POMERANCHUK THEOREMS AT FINITE ENERGIES

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Rigorous relations connecting the behaviour of real and imaginary parts of antisymmetric (symmetric) amplitude at arbitrary energies are obtained. Diverse versions of Pomeranchuk theorem and their generalizations at finite energies follow from these relations.

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1. Introduction

It is generally known that analyticity and crossing-symmetry of scattering amplitudes lead to the existence of nontrivial connection between their real and imaginary parts. The most famous result of such a type is the Pomeranchuk theorem. After the classical paper [1], a lot of works were devoted to the various generalizations of the Pomeranchuk theorem. Relations between total (differential) cross sections of particle and antiparticle for various regimes of amplitude's behaviour were established in [2–14]. A review of Pomeranchuk type results can be found in [15]. However, most of the papers in this field have dealt only with the asymptotic case $E \to \infty$ (E is the energy in laboratory system).

Generalizations of Pomeranchuk theorem to the finite energies have been obtained in the papers [16, 17]. In the present paper which is the development and the extension of the previous ones we want to stress that the Pomeranchuk theorem is only particular case of more general relations that connect the behaviours of real and imaginary parts of scattering amplitude. These finite energy relations exist both for antisymmetric and symmetric amplitudes. They are valid for arbitrary behaviour of real and imaginary parts of amplitude with respect to the growth of energy, and do not require any additional assumptions about lack of oscillations. The possibility of considering not only asymptotic but finite energies as well is connected with the rigorous estimations of integrals of imaginary or real part of amplitude over the region of super-high energies. These estimations show for example, that the bound on the difference of total cross sections in some interval $(E, \nu E)$ is determined by the behaviour of $\mathrm{Re}\, f_a(E)$ at the energies that are not very far from $E(\nu E)$. The exact sense of the last statement will become clear from expressions (4)–(8) (see below).

As was already mentioned, the existence of the connection between imaginary and real parts of scattering amplitude is based to a great extent on the analytical properties of amplitude in the complex E-plane. Therefore, we will suppose below, that $f(E)$ is analytic for $\mathrm{Im}\, E > 0$. Such analyticity is proved for the broad class of processes ($\pi\pi$, πK, πN, KK, $\pi\Lambda$ and so on). The possibilities of considering the processes for which analyticity in the whole upper half-plane is not proved will be discussed in the concluding part of the paper.

2. Generalization of the classical Pomeranchuk theorem

To begin with, let us derive the integral relation between imaginary and real parts of antisymmetric amplitude $f_a(E)$. This relation will serve as a basis for the bound on the difference of cross sections.

Let us consider

$$\int_C \frac{f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{E'^2-\nu^2E^2}},$$

where C is the contour consisting of the real axis and the half circle of the infinite radius in the upper half-plane (we have chosen the system of units in which $m_\pi^2 = 1$). With the help of the Cauchy theorem and crossing-symmetry condition $f_a(-E'+i0) = -f_a^*(E'+i0)$ we get the following relation:

$$\int_E^{\nu E} \frac{\mathrm{Im}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{\nu^2E^2-E'^2}}$$

$$= \int_1^E \frac{\mathrm{Re}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E^2-E'^2}\,\sqrt{\nu^2E^2-E'^2}} - \int_{\nu E}^{\infty} \frac{\mathrm{Re}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{E'^2-\nu^2E^2}}. \tag{1}$$

(1) is valid for arbitrary values of $E > 1$ and $\nu > 1$. This equality is a generalization of the expression found by Wit [18]. We should like to emphasize, that the choice of relation (1) as basic is not important. The crucial point for derivation of finite energy Pomeranchuk type theorem is the possibility of rigorous estimate of the high energy part of the dispersion

integral [19]. So the specific form of the integral relation connecting real and imaginary part of the scattering amplitude is not essential for us.

