STATIC CYLINDRICALLY SYMMETRIC CHARGE DISTRIBUTION IN GENERAL SCALAR TENSOR THEORY OF GRAVITATION

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Static cylindrically symmetric spacetime with an electromagnetic field is discussed in Nordtvedt's generalized scalar tensor theory of gravitation where the parameter ω is a function of the scalar field. Complete solutions are obtained in Dicke's revised units where the cylindrically symmetric line element can be written in Weyl's canonical form.

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1. *Introduction*

In Nordtvedt's [1] modification of Brans-Dicke theory [2], the parameter ω is taken to be a function of the scalar field ψ instead of being a constant. The physical implications of this generalization had been discussed by many authors (e.g. Nordtvedt [1], Barker [3]). In this modified theory, a fairly large number of static and non-static solutions are obtained (Banerjee and Duttachoudhury [4], Banerjee and Santos [5], Barker [3], Rao and Reddy [6], Van den Bergh [7, 8], Banerjee, Duttachoudhury and Banerjee [9], [10]). Very recently Duttachoudhury and Banerjee [11] discussed a static axially symmetric space time in Nordtvedt's theory and obtained vacuum solutions in prolate and oblate spheroidal coordinates. Static cylindrically symmetric vacuum solutions had also been given by them. In this context, we discuss in this paper the coupled Einstein-Maxwell-Nordtvedt field equations and obtain the solutions for a charge distribution giving rise to a cylindrically symmetric electrostatic potential. In Brans-Dicke theory, static cylindrically symmetric vacuum solutions for the uncharged case had been studied earlier by Banerjee and Bhattacharya [12].

The field equations are written in Dicke's revised units [13] where the rest masses of elementary particles vary and the so-called constant of gravitation G remains fixed. The transformed equations can be obtained by a conformal transformation given by

$$g_{\mu\nu} = \psi\bar{g}_{\mu\nu}$$

where variables with a bar are in the original atomic units of Brans and Dicke where particle masses are fixed and G varies. Variables without bar are in the revised units and ψ is the scalar field. In the revised units, the test particles do not satisfy geodesic equations of motion and hence this version is physically less interesting. But the field equations look much simpler and for static axially symmetric vacuum case one can use Weyl's canonical form of the metric in view of the fact that $R_3^3+R_0^0 = 0$ in the revised units. Moreover, once the solutions are known, one can obtain the solutions in the atomic units via the relation $g_{\mu\nu} = \psi\bar{g}_{\mu\nu}$.

In Section 2, the field equations are written in the new units. In Section 3, the equations are completely solved for static cylindrical symmetry and their properties are discussed.

2. *Field equations*

In Dicke's revised units, the field equations for Nordtvedt's general scalar tensor theory are

$$\begin{aligned} G_{\alpha\beta} &= R_{\alpha\beta}-\tfrac{1}{2}\, g_{\alpha\beta}R \\ &= -kT_{\alpha\beta}-\frac{(2\omega+3)}{2}\,(\psi_\alpha\psi_\beta-\tfrac{1}{2}\, g_{\alpha\beta}\psi_\mu\psi^\mu), \end{aligned} \tag{2.1}$$

where $k = \dfrac{8\pi G_0}{c^4}$, $\psi_\alpha = \dfrac{\partial\psi}{\partial x^\alpha}$ and $T_{\alpha\beta}$ represents the energy momentum tensor for matter or any other field e.g., the electromagnetic field. Equation (2.1) can also be written as

$$R_{\alpha\beta} = -k(T_{\alpha\beta}-\tfrac{1}{2}\, g_{\alpha\beta}T)-\frac{(2\omega+3)}{2\psi^2}\,\psi_\alpha\psi_\beta. \tag{2.2}$$

The wave equation for the scalar field is given by

$$\Box\,(\ln\psi) = (\ln\psi)^{;\mu}{}_{;\mu} = \frac{1}{(2\omega+3)}\left[T-\frac{1}{\psi}\,\psi^\mu\psi_\mu\,\frac{d\omega}{d\psi}\right]. \tag{2.3}$$

The general static cylindrically symmetric line element is

$$ds^2 = e^{2\alpha}dt^2-e^{2\beta}(d\varrho^2+dz^2)-\varrho^2e^{2\gamma}d\Phi^2, \tag{2.4}$$

where α, β, γ are functions of the radial coordinate ϱ alone.

