ON THE SEMI-CLASSICAL QUANTIZATION OF THE LUND--REGGE SYSTEM

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We have made a semiclassical quantization of solitary wave solutions in the coupled Lund-Regge model. The basic technique of calculation is that of functional integration. The energy levels are deduced corresponding to the periodic boundary conditions. Due to the differential geometric origin of the model, the quantum version may be thought of as a two--dimensional model of quantum gravity.

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In recent times, many nonlinear systems in two dimensions have been seen to be completely integrable by means of the inverse scattering technique [1]. Due to the manifestation of nonlinearity in many branches of natural science, the concept of soliton is finding application in these areas also. But to explain many microscopic properties sometimes we need the quantum version of such nonlinear systems [2]. Since the search for the canonical variables is different for a nonlinear system, the most general approach is that of functional or path integral initiated by Dashen and Neveu [3]. In this paper we consider the quantization of the soliton like configurations in a coupled Lund-Regge system [4] solvable by inverse scattering transform (IST).

Introduction

The equations under consideration were [4],

$$u_{\xi\eta}-\frac{v_\xi v_\eta \sin\frac{u}{2}}{2\cos^3\frac{u}{2}}+\sin u=0,$$

$$v_{\xi\eta}-\frac{u_\xi v_\eta+v_\xi u_\eta}{\sin u}=0. \tag{1}$$

These equations have been shown to be completely integrable with the help of IST. But a more elegant and algebraic approach has been invented by Ueno [5] and Date [6], which is called a direct method. The spirit of the method is the same as that of Krichever's [7] approach to the ecoidal waves. Since we shall be following this second line of thinking in the construction of the exact solutions, we shall devote a few words to this technique.

Equation (1) is seen to be the compatibility condition between the following linear equations:

$$\left(\frac{\partial}{\partial \xi} - i\begin{bmatrix} 0 & -a^* \\ -a & 0 \end{bmatrix} - \frac{ik}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) Y = 0 \tag{2}$$

$$\left(\frac{\partial}{\partial \eta} - \frac{ik^{-1}}{2}\begin{bmatrix} \cos u & e^{-i\omega}\sin u \\ e^{i\omega}\sin u & -\cos u \end{bmatrix}\right) Y = 0 \tag{3}$$

where k is the eigenvalue and

$$a = ie^{i\omega}\sin\frac{u}{2}\cos u,$$

$$\omega_\xi = v_\xi \cos\frac{u}{2}\cos^2\frac{u}{2}, \qquad \omega_\eta = \frac{v_\eta}{2\cos^2\frac{u}{2}}. \tag{4}$$

The construction of a soliton solution proceeds through the explicit realization of the eigenfunction Y in the form:

$$Y(k, \xi, \eta) = \hat{Y}(k, \xi, \eta) k^N \begin{bmatrix} e^\theta & 0 \\ 0 & e^{-\theta} \end{bmatrix}$$

where,

$$\hat{Y}(k, \xi, \eta) = I + \sum_{j=1}^{N} Y_j(\xi, \eta) k^{-j}$$

$$Y_j = \begin{bmatrix} y_{1,N-j} & y^*_{2,N-j} \\ -y_{2,N-j} & y^*_{1,N-j} \end{bmatrix} \tag{5}$$

$\theta = \frac{i}{2}\left(\xi k + \eta\frac{1}{k}\right)$ and the vectors Y_j are subjected to degeneracy conditions

$$Y(\alpha_j, \xi, \eta)\begin{pmatrix} 1 \\ -c_j \end{pmatrix} = 0, \qquad Y(\alpha_j^*, \xi, \eta)\begin{pmatrix} c_j^* \\ 1 \end{pmatrix} = 0 \tag{6}$$

where α_j, c_j are arbitrary complex constants. Further on it will be seen that c_j are proportional to the amplitudes of solitons. The above equations hold for all $j = 1, 2 \ldots N$. Furthermore the constants α_j are proportional to the speed of the solitons.