Now we must set the bound on the $\mathrm{Re}\, f_a(E)$ (in some papers the initial bound is imposed on $|f_a(E)|$ or even on $|f_\pm(E)|$, but the condition for $\mathrm{Re}\, f_a(E)$ is the weakest of all possible ones). For the validity of the Pomeranchuk theorem for $E \to \infty$, the condition $\mathrm{Re}\, f_a(E)/E \ln E \to 0$ is necessary [19]. But from the Froissart bound [20] it follows that $|f_a(E)| \leqslant CE(\ln E)^2$. To consider the generalizations of different variants of the Pomeranchuk theorem in one and the same manner, we will proceed from the following condition of sufficiently general form being fulfilled in some interval $E_1 < E' < E_2$:

$$c_1 \sqrt{E'^2-1}\,(\ln E')^{\gamma_1} < \mathrm{Re}\, f_a(E') < c_2 \sqrt{E'^2-1}\,(\ln E')^{\gamma_2}, \qquad \gamma_{1,2} \leqslant 2 \tag{2}$$

($c_1, c_2, \gamma_1, \gamma_2$ can be negative as well as positive; if $\gamma_{1,2} < 0$ then we will have as a result, that $\Delta\sigma(E) \equiv \sigma_+(E) - \sigma_-(E)$ decreases more rapidly than $\mathrm{Re}\, f_a(E)/E$).

With the help of the mean value theorem we have the obvious equality:

$$\int\limits_E^{\nu E} \frac{\mathrm{Im}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{\nu^2E^2-E'^2}} = \frac{\Delta\sigma(\bar{E})}{\nu E} F\left(\frac{\pi}{2}, \sqrt{1-\frac{1}{\nu^2}}\right),$$

where $F\left(\frac{\pi}{2}, k\right)$ is the full elliptic integral of the first type ($f_a(E)$ is assumed to be normalized in such a way that $\Delta\sigma(E) = \mathrm{Im}\, f_a(E)/\sqrt{E^2-1}$).

Now we must estimate the integrals of $\mathrm{Re}\, f_a(E')$ in the right-hand side of (1). To avoid the necessity of writing out the unwieldy expressions, we will cite the estimations of these integrals (and, correspondingly, the final inequalities) only for the case $E_2 \gg \nu E \gg E \gg E_1$, omitting the terms of the order of $1/\ln E$, $1/\nu^2$, E_1/E, E/E_2. More precise expressions that take such corrections into account are given in [20] and on the whole have the same structure.

Using the condition (2), we have:

$$\frac{\pi c_1}{2\nu E}(\ln E)^{\gamma_1} < \int\limits_{E_1}^{E} \frac{\mathrm{Re}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E^2-E'^2}\,\sqrt{\nu^2E^2-E'^2}} < \frac{\pi c_2}{2\nu E}(\ln E)^{\gamma_2},$$

$$\frac{\pi c_1}{2\nu E}(\ln E)^{\gamma_1} < \int\limits_{\nu E}^{E_2} \frac{\mathrm{Re}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{E'^2-\nu^2E^2}} < \frac{\pi c_2}{2\nu E}(\ln E)^{\gamma_2}. \tag{3}$$

For the estimation of the integral over the super-high energy region,

$$\tilde{I}(E_2) \equiv \int\limits_{E_2}^{\infty} \frac{\mathrm{Re}\, f_a(E')dE'}{\sqrt{E'^2-1}\,\sqrt{E'^2-E^2}\,\sqrt{E'^2-\nu^2E^2}},$$

let us use the result of [17], in which it is shown that, for $\pi\pi$-scattering,

$$\left|\int\limits_{E_2}^{\infty}\frac{\mathrm{Re}\,f_a(E')dE'}{(E')^3}\right| < \frac{\ln^2(\tilde{a}_2^t E_2)}{E_2}, \qquad \tilde{a}_2^t = \frac{60\pi^{3/2}\sqrt{2}}{7}\,a_2^t;$$

a_2^t is the t-channel D-wave scattering length. Relations of the same type are also valid for the amplitudes of other processes with similar analytical properties, if only one replaces $\tilde{\alpha}_2^t$ by the corresponding quantity characterizing the process under consideration (for example, see [21, 22] on the analogs of $\tilde{\alpha}_2^t$ for πN-scattering).

