THE CLASSIFICATION OF COMPLEX BIVECTORS IN SPACE-TIME

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A detailed classification of complex bivectors in space-time is given. The classification is achieved in several ways, with emphasis being laid on the algebraic (Segré type) structure and the geometrical interpretation in complex projective 3-space.

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1. *Introduction*

This paper is intended as a contribution to the recent work on complex general relativity theory and presents a detailed discussion of the algebraic classification of complex second order skew-symmetric tensors (bivectors). Some of the results given here have been quoted, without proof, in a recent paper [1]. The approach will be largely geometrical and is centred round the concept of the fundamental metric quadric. A brief introduction to this quadric and its properties is given in Section 2. Throughout this paper, M will denote a *real* space-time manifold and g the Lorentz metric on M whose components in some coordinate system are denoted by g_{ab}. There will be no loss of generality in taking M to be a real manifold, since the algebraic work of this paper involves the complexified tangent space to M at some point a. However, care must be taken with certain "real" aspects of the classification (to be clarified later) which are preserved under (real) transformations of the (real) tangent space to M at a but not, in general, under general transformations of the complexified tangent space. The usual notation will be followed: square brackets around indices denote skew-symmetrisation, spinor notation will be taken directly from [2], a star on a bivector denotes the usual duality operator and a complex bivector F will be called *self-dual* (*anti self-dual*) if $\overset{*}{F} = -iF$ ($\overset{*}{F} = iF$).

It is convenient here to recall a well-known result for later use. If $a \in M$, let $T_a(M)$ denote the (real) tangent space to M at a. A *real* 2-*space* at a will always mean a two-

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dimensional subspace of $T_a(M)$ and such a 2-space is called a *real timelike* (respectively, *null* or *spacelike*) *2-space* if the number of distinct null directions it contains is two (respectively, one or zero).

2. *Quadric surfaces and PC*^{3}

Let $a \in M$ and choose a coordinate domain containing a. Construct the complexified tangent space to M at a and consider those non-zero members of it, with components x^a, which satisfy the equation

$$g_{ab}x^a x^b = 0. \tag{2.1}$$

The x^a may now be considered as the homogeneous coordinates of a point in projective complex three-dimensional space $\mathbf{PC}^3$ and then (2.1) defines a *quadric surface* $\mathscr{B} \subseteq \mathbf{PC}^3$. (A general discussion of quadric surfaces in $\mathbf{PC}^3$ can be found in Semple and Kneebone [3].) The quadric $\mathscr{B}$ will be referred to as the *fundamental quadric*. It is a *proper* quadric because g_{ab} is a non-singular matrix and comprises all complex null directions at a. As a consequence, the points of $\mathscr{B}$ can be labelled by pairs of (projective) 1-spinors [4] and this feature will be returned to in Section 3(b). Many of the algebraic concepts related to the complexified tangent space to M at a can be described rather easily in terms of $\mathscr{B}$ and it is for this reason that it is introduced here.

A complex two-dimensional subspace of $\mathbf{C}^4$ (a complex 2-space) is represented by a line in $\mathbf{PC}^3$ and if p and q are distinct members of $\mathbf{PC}^3$ lying on this line (or complex vectors spanning the complex 2-space) then the line (or the associated complex 2-space) will be denoted by pq. Any such line either intersects $\mathscr{B}$ in two points (possibly coincident) or else lies entirely in $\mathscr{B}$. A line lying entirely in $\mathscr{B}$ is called a *generator* of $\mathscr{B}$ and there are exactly two families of generators lying in $\mathscr{B}$ each of which is called a *regulus* of $\mathscr{B}$. Any two members of the same regulus are disjoint whereas two generators from different reguli intersect in a unique point. There is exactly one generator from each regulus through each point of $\mathscr{B}$. The subset of $\mathscr{B}$ consisting of complex null directions at a which are complex multiples of real null directions at a is a real quadric called the *reality section* by Penrose [4] and is denoted by $\mathscr{R}$. The reality section is determined through the initial choice of chart domain about a in the real manifold M and is, of course, invariant under coordinate transformations in (the real manifold) M. Each generator of $\mathscr{B}$ is homeomorphic to $\mathbf{PC}^1$ ($\approx S^2$) and $\mathscr{B}$ has the (compact) topological structure $\mathbf{PC}^1 \times \mathbf{PC}^1$ ($\approx S^2 \times S^2$). The reality section $\mathscr{R}$ has the topological structure $\mathbf{PC}^1$ (the "celestial sphere") as expected.