For the two-soliton solution, the matrix $\hat{Y}$ is written in the form

$$\hat{Y}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+\begin{pmatrix}y_{11} & y_{21}^{*}\\ -y_{21} & y_{11}\end{pmatrix}k^{-1}+\begin{pmatrix}y_{10} & y_{20}^{*}\\ -y_{20} & y_{10}\end{pmatrix}k^{-2}$$

$$=\begin{pmatrix}1+y_{11}k^{-1}+y_{10}k^{-2} & y_{21}^{*}k^{-1}+y_{20}^{*}k^{-2}\\ -y_{21}k^{-1}-y_{20}k^{-2} & 1+y_{11}^{*}k^{-1}+y_{10}^{*}k^{-2}\end{pmatrix} \tag{7}$$

and for the 1-soliton case

$$\hat{Y}=\begin{pmatrix}k+y_{10} & y_{20}^{*}\\ -y_{20} & k+y_{10}^{*}\end{pmatrix}. \tag{8}$$

For the 2-soliton situation, one of the degeneracy conditions written in full is

$$\begin{pmatrix}\alpha_1^2+\alpha_1 y_{11}+y_{10} & y_{20}^{*}+y_{21}^{*}\alpha_1\\ -y_{21}\alpha_1-y_{20} & \alpha_1^2+y_{11}^{*}\alpha_1+y_{10}^{*}\end{pmatrix}\begin{pmatrix}e^{\theta_1} & 0\\ 0 & e^{-\theta_1}\end{pmatrix}\begin{pmatrix}1\\ -c_1\end{pmatrix}=0 \tag{9}$$

where

$$\theta_1=\frac{i}{2}\left(\alpha_1\xi+\frac{\eta}{\alpha_1}\right).$$

The complementary condition is

$$\begin{pmatrix}\alpha_1^{*2}+\alpha_1^{*}y_{11}+y_{10} & y_{20}^{*}+y_{21}^{*}\alpha_1^{*}\\ -y_{21}\alpha_1^{*}-y_{20} & \alpha_1^2+y_{11}^{*}\alpha_1+y_{10}^{*}\end{pmatrix}\begin{pmatrix}e^{\theta_1*} & 0\\ 0 & e^{-\theta_1*}\end{pmatrix}\begin{pmatrix}c_1^{*}\\ 1\end{pmatrix}=0 \tag{10}$$

with

$$\theta_1^{*}=\frac{i}{2}\left(\alpha_1^{*}\xi+\frac{\eta}{\alpha_1^{*}}\right).$$

Equations similar to (9) and (10) can be obtained for $Y^2(\alpha_2)$ and $Y^2(\alpha_2^{*})$ corresponding to $\theta_2=\frac{i}{2}\left(\alpha_2\xi+\frac{\eta}{\alpha_2}\right)$; $\theta_2^{*}=\frac{i}{2}\left(\alpha_2^{*}\xi+\frac{\eta}{\alpha_2^{*}}\right)$. Usually the degeneracy conditions lead to two sets of four coupled linear equations for y_{ij} and y_{ij}^{*}, which can be solved by Cramer's rule. One such set is

$$(\alpha_1^2+\alpha_1 y_{11}+y_{10})e^{\theta_1}-c_1(y_{21}^{*}\alpha_1+y_{20}^{*})e^{-\theta_1}=0$$

$$(\alpha_2^2+\alpha_2 y_{11}+y_{10})e^{\theta_2}-c_2(y_{21}^{*}\alpha_2+y_{20}^{*})e^{-\theta_2}=0$$

$$c_1^{*}(\alpha_1^{*2}+\alpha_1^{*}y_{11}+y_{10})e^{\theta_1*}+(y_{21}^{*}\alpha_1^{*}+y_{20}^{*})e^{-\theta_1*}=0$$

$$c_2^{*}(\alpha_2^{*2}+\alpha_2^{*}y_{11}+y_{10})e^{\theta_2*}+(y_{21}^{*}\alpha_2^{*}+y_{20}^{*})e^{-\theta_2*}=0 \tag{11}$$