We shall consider only a cylindrically symmetric electrostatic field along with the scalar field ψ. The only nonvanishing components of the Maxwell tensor $F_{\mu\nu}$ will be $F_{10} = -F_{01} = \varphi'$ where φ is the electric potential and a prime denotes differentiation with respect to ϱ. The energy momentum tensor for an electromagnetic field is

$$T_{\mu\nu} = -\frac{1}{4\pi}\,[F_{\mu\sigma}F^\sigma_\nu-\tfrac{1}{4}\, g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}]. \tag{2.5}$$

From (2.5), one obtains

$$T_3^3 = -T_0^0 = \frac{1}{8\pi} g^{11}g^{00}\varphi'^2,$$

i.e.,

$$T_3^3 + T_0^0 = 0. \tag{2.6}$$

Again, for an electromagnetic field, the trace T of $T_{\mu\nu}$ is always zero. So, in view of the fact that the scalar field ψ is a function of ϱ alone in the static cylindrically symmetric case, equation (2.2) and (2.6) together yield

$$R_3^3 + R_0^0 = 0, \tag{2.7}$$

which enables one to write the line element (2.4) in Weyl's canonical form,

$$ds^2 = e^{2\lambda}dt^2 - e^{2(\nu-\lambda)}(d\varrho^2 + dz^2) - \varrho^2 e^{-2\lambda}d\Phi^2. \tag{2.8}$$

Here λ and ν are functions of ϱ alone.

Ricci tensors are calculated from this metric and their suitable combinations (Synge [14]) yield the following explicit field equations,

$$\lambda'' + \lambda'/\varrho = e^{-2\lambda}\phi'^2, \tag{2.9}$$

$$\nu' = \varrho\lambda'^2 + \tfrac{1}{4}(2\omega+3)\varrho u'^2 - e^{-2\lambda}\phi'^2, \tag{2.10}$$

$$\nu'' + \lambda'^2 = -\tfrac{1}{4}(2\omega+3)u'^2 + e^{-2\lambda}\phi'^2, \tag{2.11}$$

where $u = \ln \psi$.

The wave equation becomes

$$u'' + \frac{u'}{\varrho} = -\frac{\omega' u'}{(2\omega+3)}. \tag{2.12}$$

The Maxwell equation,

$$F^{\mu\nu}{}_{;\nu} = 0,$$

yields explicitly

$$\phi'' - 2\lambda'\phi' + \phi'/\varrho = 0. \tag{2.13}$$

We have five unknowns — λ, ν, ϕ, u and ω. The number of independent equations is only four — (2.9) to (2.11) and (2.13). With $T = 0$ for the electromagnetic field, the wave equation is a consequence of the field equations in view of the Bianchi identity. For the complete set of solutions, one has to assume the explicit functional relationship between ω and ϕ. In the present paper, some choices of $\omega(\phi)$ will be taken up as examples.

3. *Static cylindrically symmetric solutions*

Equation (2.13) can be written as

$$\frac{\phi''}{\phi'}+\frac{1}{\varrho}=2\lambda'$$

which can be integrated to the form

$$\varrho\phi' = qe^{2\lambda} \tag{3.1}$$

where q is a constant of integration. Multiplying Eq. (2.9) by ϱ and using (3.1) one obtains

$$(\varrho\lambda')' = (q\phi)'$$

which integrates to the form

$$\varrho\lambda' = q\phi + A_1 \tag{3.2}$$

A_1 being a constant of integration. Using (3.1) and (3.2), one obtains

$$e^{2\lambda}\lambda' = (\phi + a/2)\phi', \tag{3.3}$$

where $a = 2A_1/q$. From this, one obtains after integration

$$e^{2\lambda} = \phi^2 + a\phi + b, \tag{3.4}$$

where b is another constant of integration. From (3.2) and (3.4), one can write

$$\frac{\phi'}{(\phi^2+a\phi+b)} = q/\varrho, \tag{3.5}$$

which can be integrated for the solution of the electric potential ϕ.

Now we divide the equation (2.10) by ϱ and add the result to (2.11) to obtain

$$(\varrho\nu')' = 0,$$

which integrates to the form

$$\nu = \ln C_2\varrho^{C_1}, \tag{3.6}$$

where C_1 and C_2 are arbitrary constants. From this we obtain

$$e^{2\nu} = C_2^2\varrho^{2C_1}. \tag{3.7}$$

So the metric can be obtained explicitly once the solution for ϕ is at hand. Equation (3.5) can be integrated in three different cases to give the solution for ϕ.