A complex 2-space represented by a generator of $\mathscr{B}$ will be called a (*complex*) *totally null 2-space*. Each three-dimensional subspace of $\mathbf{C}^4$ is represented by a plane in $\mathbf{PC}^3$ and intersects $\mathscr{B}$ in a (plane) conic, a degenerate case occurring when the plane is tangent to $\mathscr{B}$ at some point m. In this case the plane intersects $\mathscr{B}$ in the two generators passing through m. If m has homogeneous coordinates m^a, the members of the tangent plane to $\mathscr{B}$ at m comprise all those points of $\mathbf{PC}^3$ whose homogeneous coordinates x^a satisfy $g_{ab}m^a x^b = 0$. As a consequence, the two generators through $m \in \mathscr{B}$ comprise those complex null directions at a orthogonal to m. It easily follows that any generator of $\mathscr{B}$ intersects $\mathscr{R}$

in exactly one point and so any complex totally null 2-space contains exactly one real (necessarily null) direction and any two independent members of it are complex null orthogonal directions. If $l \in \mathscr{R}$ and if L and $\bar{L}$ denote the two generators of $\mathscr{B}$ through l then the members of $\bar{L}$ are the conjugates of those in L. The generators L and $\bar{L}$ will be referred to as *conjugate generators*.

A line in PC^3 which does not lie in $\mathscr{B}$ is either tangent to $\mathscr{B}$ (in which case the corresponding complex 2-space is called a (*complex*) *null 2-space*) or else intersects $\mathscr{B}$ in two distinct points (in which case the corresponding complex 2-space is called a (*complex*) *non-null 2-space*). Each member of a complex null 2-space represented by a line tangent to $\mathscr{B}$ at m (and hence lying in the tangent plane to $\mathscr{B}$ at m) is orthogonal to the complex null direction m and, consequently, each *real* null 2-space containing the *real* null direction l is a subset of a complex null 2-space tangent to $\mathscr{B}$ at the point $l \in \mathscr{R}$. A real timelike 2-space containing real null directions l and n is a subset of a complex non-null 2-space whose representative line intersects $\mathscr{B}$ in the two distinct points of $\mathscr{R}$ determined by l and n. A real spacelike 2-space spanned by real orthogonal unit spacelike vectors x and y at a is a subset of a complex non-null 2-space whose intersections with $\mathscr{B}$ are the conjugate complex null directions spanned by $x \pm iy$. Any complex 2-space other than the ones considered here cannot contain a real 2-space and so contains at most one real direction.

The orthogonal complement of a complex non-null 2-space is itself a complex non-null 2-space whilst the orthogonal complement of a complex null 2-space tangent to $\mathscr{B}$ at m is itself a complex null 2-space tangent to $\mathscr{B}$ at m. The orthogonal complement of a complex totally null 2-space is the same 2-space, from which one has the useful result that if pq is a complex totally null 2-space and if m is a complex vector orthogonal to p and q then $m \in pq$.

The above paragraph shows that each point $m \in \mathscr{B} \backslash \mathscr{R}$ uniquely determines a real spacelike 2-space at a spanned by the real and imaginary parts of m (and independent of the representative chosen for m). The correspondence is $2:1$ since a real spacelike 2-space at a only determines m to within conjugation. In particular, if L is any generator of $\mathscr{B}$ and $L \cap \mathscr{R} = \{l\}$ then the above paragraph shows that each member of $L \backslash \{l\}$ uniquely determines a real spacelike 2-space at a orthogonal to l. Distinct members of $L \backslash \{l\}$, corresponding to complex null directions m and m', determine distinct real spacelike 2-spaces according to the formal change $m \to m' = m + \alpha l$ ($\alpha \in \mathbf{C}$). These 2-spaces are interpreted as the *wave-surfaces* to l at a — the instantaneous wave surfaces of observers with all possible 4-velocities at a. Similar comments apply to the set $\bar{L} \backslash \{l\}$ since this set generates the same 2-spaces but with different parity. Each of the sets $L \backslash \{l\}$ and $\bar{L} \backslash \{l\}$ is diffeomorphic to $\mathbf{C}$ ($\approx \mathbf{R}^2$) and the isomorphism between the proper null notation subgroup of the Lorentz group about l and the Euclidean translation group is now clearly displayed. Now choose any $m \in L \backslash \{l\}$ and the corresponding $\bar{m} \in \bar{L} \backslash \{l\}$. The second generators through m and $\bar{m}$ meet on $\mathscr{R}$ and thus determine another real null direction n distinct from l. With an abuse of notation, if one thinks of l, m, $\bar{m}$ and n as vectors at a spanning the associated members of $\mathscr{B}$ and restricted by $l_a n^a = m_a \bar{m}^a = 1$, one determines a complex null tetrad l, m, $\bar{m}$, n up to the changes $l \to l' = Al$ ($A \in \mathbf{R}$, $A \neq 0$), $n \to n' = A^{-1} n$ and $m \to m' = e^{i\theta} m$ ($\theta \in \mathbf{R}$) together with the interchange $m \leftrightarrow \bar{m}$. The full group of null rotations about