So, for $E_2 \gg E$ we have:

$$|\tilde{I}(E_2)| < \frac{\pi}{2\nu E}\tilde{R}(E_2); \qquad \tilde{R}(E_2) \equiv \frac{2\nu E}{\pi E_2}\ln^2(\tilde{a}_2^t E_2). \tag{4}$$

We are still to consider

$$I(E_1) = \int\limits_1^{E_1}\frac{\mathrm{Re}\,f_a(E')dE'}{\sqrt{E'^2-1}\sqrt{E^2-E'^2}\sqrt{\nu^2E^2-E'^2}} \equiv \frac{\pi}{2\nu E}R(E_1). \tag{5}$$

We shall assume that we have the information about $\mathrm{Re}\,f_a(E')$ at "low energies", i.e. $R(E_1)$ is known from experimental data. Certainly, the absolute bound of the same type as (4) can be obtained for the integral over "low energies" as well.

Now, substituting (3)–(5) in (1) and taking into account that $F\left(\frac{\pi}{2}, \sqrt{1-\frac{1}{\nu^2}}\right) \approx \ln 4\nu$ for $\nu \gg 1$, we shall get the searched bound:

$$\frac{\pi}{2\ln 4\nu}[c_1(\ln E)^{\gamma_1} - c_2(\ln E)^{\gamma_2} + R(E_1) - \tilde{R}(E_2)] < \Delta\sigma(\bar{E}) < \frac{\pi}{2\ln 4\nu}$$
$$\times[c_2(\ln E)^{\gamma_2} - c_1(\ln E)^{\gamma_1} + R(E_1) + \tilde{R}(E_2)]. \tag{6}$$

Particularly, if ν is sufficiently big ($\nu \sim E^\alpha$, α is some constant), then (6) transforms into

$$\frac{\pi}{2\alpha}\left[c_1(\ln E)^{\gamma_1-1} - c_2(\ln E)^{\gamma_2-1} + \frac{R(E_1)-\tilde{R}(E_2)}{\ln E}\right] < \Delta\sigma(\bar{E}) < \frac{\pi}{2\alpha}$$
$$\times\left[c_2(\ln E)^{\gamma_2-1} - c_1(\ln E)^{\gamma_1-1} + \frac{R(E_1)+\tilde{R}(E_2)}{\ln E}\right]. \tag{7}$$

Thus, $\Delta\sigma(\bar{E}) \to 0$ if only $\mathrm{Re}\,f_a(E)/E\ln E \to 0$. We see that if $\mathrm{Re}\,f_a(E')$ change substantially with the passage from $E' < E$ to $E' > \nu E$ (i.e. $\gamma_1 \neq \gamma_2$), then the effect of suppression of $\Delta\sigma$ with respect to $\mathrm{Re}\,f_a(E)/E$ is revealed only for rather big intervals of energies. Yet, if

$\operatorname{Re} f_a(E)$ does not oscillate violently (for example, if

$$1-c\,\frac{\ln\ln E}{\ln E} < \frac{\operatorname{Re} f_a(E+\eta E)}{(E+\eta E)}\cdot\frac{E}{\operatorname{Re} f_a(E)} < 1+c\,\frac{\ln\ln E}{\ln E},$$

where $\eta \leqslant c' \ln^2 E$; c, c' are constants), then the bounds of the type of double inequality (7) are valid for the fixed values of ν.

Let us note also that, in accordance with (4), the contribution of $\tilde{R}(E_2)$ in the inequalities (6), (7) is unessential regardless of the behaviour of $\operatorname{Re} f_a(E')$ for $E' > \nu E$ if only $E_2 > \nu E(\ln E)^{2-\gamma_2}$. Analogously, the contribution of $E' < E/(\ln E)^{2-\gamma_1}$ is unessential as well. Thus, the behaviour of $\Delta\sigma(E')$ in the interval $(E, \nu E)$ is essentially influenced only by the behaviour of $\operatorname{Re} f_a(E')$ in the interval

$$\frac{E}{(\ln E)^{2-\gamma_1}} < E' < \nu E(\ln E)^{2-\gamma_2}. \tag{8}$$

It is also interesting to note the following consequence of the relation (1): if $\operatorname{Re} f_a(E')/E'$ increases monotonously, then $\Delta\sigma(\bar{E}) < 0$; and if $\operatorname{Re} f_a(E')/E'$ monotonously decreases, then $\Delta\sigma(\bar{E}) > 0$.

So we see that for any γ_1 and γ_2 the nontrivial bounds on $\Delta\sigma(E')$ exist in the interval $(E, \nu E)$ with arbitrary value of E (as was already mentioned above, the rejection of the condition $E \gg 1$ only makes the expressions somewhat more complicated, but does not change them principally [20]). These bounds are the best of those that can be obtained only from analyticity in E and crossing-symmetry (the inequalities (6), (7) transform into equalities with the saturation of the condition (2) for $\operatorname{Re} f_a(E')$).