for the two-soliton sectors whose explicit solutions have been presented in equation (12). Then, the soliton solutions for the fields u and v are written as:

$$\cos u = \frac{|y_{10}|^2 - |y_{20}|^2}{|y_{10}|^2 + |y_{20}|^2} \tag{11a}$$

and

$$e^{i\omega} \sin u = - \frac{2y_{10}^* y_{20}}{|y_{10}|^2 + |y_{20}|^2},$$

$$v = -i \log (y_{20} y_{20}^*)^{-1}. \tag{11b}$$

The number of solitons depends on the number of terms kept in the series for $\hat{Y}(k, \xi, \eta)$.

The semi-classical approach

In the semi-classical approach to quantization, it is customary to construct the fluctuations about the classical solutions $u_{\rm cl}$, $v_{\rm cl}$ and then to evaluate the functional integral of $e^{\int L dt}$ over these. For the construction of the fluctuations around the classical solutions, we may set $u = u_{\rm cl} + \sigma$, $v = v_{\rm cl} + \nu$ and linearize around $u_{\rm cl}$ and $v_{\rm cl}$. Here, we follow a celebrated technique followed by almost all the authors for the evaluation of the fluctuations around a soliton solution. Such a procedure tacitly exploits the complete integrability of the system. The basic idea is simple. Since the N-soliton solution is obtained when N pairs of parameters α_j, c_j are taken, therefore we denote such a soliton solution as $\cos u(\alpha_1 c_1, \alpha_2 c_2, \ldots, \alpha_N c_N)$ and $v(\alpha_1 c_1, \alpha_2 c_2, \ldots, \alpha_N c_N)$ which satisfies the nonlinear equation for all values of these parameters. Furthermore, the solution constructed in equations (10) and (11, 12) with two such sets of parameters (α_1, c_1) and (α_2, c_2) reduces immediately to one soliton solution as c_2 is set to zero.

So, if we can say that both $\cos u(\alpha_1 c_1, \alpha_2 c_2)$ and $v(\alpha_1 c_1, \alpha_2 c_2)$ are solutions for all values of c_2, then they are solutions for small values of c_2 also. But when we expand a general solution around the one soliton solution $\cos u(\alpha_1, c_1)$ and $v(\alpha_1, c_1)$ and want an explicit realization of these fluctuations, then one possible way is to get a two-soliton solution and expand in the parameters pertaining to the second one. Actually it can be demonstrated that a fluctuation constructed in such a manner does solve the differential equation obtained by linearization about the one-soliton solution. So that in fact we are using the idea that a small deformation of any soliton solution is a solution of the fluctuation equation. Further details of this method can be found in the works of Dashen, Hasslacher, Neveu, Maillet and de Vega [8]. But here we may exploit the complete integrability of the system (1) to construct a two-soliton solution and then expand in the second parameter up to the first order to construct σ and ν, instead of solving the linearized equations.

The elements $y_{10}, y_{10}^*, y_{20}, y_{20}^*$ pertaining to the two soliton solution are:

$$y_{10} = \frac{D_{10}}{D}; \quad y_{10}^* = \frac{D_{10}^*}{D}; \quad y_{20} = \frac{D_{20}}{D}; \quad y_{20}^* = \frac{D_{20}^*}{D} \tag{12}$$

where

$$D = \begin{vmatrix} -\alpha_1 f_1 & -f_1 & -c_1\alpha_1/f_1 & -c_1/f_1 \\ -\alpha_2 f_2 & -f_2 & -c_2\alpha_2/f_2 & -c_2/f_2 \\ \alpha_1^* c_1^* f_1^* & c_1^* f_1^* & -\alpha_1^*/f_1^* & -1/f_1^* \\ \alpha_2^* c_2^* f_2^* & c_2^* f_2^* & -\alpha_2^*/f_2^* & -1/f_2^* \end{vmatrix}$$