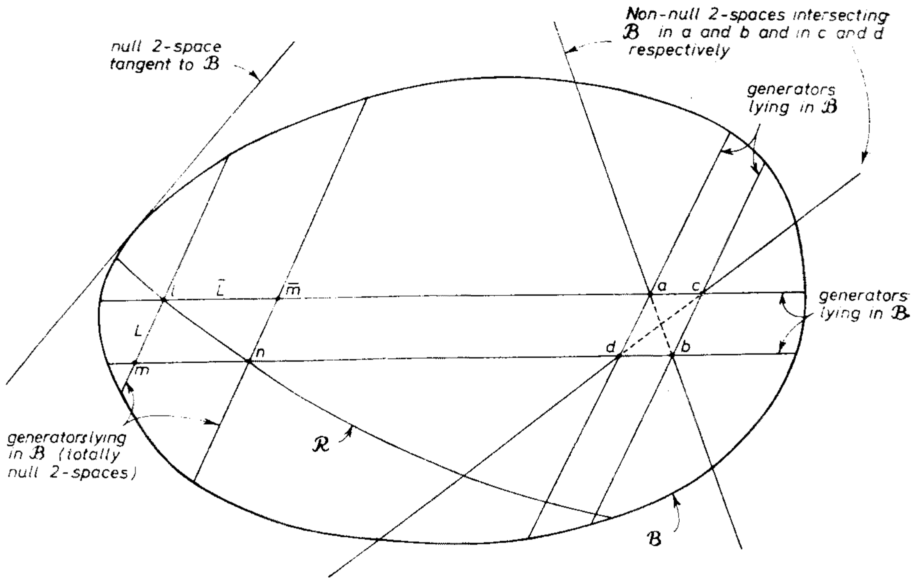

Diagram 1. The quadric $\mathscr{B}$, indicating $\mathscr{R}$ and examples of generators and 2-spaces

l is then recovered by insisting that the changes be future — and parity — preserving so that $A > 0$ and the interchange $m \leftrightarrow \bar{m}$ is forbidden.

Some of the more important features of the quadric $\mathscr{B}$ are represented in diagram 1. It is remarked also that the geometry of quadric surfaces has been used to discuss the classification of symmetric tensors in space-time [4–6].

3. The classification of complex bivectors

In this section, the algebraic structure of a non-zero complex bivector F at $a \in M$, whose components in some coordinate system of M about a are represented by F_{ab} ($= -F_{ba}$), will be considered. The algebraic structure will be described in two ways: (*i*) by considering the associated linear map from the complexified tangent space to M at a to itself represented by the complex matrix $F^a{}_b$ ($= g^{ac}F_{cb}$) and then studying the eigenvector-eigenvalue problem

$$F^a{}_b k^b = \lambda k^a \tag{3.1}$$

for the eigenvector-eigenvalue pairs (k, λ) where $k \in \mathbf{C}^4$ ($k \neq 0$) and $\lambda \in \mathbf{C}$, giving the result in terms of the well-known Segré notation and convenient canonical forms (*ii*) by considering the spinor equivalent of F and performing a spinor classification. A few brief comments will also be made concerning the invariant 2-space structure of F and the concepts of self- and anti self-duality. For completeness, the case when F is a real bivector will be included specifically. The algebraic results will be given a convenient geometrical interpretation by using the quadric surface $\mathscr{B}$ introduced in Section 2.

(a) The eigenvector-eigenvalue structure — Segré types

The following simple comments follow directly from the skew-symmetry of F:

(*i*) if (k, λ) is an eigenvector-eigenvalue pair satisfying (3.1) then either $\lambda = 0$ or k is null,

(*ii*) if (k_1, λ_1) and (k_2, λ_2) are eigenvector-eigenvalue pairs satisfying (3.1) then either k_1 and k_2 are orthogonal or $\lambda_1 + \lambda_2 = 0$,

(*iii*) the sum of the eigenvalues of F (counted properly) is zero,

(*iv*) the rank of the matrix $F^a{}_b$ is 2 or 4.