It should be stressed that the obtained double inequalities include both the classical Pomeranchuk theorem and its various generalizations as particular cases for $E \to \infty$. For instance, if the condition (2) is replaced by $|\operatorname{Re} f_a(E')| < cE'(\ln E')^{\gamma}$ for $E' \to \infty$, then (7) transforms into

$$|\Delta\sigma(\bar{E})| < \frac{\pi c}{2\alpha}(\ln E)^{\gamma-1}.$$

In particular, if $\sigma_+(E') \to c_+$, $\sigma_-(E') \to c_-$ and $\operatorname{Re} f_a(E')/E' \ln E' \to 0$ for $E' \to \infty$, then $c_+ = c_-$.

3. *Finite energy analogs of the inverse Pomeranchuk theorem*

In this section we will consider the finite energy analog of the inverse Pomeranchuk theorem, that is we will obtain the bound on $\operatorname{Re} f_a(\bar{E})$ with the given restriction of the growth of $\Delta\sigma(E')$, i.e. $\operatorname{Im} f_a(E')$.

The base for the deriving of the searched bound is the analog of the relation (1) that can be obtained by consideration of the integral

$$\int_C \frac{f_a(E')\,dE'}{(E')^2\sqrt{E'^2-E^2}\sqrt{E'^2-\nu^2E^2}}\,.$$

The Cauchy theorem and the crossing-symmetry condition lead to the necessary equality:

$$\int_{E}^{\nu E} \frac{\mathrm{Re} f_a(E')dE'}{(E')^2 \sqrt{E'^2-E^2}\sqrt{\nu^2E^2-E'^2}}$$

$$= -\int_{1}^{E} \frac{\mathrm{Im} f_a(E')dE'}{(E')^2 \sqrt{E^2-E'^2}\sqrt{\nu^2E^2-E'^2}} + \int_{\nu E}^{\infty} \frac{\mathrm{Im} f_a(E')dE'}{(E')^2 \sqrt{E'^2-E^2}\sqrt{E'^2-\nu^2E^2}} + \frac{\pi f_a'(0)}{2\nu E^2}. \tag{9}$$

The last term in (9) appears because the integrated expression has a pole at $E' = 0$. It is clear that the contribution of the pole term in (9) is unessential for sufficiently large values of E.

Let us assume that the following condition is fulfilled in the interval $E_1 < E' < E_2$:

$$c_1(\ln E')^{\gamma_1} < \Delta\sigma(E') < c_2(\ln E')^{\gamma_2}. \tag{10}$$

Then the estimations of integrals on the right-hand side of the relation (2) give us (for the sake of simplicity we assume again $E_2 \gg \nu E \gg E \gg E_1$; more precise estimations for arbitrary values of parameters can be found in [20]):

$$\frac{c_1}{\nu^2E^2}(\ln E)^{\gamma_1} < \int_{\nu E}^{E_2} \frac{\mathrm{Im} f_a(E')dE'}{(E')^2 \sqrt{E'^2-E^2}\sqrt{E'^2-\nu^2E^2}} < \frac{c_2}{\nu^2E^2}(\ln E)^{\gamma_2},$$

$$\frac{c_1}{\nu E^2}\ln\frac{2E}{E_1}(\ln E)^{\gamma_1} < \int_{E_1}^{E} \frac{\mathrm{Im} f_a(E')dE'}{(E')^2 \sqrt{E^2-E'^2}\sqrt{\nu^2E^2-E'^2}} < \frac{c_2}{\nu E^2}\ln\frac{2E}{E_1}(\ln E)^{\gamma_2}.$$