$$D_{10} = \begin{vmatrix} \alpha_1 f_1 & -c_1\alpha_1/f_1 & -c_1/f_1 & -\alpha_1^2 f_1 \\ \alpha_2 f_2 & -c_2\alpha_2/f_2 & -c_2/f_2 & -\alpha_2^2 f_2 \\ c_1^*\alpha_1^* f_1^* & \alpha_1^*/f_1^* & 1/f_1^* & -c_1^*\alpha_1^{*2} f_1^* \\ c_2^*\alpha_2^* f_2^* & \alpha_2^*/f_2^* & 1/f_2^* & -c_2\alpha_2^{*2} f_2^* \end{vmatrix} \tag{13}$$

where $f_1 = e^{\theta_1}, f_2 = e^{\theta_2}$, with similar expressions for D_{20}, D_{20}^* and D_{10}^*. Here the important point is to note that $(\alpha_1, \theta_1, c_1)$ and $(\alpha_2, \theta_2, c_2)$ respectively refer to the two-soliton. It can be shown that if $c_2 = c_2^* = 0$ the Y's revert back to their corresponding values for the 1-soliton case. Expanding each of these determinants around $c_2 = c_2^* = 0$ up to the first order, we obtain the linearized forms of y_{ij} which will yield via formulae (11a) and (11b) the fluctuations around the classical solutions in the following form:

$$\cos u \approx (\cos u)_{1s} + Pc_2 + Qc_2^* = (\cos u)_{1s} + \sigma \tag{14}$$

where $(\cos u)_{1s}$ represents the one-soliton solution for the u field and P, Q are given as

$$P = -\frac{2c_1^*(\alpha_1-\alpha_1^*)(\alpha_2-\alpha_2^*)(\alpha_1^*-\alpha_2)}{(\alpha_1-\alpha_2)(\alpha_1^*-\alpha_2^*)\alpha_1^*\alpha_2}\, e^{2(\theta_1^*-\theta_2)},$$

$$Q = -\frac{2c_1(\alpha_1-\alpha_1^*)(\alpha_2-\alpha_2^*)(\alpha_1-\alpha_2^*)}{(\alpha_1-\alpha_2)(\alpha_1^*-\alpha_2^*)\alpha_1\alpha_2^*}\, e^{2(\theta_2^*-\theta_1)} \tag{15}$$

For the field v we obtain

$$v = v_{1s} + Rc_2 + Sc_2^* \tag{16}$$

v_{1s} represents the one-soliton solution corresponding to the field v.

$$R = -i\,\frac{\alpha_1(\alpha_2-\alpha_2^*)(\alpha_1^*-\alpha_2)}{c_1\alpha_2(\alpha_1-\alpha_1^*)(\alpha_1-\alpha_2^*)}\, e^{2(\theta_1-\theta_2)},$$

$$S = i\,\frac{\alpha_1^*(\alpha_2-\alpha_2^*)(\alpha_1-\alpha_2^*)}{c_1^*\alpha_2^*(\alpha_1-\alpha_1^*)(\alpha_1^*-\alpha_2)}\, e^{-2(\theta_1^*-\theta_2^*)}. \tag{17}$$

Now, the spectrum of the system is obtained from the poles of the propagator [8].

$$G(E) = \mathrm{tr}\left(\frac{1}{H-E}\right) = i\int_0^\infty dT\, \mathrm{tr}\,(e^{-iHT})e^{iET}.$$

In the semi-classical approximation $\mathrm{tr}\,(e^{-iHT})$ is given as a sum over the stationary phase points which are periodic in time. The important saddle point configurations are the one--soliton solutions $(\cos u)_{1s}$ and v_{1s}. The Lagrangian for the system is written as

$$\mathscr{L} = \tfrac{1}{8}\int d\xi d\eta \left\{u_\xi u_\eta - 2(1+\cos u) + \frac{1-\cos u}{1+\cos u}\, v_\xi v_\eta\right\}.$$