If F has rank 2 it can be written as $F_{ab} = 2u_{[a}v_{b]}$ and the associated complex vectors u and v span a two-dimensional subspace of the complexified tangent space at a called the (complex) *blade* of F. Such bivectors are called *simple*, otherwise *non-simple*. The possible Segré types for a general 4×4 complex matrix are, in the usual notation, $\{4\}$, $\{3, 1\}$, $\{2, 2\}$, $\{2, 1, 1\}$ and $\{1, 1, 1, 1\}$ and their degeneracies and their canonical Jordan matrices are, respectively,

$$\begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 1 \\ 0 & 0 & 0 & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 1 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \quad (3.2)$$

where $\alpha, \beta, \gamma, \delta \in \mathbf{C}$ are the associated eigenvalues. Eigenvalue degeneracies will, as usual, be indicated by enclosing the appropriate digits in the Segré symbol inside round brackets. It is then an immediate consequence of (*iii*) and (*iv*) above that type $\{4\}$ is impossible for F. The Segré type $\{3, 1\}$ *with non-zero eigenvalues* for F is also impossible for, in this case, the appropriate matrix in (3.2) shows that independent complex vectors p, q, r, s exist satisfying

$$F_{ab}p^b = \alpha p_a \quad (3.3)$$

$$F_{ab}q^b = \alpha q_a + p_a \quad (3.4)$$

$$F_{ab}r^b = \alpha r_a + q_a \quad (3.5)$$

$$F_{ab}s^b = -3\alpha s_a. \quad (3.6)$$

Now $\alpha \neq 0$ and, as a consequence of results (*i*) and (*ii*) above, p and s are null (and orthogonal). Then a contraction of (3.3) with q^a and (3.4) with p^a shows that p and q are orthogonal whilst a contraction of (3.4) with q^a shows that q is null. Hence pq is totally null. Finally, a contraction of (3.3) with r^a and (3.5) with p^a shows that $p_a r^a = 0$ whilst a contraction of (3.4) with r^a and (3.5) with q^a shows that $q^a r_a = 0$. It follows that $r \in pq$, contradicting the fact that p, q and r are independent.

Consider next the case of Segré type $\{(3, 1)\}$ with all eigenvalues zero. The bivector F is then simple and (3.3) to (3.6) hold with $\alpha = 0$. Contractions similar to those of the previous paragraph then give $p_a p^a = p_a q^a = s_a q^a = r_a q^a = s_a p^a = 0$. It then follows that pq is *not* totally null for if it is (and hence q is null) the previous results show that $s \in pq$

and one has a contradiction as before. As a result, $r_a p^a \ (= -q_a q^a) \neq 0$. Similarly, ps is not totally null and so s is not null. This case is then possible and F has a canonical form

$$F_{ab} = 2p_{[a}q_{b]}, \tag{3.7}$$

where $p_a p^a = p_a q^a = 0$ and p is normalised so that $q_a q^a = -r_a p^a = 1$. There is a uniquely determined complex null eigendirection represented by p and q is determined to within an additive multiple of p. The totality of eigendirections of F lies in the null 2-space through p orthogonal to pq. A bivector of this type will be called *null* because its blade pq is a complex null 2-space.

For the Segré type $\{2, 2\}$ with eigenvalues $\neq 0$ the rank is 4 and one can choose independent vectors p, q, r, s such that

$$F_{ab}p^b = \alpha p_a \tag{3.8}$$

$$F_{ab}q^b = \alpha q_a + p_a \tag{3.9}$$

$$F_{ab}r^b = -\alpha r_a \tag{3.10}$$

$$F_{ab}s^b = -\alpha s_a + r_a. \tag{3.11}$$

Similar contractions to those above give $p_a p^a = p_a q^a = q_a q^a = r_a r^a = r_a s^a = s_a s^a = 0$ and so pq and rs are totally null. Equations (3.9) and (3.10) also show that $p_a r^a = 0$. The independence of p, q and r then shows that $r_a q^a \neq 0$. Now the equations (3.8) to (3.11) are unchanged if s is replaced by $s' = s + \lambda r$ ($\lambda \in \mathbb{C}$) and, since $r_a q^a \neq 0$, λ may be chosen such that $q_a s'^a = 0$. Equations (3.9) and (3.11) contracted with s^a and q^a respectively then show that $p_a s^a = -q_a r^a$ which, since non-zero, may be set equal to unity by an appropriate scaling. A canonical form for F is then (with the prime on s omitted)

$$F_{ab} = 2\alpha p_{[a}s_{b]} - 2\alpha q_{[a}r_{b]} - 2p_{[a}r_{b]}. \tag{3.12}$$

In this canonical form pq, qs, sr and rp are totally null complex 2-spaces and are easily represented on the quadric $\mathscr{B}$. The null directions represented by p and r are the only eigendirections of F whilst there is an obvious freedom in the choice of q and s. The above set-up including the canonical form (3.12), is preserved by the changes $q \to q' = q + \mu p$, $s \to s' = s + \mu r$ ($\mu \in \mathbb{C}$).