For the rough estimation of the integral over the region of super-high energies (precise estimation is not necessary because the contribution of asymptotic energies in (9) is strongly supressed) it is sufficient to make use of the Froissart bound which is valid at finite energies as well as at asymptotic ones [22, 23], i.e. $|\Delta\sigma(E')| < \pi \ln^2 \frac{E}{E_0}$. For the definition of E_0 in the finite energy bounds on total cross sections of some processes see for example [21–23]. Using the Froissart bound, we will have the following estimation:

$$\left|\int_{E_2}^{\infty} \frac{\mathrm{Im} f_a(E')dE'}{(E')^2 \sqrt{E'^2-E^2}\sqrt{E'^2-\nu^2E^2}}\right| < \frac{1}{\nu E^2}\tilde{A}(E_2),$$

$$\tilde{A}(E_2) \equiv \frac{\pi\nu E^2}{2E_2^2}\left(\ln\frac{\sqrt{e}\,E_2}{E_0}\right)^2. \tag{11}$$

As regards the integral over the region of low energies (from the threshold to the E_1), we will assume that its value is known from the available data on the total cross sections (of course, the estimations of the type (4) or (11) can be obtained for this integral as well).

Uniting the integral over the "low energies" with the pole term in (9) which has the similar dependence of E for $E \gg 1$, let us denote:

$$\int_1^{E_1} \frac{\operatorname{Im} f_a(E')dE'}{(E')^2 \sqrt{E^2-E'^2}\sqrt{v^2E^2-E'^2}} - \frac{\pi f_a'(0)}{2vE^2} \equiv \frac{1}{vE^2} A(E_1). \tag{12}$$

Writing out the left-hand side of (9) as

$$\int_E^{vE} \frac{\operatorname{Re} f_a(E')dE'}{(E')^2 \sqrt{E'^2-E^2}\sqrt{v^2E^2-E'^2}} = \frac{\pi \operatorname{Re} f_a(\bar{E})}{2vE^2\bar{E}},$$

we obtain that the searched bound on $\operatorname{Re} f_a(\bar{E})$ has the following form (if $\Delta\sigma(E')$ satisfies the condition (10)):

$$\frac{2\bar{E}}{\pi}\left[\frac{c_1}{v}(\ln E)^{\gamma_1} - c_2 \ln \frac{2E}{E_1}(\ln E)^{\gamma_2} - A(E_1) - \tilde{A}(E_2)\right] < \operatorname{Re} f_a(\bar{E})$$

$$< \frac{2\bar{E}}{\pi}\left[\frac{c_2}{v}(\ln E)^{\gamma_2} - c_1 \ln \frac{2E}{E_1}(\ln E)^{\gamma_1} - A(E_1) + \tilde{A}(E_2)\right]. \tag{13}$$

Let us note that if v is sufficiently large $\left(\left(v \gg \left|\frac{c_2}{c_1}\right| (\ln E)^{\gamma_2-\gamma_1-1}\right) \right)$, then the upper bound on $\operatorname{Re} f_a(\bar{E})$ is determined in fact only by the values of c_1, γ_1 — i.e. by the lower bound on $\Delta\sigma$. And the lower bound on $\operatorname{Re} f_a(\bar{E})$ is determined by the upper bound on $\Delta\sigma$ (by the values of c_2, γ_2), if only $v \gg \left|\frac{c_2}{c_1}\right| (\ln E)^{\gamma_1-\gamma_2-1}$.

Considering the contributions of different terms in (13), it is also clearly seen that the term $\tilde{A}(E_2)$ is practically of no importance if $E_2 > \sqrt{v}E(\ln E)^{1-\gamma_1/2}$, and the term $A(E_1)$ is negligible with respect to the others for $E_1 < \sqrt{E}$. It means in practice that the behaviour of $\operatorname{Re} f_a(E')$ in the interval (E, vE) is influenced only by the behaviour of $\Delta\sigma/E')$ in the interval

$$\sqrt{E} < E' < \sqrt{v}\, E(\ln E)^{1-\gamma_1/2}. \tag{14}$$

4. *Inequalities for real and imaginary parts of symmetric amplitude*

Up to this point we considered only the antisymmetric amplitude. But it is quite clear that the bounds on the symmetric amplitude (which satisfies the crossing-symmetry condition $f_s(-E'+i0) = f_s^*(E'+i0)$) can be obtained by the same method. Not dwelling on the intermediate calculations that are absolutely analogous to those previously carried out, let us give the final bounds for the case $E_2 \gg vE \gg E \gg E_1$ (detailed proofs and corrections of the order of $1/\ln E$, $1/v^2$, E_1/E, E/E_2 are given in [20]).

