THE EFFECT OF INCLUDING TENSOR FORCES IN NUCLEON-NUCLEON INTERACTION ON THREE-NUCLEON BINDING ENERGY

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Separable two-body interactions are used in considering the three-nucleon problem. The nucleon-nucleon potentials are taken to include attraction and repulsion as well as tensor forces. The separable approximation is used in order to investigate the effect of the tensor forces. The separable expansion is introduced in the three-nucleon problem, by which the Faddeev equations are reduced to a well-behaved set of coupled integral equations. Numerical calculations are carried out for the obtained integral equations using potential functions of the Yamaguchi, Gaussian, Takabin, Mongan and Reid forms. The present calculated values of the binding energies of the ^3H and ^3He nuclei are in good agreement with the experimental values. The effect of including the tensor forces in the nucleon-nucleon interactions is found to improve the three-nucleon binding energy by about 4.490% to 8.324%.

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1. Introduction

Several authors introduced different solvable models for studying the three-nucleon problem. Faddeev [1] had given one of these model approaches. Faddeev successfully introduced an exact solution for the three-nucleon problem. The Faddeev equations are reduced to a well-behaved set of coupled integral equations following the Lovelace [2] formulation by using the separable two-body interactions.

The nucleon-nucleon interaction becomes strongly repulsive at short distances as it has been realized from phase-shift analysis. Thus, the nucleon-nucleon interaction is represented by a short-range repulsive potential surrounded by a long-range attractive potential. The few-particle problems have been studied by suggesting different forms for the two-body interactions with considerable success achieved in fitting the two-body data. The tensor forces are important and should necessarily be included in the two-body interactions to fit the two-body phase shift data. Mitra et al. [3–5] considered the effects of the short-range repulsion and of the tensor force in the three-nucleon system. Amado [6] and his collaborators investigated the effect of the tensor force and short-range repulsion by using a small but non-zero value of the Amado wave-function renormalization parameter z. The ground state of the three-body problem was treated by Phillips [7] by taking into account the effect of the tensor force and the short-range repulsion and using a phenomenological three-body force. The Faddeev-Lovelace approach had been extended by Fuda [8] to include some of the effects of the tensor force in calculations on the three-nucleon ground state. Osman [9] investigated the inclusion of the short-range repulsion in the three-nucleon system which has been found to improve the three-nucleon binding energy. In studying the few-particle problems, different forms of the two-body interactions have been suggested [10–13] which include both attraction and repulsion. Osman [14–16] applied some of the two-body interactions which include short-range repulsion and from these calculations it has been found that the short-range repulsion is important and must be included in the three-body calculations.

In the present work we are interested in studying the effect of the tensor forces in the three-nucleon problem. We follow the Faddeev-Lovelace formalism in solving the three-nucleon problem. The three-body Faddeev equations are reduced to a well-behaved set of coupled integral equations using separable two-body interactions. The two-body interactions used contain both attraction and repulsion parts. In the present work, we use potential functions of the Yamaguchi [17], Gaussian [18], Tabakin [19], Mongan [20, 21] and also of the Reid [22] forms for each of the attraction and repulsion parts of the two-body interactions. The two-body potentials used here also include tensor forces. The separable approximation is applied to the T matrices present in the three-body problem. The three-body problem is solved with the two-body interactions considering both the tensor forces and the short-range repulsion by making use of a separable expansion. The Faddeev equations are reduced to an infinite set of coupled one-dimensional integral equations by using the separable approximation in the three-body problem. These equations may be cut off for finite-range forces. The three-body Faddeev equations are reduced to a finite well-behaved set of coupled one-dimensional integral equations by using the separable approximation for the tensor forces.

In the present work, we study the three-nucleon nuclei ^{3}H and ^{3}He. Each of these nuclei is taken as a bound state of three nucleons. Five different types of interaction are used for the different parts of the two-body interactions. Each of the two-body interactions which we use includes short-range repulsion as well as tensor forces. The Faddeev equations are solved with these two-body potentials by making use of the Lovelace formulations. Direct numerical calculations are performed for the resulting integral equations. We calculate the binding energies of the ^{3}H and ^{3}He nuclei. The numerical calculations of the binding energies are carried out in three different cases to test the effects of both the short-range repulsion and the tensor forces. In the first case the two-body interactions are purely attractive. In the second case the two-body interactions are taken as a short-range repulsive potential surrounded by a long-range attractive potential. Then, in the third case the two-body interactions are taken to include attraction and repulsion and also tensor forces. For each nucleus, the numerical calculations are carried out for the three different cases, using five different forms of two-body interactions. Comparing the results calculated for each nucleus for the same potentials with and without the repulsion part respectively, we get the effect of the short-range repulsion on the binding energy. Comparing the results calculated for each nucleus for the same potential containing repulsion, respectively including and not including tensor forces, we get the effect of tensor forces on the binding energy.