If F has Segré type $\{(2, 2)\}$ with zero eigenvalues, then (3.8) to (3.11) hold with $\alpha = 0$. One then finds $p_a p^a = p_a q^a = r_a r^a = r_a s^a = r_a p^a = 0$ and $s_a p^a = -q_a r^a \neq 0$. With an appropriate scaling, so that $q_a r^a = 1$, one finds a canonical form

$$F_{ab} = 2p_{[a}r_{b]}. \tag{3.13}$$

Such a bivector is simple and called *totally null* because its blade pr is a totally null complex 2-space. It has infinitely many null eigendirections (exactly one of which is real) comprising all members of the generator represented by pr and there are no other eigendirections. Thus in the canonical form (3.13), p and r may be replaced according to $p \to p' = \mu p + \nu r$, $r \to r' = \varrho p + \sigma r$ where $\mu\sigma - \nu\varrho = 1$.

Next consider the case when F has Segré type $\{2, 1, 1\}$ or one of its degeneracies. It follows from the rank condition (iv) at the beginning of this section that all the eigenvalues must be non-zero and so one can choose independent vectors p, q, r, s such that

$$F_{ab}p^b = \alpha p_a \tag{3.14}$$

$$F_{ab}q^b = \alpha q_a + p_a \tag{3.15}$$

$$F_{ab}r^b = \beta r_a \tag{3.16}$$

$$F_{ab}s^b = \gamma s_a, \tag{3.17}$$

where $\alpha, \beta, \gamma \neq 0$ and $2\alpha+\beta+\gamma = 0$. It immediately follows that pq is totally null and that r and s are null. But it also follows that $(\beta+\gamma)r_a s^a = 0$ and so $r_a s^a = 0$ and rs is totally null. Next, one easily shows that $(\alpha+\beta)p_a r^a = (\alpha+\beta)q_a r^a + p_a r^a = 0$ and $(\alpha+\gamma)p_a s^a = (\alpha+\gamma)q_a s^a + p_a s^a = 0$ ane so, irrespective of the values of $(\alpha+\beta)$ and $(\alpha+\gamma)$, $p_a r^a = p_a s^a = 0$. This means that $p \in rs$ and contradicts the independence of p, r and s. Thus the Segré type $\{2, 1, 1\}$ for F is impossible.

Finally, consider the case when F has Segré type $\{1, 1, 1, 1\}$ or one of its degeneracies, (so that F is diagonable over $\mathbf{C}$) and suppose first that F has rank 4. Then one can choose four independent eigenvectors p, q, r, s with respective non-zero eigenvalues $\alpha, \beta, \gamma, \delta$, whose sum is zero. It follows that p, q, r, s are all null and that $(\alpha+\beta)p_a q^a = (\alpha+\gamma)p_a r^a = (\alpha+\delta)p_a s^a = (\beta+\gamma)q_a r^a = (\beta+\delta)q_a s^a = (\gamma+\delta)r_a s^a = 0$. As a result, $(\alpha+\beta)$, $(\alpha+\gamma)$ and $(\alpha+\delta)$ cannot all be non-zero because then $p_a q^a = p_a r^a = p_a s^a = 0$ and so pq, pr and ps would all be totally null, giving the usual contradiction. So suppose, say, that $\alpha+\beta = 0$. Then $\gamma+\delta = 0$ also and the eigenvalues can be written as $(\alpha, -\alpha, \gamma, -\gamma)$. By switching γ and $-\gamma$, if necessary, one can always arrange that $\alpha+\gamma \neq 0$ and then $p_a r^a = q_a s^a = 0$. Thus pr and qs are totally null. If $\alpha-\gamma \neq 0$ also (the non-degenerate case) then $p_a s^a = q_a r^a = 0$ and so ps and qr are also totally null, showing that ps, sq, qr and rp represent generators of $\mathscr{B}$ with the null eigendirections p, q, r, s uniquely determined and comprising the totality of eigendirections of F. After scaling so that $p_a q^a$ and $r_a s^a$ (which are necessarily non-zero) are unity, one achieves a canonical form for this case (Segré type $\{1, 1, 1, 1\}$)