If the condition

$$c_1(\ln E')^{\gamma_1} < \sigma_s(E') \equiv \sigma_+(E') + \sigma_-(E') < c_2(\ln E')^{\gamma_2}$$

is fulfilled for $E_1 < E' < E_2$, then

$$\frac{\pi \bar{E}}{2 \ln 4\nu}\left[c_1(\ln E)^{\gamma_1} - c_2(\ln E)^{\gamma_2} + A_s(E_1) - \tilde{A}_s(E_2)\right] < \operatorname{Re} f_s(\bar{E})$$

$$< \frac{\pi \bar{E}}{2 \ln 4\nu}\left[c_2(\ln E)^{\gamma_2} - c_1(\ln E)^{\gamma_1} + A_s(E_1) + \tilde{A}_s(E_2)\right]. \tag{15}$$

If for $E_1 < E' < E_2$:

$$c_1 E'(\ln E')^{\gamma_1} < \operatorname{Re} f_s(E') < c_2 E'(\ln E')^{\gamma_2},$$

then

$$\frac{2}{\pi}\left[c_1 \ln \frac{2E}{E_1}(\ln E)^{\gamma_1} - \frac{c_2}{\nu}(\ln E)^{\gamma_2} - R_s(E_1) - \tilde{R}_s(E_2)\right] < \sigma_s(\bar{E})$$

$$< \frac{2}{\pi}\left[c_2 \ln \frac{2E}{E_1}(\ln E)^{\gamma_2} - \frac{c_1}{\nu}(\ln E)^{\gamma_1} - R_s(E_1) + \tilde{R}_s(E_2)\right]. \tag{16}$$

The quantities $A_s(E_1)$, $\tilde{A}_s(E_2)$ and $R_s(E_1)$, $\tilde{R}_s(E_2)$ are defined and estimated in the same way as $A(E_1)$, $\tilde{A}(E_2)$ and $R(E_1)$, $\tilde{R}(E_2)$ (see expressions (11), (12) and (4), (5)), but for the symmetric amplitude only. As in the case of the bounds on $f_a(E)$, the contribution of $A_s(E_1)$ and $\tilde{A}_s(E_2)$ in (15) is unessential if E_1 and E_2 are outside the interval (8), and the contribution of $R_s(E_1)$ and $\tilde{R}_s(E_2)$ in the double inequality (16) is negligible if only E_1 and E_2 are outside the interval (14).

Let us also note that one must always remember that $\sigma_s(E') \geqslant 0$. So, if the left-hand side in (16) becomes negative (for example, if $c_1 < 0$, $c_2 > 0$), then actually there is only upper bound on $\sigma_s(\bar{E})$, and the lower bound must be replaced by zero.

5. *Conclusion*

The obtained inequalities show that nontrivial bounds connecting the behaviours of real and imaginary parts of both antisymmetric and symmetric amplitudes exist for the broad class of scattering processes, amplitudes of which are analytic in energy in the upper half-plane. Furthermore, the inequality that restricts the growth of Re $f_s(\bar{E})$ has the form analogous with the bound on Im $f_a(\bar{E})$; and the bound on Im $f_s(\bar{E})$ has the structure similar with the bound on Re $f_a(\bar{E})$. Let us also stress that all obtained bounds are the optimal ones, i.e. they cannot be improved essentially without use of some additional information besides the analyticity in E and the crossing-symmetry (the only improvement that can be done without additional assumptions is the calculations of the corrections of the order of $1/\ln E$, $1/\nu^2$, E_1/E, E/E_2; such corrections are given in [20], and they do not change the general structure of the final inequalities).

In conclusion, let us give the brief discussion of the possibility to use the method of this paper for the processes for which analyticity of scattering amplitudes in the whole upper E-half-plane is not proved (for example one of such processes is the nucleon-nucleon scattering).

It has been shown in [24] that the amplitude of the arbitrary process of scattering of particles with nonzero masses is analytic in energy in the whole upper half-plane with the only possible exception of a certain domain D of finite extension in the vicinity of the coordinate origin. The amplitude is analytic also in a certain vicinity of each physical point.

