g_+ = \frac{g_2 g_4 (1-\sin^2\xi_0)}{g_4 - g_2\sin^2\xi_0}. \tag{34}$$

Next, we proceed to determine the solution for R_-. The equation for R_- reads (neglecting terms of order ε^8):

$$\frac{d^2R_-}{d\xi^2} + 6\mu_1 R_-^5 + 4\mu_2 R_-^3 + 2\mu_{3+}^2 R_- + 2\gamma_1 R_+^2 R_- + 2\gamma_2 R_+^4 R_- = 0. \tag{35}$$

Without any loss of generality we assume the following values for the constants:

$$\underset{1}{\mu} = -\tfrac{1}{2} = -\underset{2}{\mu}, \quad \gamma_1 = 2, \quad \gamma_2 = \gamma_3 = -\tfrac{9}{2}, \quad \underset{3-}{\mu^2} = -\tfrac{5}{32}.$$

First, we consider the reduced equation for R_-, obtained through the above sets of values of the constants:

$$\frac{d^2R_-}{d\xi^2} = 3R_-^5 - 2R_-^3 + \tfrac{5}{16} R_-. \tag{36}$$

If we can obtain two independent solutions R_1 and R_2 of (36) then the solution of the full inhomogeneous equation (35) can be obtained through the technique of variation of parameter and it can be written in the form

$$R_- = V_1R_1 + V_2R_2,$$

with

$$V_1 = -\int \frac{R_2}{\Delta(R_1, R_2)} \cdot \frac{(1-\sin^2\xi_0)(1+7\sin^2\xi_0)}{(2-4\sin^2\xi_0)^2} \, d\xi_0,$$

$$V_2 = \int \frac{R_1}{\Delta(R_1, R_2)} \cdot \frac{(1-\sin^2\xi_0)(1+7\sin^2\xi_0)}{(2-4\sin^2\xi_0)^2} \, d\xi_0. \tag{37}$$

$\Delta(R_1, R_2)$ being the Wronskian of R_1 and R_2. Two independent solutions of the reduced equation (36) are found to be

$$R_1 = \sqrt{\frac{1-\sin^2\xi_0}{2(1-2\sin^2\xi_0)}}, \quad R_2 = \sqrt{\frac{1-\cos^2\xi_0}{2(1-2\cos^2\xi_0)}}, \tag{38}$$

and so we can compute $\Delta(R_1, R_2)$ which comes out to be equal to

$$\Delta(R_1, R_2) = \frac{(1-2\sin^2\xi_0)^2}{(2-4\cos^2\xi_0)^{3/2}(2-4\sin^2\xi_0)^{3/2}}. \tag{39}$$

Substituting these in formulae (37), we can obtain by actual quadrature the final forms of V_1 and V_2, but their detailed structure is too complicated to be reproduced here. On the other hand, one may study numerically their graphical behaviour.

4. *Discussion*

We have discussed in a model the interaction of solitary-wave-like excitations in nuclear matter. Our method is essentially a variant of singular perturbation analysis, done through a scaling of space-time variables. The perturbation is essentially done in the scaling variable ε, to depict in a clear way the collision of two such excitations.

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