$$F_{ab} = 2\alpha p_{[a}q_{b]} + 2\gamma r_{[a}s_{b]}. \tag{3.18}$$

If $\alpha-\gamma = 0$, one has the Segré type $\{(1, 1)(1, 1)\}$ and the usual argument shows that pr and qs are totally null and that these two complex 2-spaces comprise the (infinite) totality of all the (necessarily null) eigendirections of F. Appropriate vectors may be chosen from these 2-spaces so that (3.18) holds with $\alpha = \gamma$. If F is simple of rank 2, then the eigenvalues may be taken as $(\alpha, -\alpha, 0, 0)$. One then finds uniquely determined eigendirections p and q with eigenvalues α and $-\alpha$ such that $p_a p^a = q_a q^a = 0$, $p_a q^a \neq 0$ and a non-null complex 2-space orthogonal to (the non-null complex 2-space) pq which comprise the remainder of the eigendirections of F all with eigenvalue zero. This latter 2-space contains a unique pair of null eigendirections r and s and then (suitably labelled) ps, sq, qr and rp represent generators of $\mathscr{B}$. The Segré type is $\{1, 1(1, 1)\}$ and (3.18) holds with $\gamma = 0$.

(b) Spinor representation

A non-zero complex bivector F determines and is determined by two symmetric 2-spinors ϕ_{AB} and ψ_{AB}. The spinors ϕ_{AB} and ψ_{AB} are, in general, independent but important special cases occur when F is real (in which case $\bar{\psi}_{\dot{X}\dot{Y}}$ is conjugate to ϕ_{AB}) and when F is self--dual or anti self-dual (in which case one or the other of the spinors ϕ_{AB} and ψ_{AB} is zero). The representation of F in terms of these spinors is just the unique splitting of F into its self-dual and anti self-dual parts (see Section 3(f)). Now a non-zero symmetric 2-spinor ϕ_{AB} uniquely determines a pair of principal 1-spinor directions and is classified as *non-null* or *null* according as these principal directions are distinct or coincident. This classification can also be achieved by considering the complex 2×2 matrix $\phi^A{}_B = \varepsilon^{AC}\phi_{CB}$. It is easily shown that such a matrix must be either of Segré type $\{1, 1\}$ with equal and opposite eigenvalues, or $\{2\}$ with eigenvalue zero. These characterise, respectively, the non-null and null cases. In the non-null case ϕ_{AB} may be written in terms of its principal 1-spinor directions (eigenspinors), represented by ξ_A and η_A, as $\phi_{AB} \propto \xi_{(A}\eta_{B)}$. In the null case, with principal 1-spinor (eigenspinor) represented by ξ_A, $\phi_{AB} \propto \xi_A\xi_B$.

A complex null direction represented, say, by k determines and is determined by a pair of 1-spinor directions represented, say, by ξ_A and η_A. In this sense, one may regard the 2-spinor $\xi_A\eta_{\dot{X}}$ as the spinor equivalent of k^a. It is then easily shown that k is a complex null eigenvector of F if and only if ξ_A and η_A are, respectively, eigenspinors of ϕ_{AB} and ψ_{AB}, $\phi_{AB}\xi^B \propto \xi_A$, $\psi_{AB}\eta^B \propto \eta_A$. A *real* null eigenvector of F then occurs if and only if a certain

TABLE I

The description of a complex bivector of given Segré type in terms of its spinor decomposition (complex null eigenvector structure) and invariant 2-space structure. The notation is the same as in the text

ϕ_{AB}	ψ_{AB}	F_{ab}	Invariant 2-spaces		
			totally null	null	non-null
$\xi_A\xi_B$	0	$\{(2, 2)\}$	pr and any generator of the other regulus	lx for any $l \in pr$ and any x with $x_a l^a = 0$, $x_a x^a \neq 0$	none
$\xi_{(A}\eta_{B)}$	0	$\{(1, 1)(1, 1)\}$	the two "eigenvector generators" and any member of other regulus	none	any non-null line intersecting both "eigenvector generators"
$\xi_A\xi_B$	$\eta_A\eta_B$	$\{(3, 1)\}$	the two generators through p	px for any x with $p_a x^a = 0$, $x_a x^a \neq 0$	none
$\xi_{(A}\eta_{B)}$	$\mu_A\mu_B$	$\{2, 2\}$	pq, pr, rs	none	none
$\xi_{(A}\eta_{B)}$	$\mu_{(A}\nu_{B)}$	$\{1, 1, 1, 1\}$	ps, sq, qr, rp	none	pq, rs
		$\{1, 1(1, 1)\}$	ps, sq, qr, rp	$px(qy)$ for any $x(y)$ with $x_a p^a = 0$ $x_a x^a \neq 0$ ($y_a p^a = 0$, $y_a y^a \neq 0$)	pq, rs