Thus, if we substitute the interval $(-1, 1)$ of the contour C by the contour C' that goes in the upper half-plane around the domain of nonanalyticity D, then the relations of the type (1), (9) will remain practically unchanged. The only change of the integral relations will be addition of one more term, that corresponds to the integration over C'. For example, the additional term for the relation (1) will be as follows:

$$\int_{C'} \frac{f_a(E')dE'}{\sqrt{E'^2-1}\sqrt{E'^2-E^2}\sqrt{E'^2-\nu^2E^2}}. \tag{17}$$

Unfortunately, one cannot say anything about concrete extension of the domain D and, consequently, about the length of the contour C'. They are finite — that is all that is known. Therefore, it is difficult to determine the value of the terms of the type (17) and their contributions to the final inequalities. But the contribution of the terms corresponding to the integration over the contour C' decreases proportionally to E^{-2} for sufficiently large values of E. So, the contributions of such terms can be neglected at sufficiently high energies, and the bounds (6), (7), (13), (15), (16) remain valid.

REFERENCES

[1] I. Ya. Pomeranchuk, *Zh. Eksp. Teor. Fiz.* **34**, 725 (1958).
[2] St. Weinberg, *Phys. Rev.* **124**, 2049 (1961).
[3] N. N. Meiman, *Zh. Eksp. Teor. Fiz.* **43**, 2277 (1962); **46**, 1039 (1964); **68**, 791 (1976).
[4] A. A. Logunov, Nguen van Hieu, I. T. Todorov, O. A. Khrustalev, *Phys. Lett.* **7**, 69 (1963).
[5] L. Van Hove, *Rev. Mod. Phys.* **36**, 655 (1964).
[6] A. Martin, *Nuovo Cimento* **39**, 704 (1965).
[7] G. G. Volkov, A. A. Logunov, M. A. Mestvirishvily, *Theor. Mat. Fiz.* **4**, 196 (1970).
[8] T. Kinoshita, *Phys. Rev.* **D2**, 2346 (1970).
[9] Yu. S. Vernov, *Theor. Mat. Fiz.* **4**, 3 (1970).
[10] A. A. Logunov, M. A. Mestvirishvili, O. A. Khrustalev, *Particles and Nuclei* (USSR) **3**, 515 (1972).
[11] R. Wit, *Acta Phys. Slovaca* **23**, 203 (1973).
[12] G. Grunberg, Tran N. Truong, *Phys. Rev.* **D9**, 2874 (1974); G. Grunberg, T. N. Pham, Tran N. Truong, *Phys. Rev.* **D10**, 3829 (1974).
[13] J. Fischer, P. Kolář, *Phys. Lett.* **64B**, 45 (1976); *Phys. Rev.* **D17**, 2168 (1978).
[14] Yu. M. Lomsadze, *Yad. Fiz.* **30**, 754 (1979).
[15] J. Fischer, *Phys. Rep.* **76**, 159 (1981).
[16] Yu. S. Vernov, M. N. Mnatsakanova, *Theor. Mat. Fiz.* **34**, 153 (1978).
[17] Yu. S. Vernov, M. N. Mnatsakanova, Proceedings of the II International Seminar "Problems of High Energy Physics and Quantum Field Theory", Protvino 1979, p. 56.
[18] R. Wit, *Nucl. Phys.* **79**, 664 (1966).

[19] Yu. S. Vernov, M. N. Mnatsakanova, M. S. Chubarov, Proceedings of the VI International Seminar "Problems of High Energy Physics and Quantum Field Theory", Protvino 1983, p. 154.
[20] M. Froissart, *Phys. Rev.* **123**, 1053 (1961); A. Martin, *Nuovo Cimento* **42**, 930 (1966).
[21] A. K. Common, F. I. Yndurain, *Nucl. Phys.* **B26**, 167 (1971).
[22] M. S. Chubarov, M. N. Mnatsakanova, Yu. S. Vernov, *Acta Phys. Pol.* **B13**, 427 (1982).
[23] F. I. Yndurain, *Phys. Lett.* **B31**, 368 (1970); S. M. Roy, *Phys. Reports* **5**, 125 (1970); Yu. S. Vernov, M. N. Mnatsakanova, Proceedings of the International Seminar on Multiple Production and Inclusive Processes at High Energies, Protvino 1977, p. 168.
[24] J. Bros, H. Epstein, V. Glaser, *Nuovo Cimento* **31**, 1265 (1964); *Commun. Math. Phys.* **1**, 240 (1965).