Introducing the fluctuations and retaining only up to quadratic terms, we can represent $\mathrm{tr}\,(e^{-iHT})$ as Gaussian integral over fluctuations and we get an expression of $\mathrm{tr}\,(e^{-iHT})$ in the form:

$$\mathrm{tr}\,(e^{-iHT}) = \sum e^{ils_0}(\det D_{ln})^{-1/2}, \tag{18}$$

so being the value of the action for the one-soliton solution. The quadratic part is the fluctuation σ, v is written as:

$$D_{ln} = \int\{\sigma D_1\sigma + 4\sigma D_2\sigma + vD_3\sigma\}d\xi d\eta, \tag{19}$$

where D_1, D_2, D_3 are defined through

$$D_1 = \frac{1}{1-u_1^2}\partial_x^2 - \frac{2u_1u_{1x}}{(1-u_1^2)^2}\partial_x - \frac{1}{1-u_1^2}\partial_t^2$$

$$+2\frac{u_1u_{1t}}{(1-u_1^2)^2}\partial_t + \frac{u_{1t}^2-u_{1x}^2}{(1-u_1^2)^2} + \frac{v_{1x}^2-v_{1t}^2}{(1+u_1)^2}$$

$$D_2 = \frac{v_{1x}}{(1+u_1)^2}\partial_x - \frac{v_{1t}}{(1+u_1)^2}\partial_t$$

$$D_3 = \frac{1-u_1}{1+u_1}\partial_t^2 - \frac{1-u_1}{1+u_1}\partial_x^2 + \frac{2u_{1x}}{(1+u_1)^2}\partial_x - \frac{2u_{1t}}{(1+u_1)^2}\partial_t \tag{20}$$

where the coefficients are to be determined by the one-soliton solutions u_1, v_1.

The actual evaluation of the determinant is done with the help of the stability angles ϵ_j and the periodicity condition. Now the periodicity condition for the fluctuations in a finite length of dimension L is

$$(\alpha_1^*-\alpha_2)L + 2\theta + 2\delta_1 = 2n\pi \tag{21}$$

where

$$2i\delta_1 = \log\frac{\alpha_1(\alpha_2-\alpha_2^*)(\alpha_1^*-\alpha_2)}{\alpha_2(\alpha_1-\alpha_1^*)(\alpha_1-\alpha_2^*)}\cdot\frac{1}{c_1}$$

and

$$(\alpha_1-\alpha_2)L + 2\theta + 2\delta_2 = 2m\pi \tag{22}$$

with

$$2i\delta_2 = \log\frac{\alpha_1(\alpha_2-\alpha_2^*)}{\alpha_2(\alpha_1-\alpha_1^*)}\cdot\frac{1}{c_1}$$

where m and n are integers running over 0, 1, 2 ... etc. From the asymptotic expressions of the fluctuations σ and v as $t \to \pm\infty$ we find that

$$\varepsilon_j = \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_1^*} - \frac{2}{\alpha_2}\right) \quad \text{and} \quad \varepsilon_j' = \left(\frac{1}{\alpha_1} + \frac{1}{\alpha_1^*} - \frac{2}{\alpha_2^*}\right)$$

where the index j stands for the j-th value of α_2, α_2^* to be determined from the equations (21) and (22).

Now, by a suitable linear transformation it is possible to diagonalize the quadratic form in Eqs. (19, 20); so that the determinant factorizes and is equal to

$$D = (\prod_{j=1}^{N} \varepsilon_j \prod_{i=1}^{M} \varepsilon_i')$$

or, $\log D = \sum_{j=1}^{N} \log \varepsilon_j + \sum_{i=1}^{M} \log \varepsilon_i'$ where the summations are to be performed over these values of α_2, α_2^* determined in a set of discrete forms from equations (15) and (16). This discretization of the energy values is actually the first step to quantization. But in reality the sum over (i, j) referred to above is a divergent one and needs a proper renormalization by subtracting the vacuum energy. Such considerations and other details will be the subject matter of another communication.

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