1-spinor is simultaneously an eigenspinor of ϕ_{AB} and ψ_{AB}. If F is self-dual so that, say, $\psi_{AB} = 0$, and if ξ_A is an eigenspinor of ϕ_{AB} then $\xi_A \eta_{\dot{X}}$ corresponds to a complex eigenvector of F for all non-zero choices of η_A. Similar comments apply in the anti self-dual case. The classification of F can be put together by considering the separate algebraic types of ϕ_{AB} and ψ_{AB} and comparing with the Segré type of F. This is recorded in Table I, and is essentially the classification of F according to its null eigenvector structure.

(c) Invariant 2-spaces

Let $\tilde{F}$ be the linear map from the complexified tangent space to M at a to itself associated with the bivector F as mentioned in Section 3(a). A subspace V of this complexified tangent space is called a (*complex*) *invariant 2-space* of $\tilde{F}$ if $\tilde{F}(V) \subset V$. An algebraic classification of complex bivectors can be achieved by considering the number and nature (totally null, null and non-null) of the associated invariant 2-spaces. In fact, if pq is an invariant 2-space of F at a, one has in an obvious notation

$$F_{ab}p^b = \alpha p_a + \beta q_a, \qquad F_{ab}q^b = \gamma p_a + \delta q_a \tag{3.19}$$

where $\alpha, \beta, \gamma, \delta, \in \mathbf{C}$. By choosing appropriate spanning vectors in an invariant 2-space of a given type and extending to a convenient basis of $\mathbf{C}^4$ one can then expand F_{ab} in terms of skew products of basis members. The equations (3.19) can then be used to show that [7]

(*v*) every totally null invariant 2-space of F contains at least one null eigendirection of F,

(*vi*) every null invariant 2-space of F contains exactly one null eigendirection of F and if a null invariant 2-space is admitted, F is necessarily simple,

(*vii*) every non-null invariant 2-space contains exactly two null eigendirections of F and if a non-null invariant 2-space is admitted, F has four independent null eigendirections and is thus one of the diagonable Segré types $\{1, 1, 1, 1\}$, $\{(1, 1)(1, 1)\}$ or $\{1, 1(1, 1)\}$,

(*viii*) in every case the orthogonal complement of an invariant 2-space of F is also an invariant 2-space of F.

As a consequence of those observations, the distinct algebraic types so far encountered can be distinguished according to invariant 2-space structure. The details are listed in Table I.

(d) Geometrical interpretation

In performing the algebraic classification of complex bivectors, the geometry of the quadric $\mathscr{B}$ has been used extensively. In fact, one can very quickly summarise the situation in terms of the elementary geometrical structure of $\mathscr{B}$. A complex bivector of type $\{(3, 1)\}$ is characterised by a tangent line to $\mathscr{B}$ (its blade). The orthogonal 2-space to this blade, also a tangent line, represents all the eigendirections of the bivector. A bivector of type $\{(2, 2)\}$ is characterised by a generator of $\mathscr{B}$ which is simultaneously its blade and all its eigendirections. A bivector of type $\{2, 2\}$ is characterised by a unique pair of points on $\mathscr{B}$ which lie on the same generator and are the only eigendirections in this case. A bivector of type $\{1, 1, 1, 1\}$ is characterised by four distinct points p, r, q, s on $\mathscr{B}$ such that pr, rq, qs and sp are generators. These points represent the only eigendirections in this case.

A bivector of type $\{(1, 1)(1, 1)\}$ is characterised by two distinct generators from the same regulus on $\mathscr{B}$ and they comprise the totality of eigendirections for this type, each generator giving the eigendirections resulting from one of the two distinct eigenvalues. Finally, the type $\{1, 1, (1, 1)\}$ is characterised by two points on $\mathscr{B}$ not on the same generator, and which correspond to the eigendirections resulting from the two distinct non-zero eigenvalues. The orthogonal complement of the 2-space spanned by these points constitutes the remaining eigendirections all with zero eigenvalue and two of which are null.