Case I: $a^2 > 4b$,

$$\phi = \frac{\sqrt{a^2-4b}}{2}\left[\frac{1+(\varrho/\varrho_0)^{q/\sqrt{a^2-4b}}}{1-(\varrho/\varrho_0)^{q/\sqrt{a^2-4b}}}\right]-\frac{a}{2}. \tag{3.8}$$

Case II: $a^2 = 4b$,

$$\phi = [q \ln (\varrho_0/\varrho)]^{-1} - \frac{a}{2}. \tag{3.9}$$

Case III: $a^2 < 4b$

$$\phi = \frac{\sqrt{4b-a^2}}{2} \tan \left\{ \frac{q\sqrt{4b-a^2}}{2} \ln (\varrho/\varrho_0) \right\} - \frac{a}{2}. \tag{3.10}$$

Here ϱ_0 is a constant of integration. With these solutions for ϕ, one can obtain $e^{2\lambda}$ and $e^{2\nu}$ from equations (3.4) and (3.7).

For the complete solution of the problem, the solutions for the scalar field should also be specified. The wave equation (2.12) can be written as

$$\frac{u''}{u'} + \frac{1}{\varrho} + \frac{(2\omega+3)'}{2(2\omega+3)} = 0,$$

which yields after integration

$$(2\omega+3)^{1/2} u' = \frac{m}{\varrho}, \tag{3.11}$$

where m is a constant of integration. This equation can be integrated if the exact functional form of $\omega(\phi)$ is known. We shall take up a few choices as examples.

(1) Brans-Dicke theory [2]: $\omega = \omega_0$, a constant.

$$\psi = e^u = A_1 \varrho^{m/\sqrt{2\omega_0+3}}. \tag{3.12}$$

(2) Barker theory [3]: $(2\omega+3) = 1/(\psi-1) = 1/(e^u-1)$.

$$\psi = e^u = \sec^2 [\ln (A_2 \varrho^{m/2})]. \tag{3.13}$$

(3) Schwinger theory [7]: $2\omega+3 = 1/\alpha\psi = 1/\alpha e^u$.

$$\psi = e^u = (\ln A_3 \varrho^{\frac{1}{2}m\sqrt{\alpha}})^{-2}. \tag{3.14}$$

(4) Curvature coupling [7]: $2\omega+3 = \dfrac{3}{(1-\psi)} = \dfrac{3}{(1-e^u)}$.

$$\psi = e^u = \frac{4A_4\varrho^m}{(1+A_4\varrho^m)^2}. \tag{3.15}$$

Here A_1, A_2, A_3 and A_4 are constants of integration.

With the solutions for ψ at hand, one can transform the metric into the atomic units of Brans and Dicke via the transformation equation

$$\bar{g}_{\mu\nu} = \frac{1}{\psi} g_{\mu\nu},$$

where variables with a bar are in the atomic units whereas those without a bar are in the revised atomic units.

It should be noted that solutions of the scalar field given in the equations (3.12) to (3.15) are the same as those given in the uncharged case by Duttachoudhury and Banerjee [11]. This is to be expected as the scalar field depends on the trace of the energy momentum tensor and $T_{\mu\nu}$ for the electromagnetic field is traceless.

Equations (3.4) and (3.7) show that the metric components exhibit singularity when $(\phi^2+a\phi+b) = 0$ or ∞. As for the electric potential, we see that for the first two cases, i.e. for $a^2 > 4b$ and $a^2 = 4b$, $\phi \to \infty$ when $\varrho = \varrho_0$. So $\varrho = \varrho_0$ defines the cylindrical surface which is the boundary of the charge distribution. In the third case, $\varrho = \varrho_0$ yields $\phi = -a/2$. In this case $\phi \to \infty$ at the cylindrical surface defined by

$$\varrho = \varrho_0 \exp\left[\frac{(2n+1)\pi}{q\sqrt{4b-a^2}}\right],$$

where n is an integer.

In the revised units, the solutions for the metric does not depend on the choice of the functional form of ω, as we have seen that equations (3.4), (3.7) to (3.10) are obtained without assuming any functional form of ω. But in the atomic units, the solutions will depend on the particular choice of ω. One can find out $\bar{g}_{\mu\nu}$ in atomic units via the relation $\bar{g}_{\mu\nu} = \dfrac{1}{\psi} g_{\mu\nu}$, and the solution for ψ is obtained by integrating the equation (3.11) which demands the explicit functional form of ω.

In absence of the scalar field, i.e., for $u = 0$, the solutions should reduce to the general relativity solutions given by Bonnor [15]. Using the solutions obtained in this paper in equation (2.10), one obtains a relation like

$$m^2 = 4C_1 + q(4b-a^2). \tag{3.16}$$

If $u = 0$, we get from (3.11) that $m = 0$. So in the absence of the scalar field, we must have

$$C_1 = \frac{q(a^2-4b)}{4}.$$

With this choice of C_1 the solutions given in the cases $a^2 = 4b$ and $a^2 < 4b$ are seen to be the same as those given by Bonnor which confirms the validity of the solutions obtained in this paper. The solution for the case $a^2 > 4b$, however, has not been given by Bonnor.

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