(e) Real bivectors

For completeness, the case when F is a non-zero *real* bivector at a will be considered briefly. In fact, the standard classification of real bivectors is not given in terms of Segré type and so the present section will tie up the usual scheme with that given here. In the conventional approach, a real bivector F at a is called *null* if there exists a (real) vector k such that $F_{ab}k^b = \overset{*}{F}_{ab}k^b = 0$. Thus F is simple and the direction spanned by k is easily shown to be null and uniquely determined by F. Clearly $\overset{*}{F}$ is null if and only if F is and F and $\overset{*}{F}$ then have orthogonal (real) null blades. Otherwise, a real bivector is called *non-null*. If F is non-null and simple it is called *spacelike* or *timelike* according as its blade is spacelike or timelike. Any non-null real bivector uniquely determines a (real) spacelike-timelike pair of orthogonal 2-spaces at a.

Suppose then that F is a real bivector at a and construct a real null tetrad at a. One can then express F in terms of skew products of these tetrad members and easily find the following results concerning the real invariant 2-space structure of F

(*ix*) if V is a real invariant 2-space of F then so is its orthogonal complement V^*; if V is spacelike (and then V^* is timelike) V^* contains two independent real null eigendirections of F and V contains two independent complex conjugate eigendirections of F with conjugate complex eigenvalues,

(*x*) if F admits a complex null eigenvector k which is not a complex multiple of a real vector then the real and imaginary parts of k span a spacelike invariant 2-space of F (independent of the representative chosen for k).

It easily follows from (*ix*) and (*x*) that any real bivector which admits a complex null eigendirection in the sense of (*x*) above must admit four independent complex null eigendirections, two of which are real, and hence must be diagonable over **C** or **R** with two real eigenvalues α and β and two conjugate complex eigenvalues $\gamma \pm i\delta$. Since independent real null directions are never orthogonal, the results (*ii*) and (*iii*) earlier show that that $\beta = -\alpha$ and $\gamma = 0$. It follows that the Segré types $\{(2, 2)\}$ and $\{2, 2\}$ cannot now occur. Constructing Segré types over **R** and using the symbol z to denote a complex (non--real) eigenvalue in those cases which are not diagonable over **R**, one finds that the possible types for a real bivector are $\{(3, 1)\}$ (the real null case), $\{1, 1(1, 1)\}$ (the real, non-null, timelike case), $\{(1, 1)z, \bar{z}\}$ (the real, non-null, spacelike case) and $\{1, 1, z, \bar{z}\}$ (the non-null, non-simple case) where, in the first three cases, the repeated eigenvalue is zero. The geometrical interpretation is readily given as before, full details being essentially contained in Table 1.

(f) Duality

A self-dual complex bivector F ($\neq 0$) may be written in the form $F = A + i\overset{*}{A}$ where the (complex) bivector A is determined up to an additive multiple of a (complex) anti self-dual bivector. The bivector A may be chosen real and is then uniquely determined. Similar comments with obvious modifications hold when F is anti self-dual. If F is self-dual and expressed as above with A a real bivector of Segré type $\{(3, 1)\}$ (a real null bivector) then F is simple and of Segré type $\{(2, 2)\}$ whereas if A is real and of any other Segré type, F is not simple and of Segré type $\{(1, 1)(1, 1)\}$. In the former case F determines a single generator of $\mathscr{B}$ whereas in the latter case F determines a pair of generators from the same regulus. Again, similar comments apply if F is anti self-dual. If F is self-dual, then $\bar{F}$ is anti self-dual (and vice versa) and the generators determined by F and $\bar{F}$ belong to different reguli.

For completeness it may be added here that any simple complex bivector has a simple dual and their blades are orthogonal complements and that any complex bivector can be written as the sum of two simple bivectors. This latter result follows from equations (3.7), (3.12), (3.13) and (3.18). Also, any complex bivector F may be decomposed uniquely into its self-dual and anti self-dual parts (see Section 3(b)). If $F = G + H$ is such a decomposition then G and H are complex bivectors with Segré type $\{(2, 2)\}$ or $\{(1, 1)(1, 1)\}$. The relationship between the Segré type of F and those of G and H can easily be inferred from the work in Section 3(b) and Table I. Since a complex bivector and its dual have essentially the same such decomposition (apart from multiplicative factors) and since such a decomposition determines the complex null directions of F (see Section 3(b)), it follows that a complex bivector and its dual will have identical null eigendirections.

Finally, it is pointed out that the classification of complex bivectors given here is *dual invariant* in the sense that F and $\overset{*}{F}$ will always have the same Segré type. This follows in a straightforward way from the work in Section 3(a). However, from the point of view of the real numbers, the same is not true for real bivectors (see Section 3(e)). Here the classification distinguishes between real and complex eigenvalues and a dual pair of simple real non-null bivectors, one timelike and one spacelike, will have different Segré types.

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