In Section 2, we introduce the expressions including the tensor forces in the three--body Faddeev equations by making use of the separable approximation. In Section 3, numerical calculations and results are presented. Section 4 is devoted to discussion and conclusions.

2. The tensor forces in the nucleon-nucleon interactions

Many models have been suggested for solving the three-body problem. One of these models is the Faddeev approach, which introduces an exact solution for the three-body problem. The Faddeev equations [1, 23] are a well-behaved set of three-body coupled integral equations, which involve the two-body T matrix rather than the potential. The T matrix plays a central role in the Faddeev approach. So, in the three-body Faddeev equations the two-body T matrix plays the part of a potential in the two-body Lippmann--Schwinger equation. Using the same notations as introduced by Lovelace [2], the two--body T matrix is introduced as,

$$T_{ij}(p, q; z) = V_{ij}(p, q)+\frac{1}{\pi}\int\frac{V_{ij}(p, q)kT_{ij}(p, q; z)}{k^2-z}\,d^2k, \tag{1}$$

where V_{ij} is the separable potential which must be strong enough to give a bound state at an energy eigenvalue $-\varepsilon$ and a bound state eigenfunction $|\phi\rangle$. For concreteness, considering the triplet state of the two-nucleon system, there is one bound state which is the deuteron. The T matrix can be defined in terms of the potential theory as

$$t(z) = V+VG(z)V, \tag{2}$$

where V is the two-body potential and

$$G(z) = (z-H)^{-1}; \tag{3}$$

$G(z)$ is the resolvant for the two-body system, z is a complex variable and H is the Hamiltonian. Since $t(z)$ is a solution of the equation

$$t(z) = V+VG_0(z)t(z), \tag{4}$$

$$G_0(z) = (z-H_0)^{-1}, \tag{5}$$

where H_0 is the two-body kinetic energy operator.

Suppose $|\beta\rangle$ is the deuteron state vector and its energy eigenvalue is $-\varepsilon$, and we have a separable potential in the form [1]

$$V_{ij}(p, q) = \lambda f(p)f(q). \tag{6}$$

Then, the approximate two-body T matrix in the triplet channel is

$$t_{\text{sep}}(z) = \frac{V|\beta\rangle\langle\beta|V}{D(z)}, \tag{7}$$

where

$$D(z) = \frac{1}{\lambda} + \langle\beta|VG_0(z)V|\beta\rangle. \tag{8}$$

Picking λ so that $D(z)$ vanishes at $z = -\varepsilon$,

$$t_{\text{sep}}(z) \xrightarrow[z\to-\varepsilon]{} \frac{V|\beta\rangle\langle\beta|V}{z+\varepsilon}, \tag{9}$$

which is a separable expression and satisfies unitarity.

Lovelace showed that the existence of a virtual bound state in the singlet channel justifies the use of separable T matrix. Putting the state vector $|g_{\text{st}}\rangle = V|\beta\rangle$, we can write the two-body T matrix in the form

$$t(z) = |g_{\text{st}}\rangle D_{\text{st}}^{-1}(z)\langle g_{\text{st}}| + |g_{\text{ss}}\rangle D_{\text{ss}}^{-1}(z)\langle g_{\text{ss}}|, \tag{10}$$

where

$$D_{\text{st}}(z) = \frac{1}{\lambda_{\text{st}}} + \langle g_{\text{st}}|G_0(z)|g_{\text{st}}\rangle, \tag{11}$$

$$D_{\text{ss}}(z) = \frac{1}{\lambda_{\text{ss}}} + \langle g_{\text{ss}}|G_0(z)|g_{\text{ss}}\rangle. \tag{12}$$

The subscripts st and ss refer to the spin-triplet state and the spin-singlet state respectively. The vector $|g_{\text{st}}\rangle$ is an $\tilde{L} = 0$ object, since in the singlet channel $\tilde{L}$ is a good quantum number. The vector $|g_{\text{st}}\rangle$ is more complicated, since the deuteron wave function is a mixture of $\tilde{L} = 0$ and $\tilde{L} = 2$ components. The interaction between nucleons in triplet-triplet and singlet-singlet states are very small and are neglected because of symmetry properties.

The nucleon-nucleon potentials used in the present work contain both attraction and repulsion represented as

$$V_{\text{NN}}(\vec{P}, \vec{P}') = \frac{2}{\pi}\bar{\lambda} \sum_{\alpha MLL'} i^{L'-L}(-g_{\alpha L}(p)g_{\alpha L'}(p') + h_{\alpha L}(p)h_{\alpha L'}(p'))Y_{\alpha L}^{M}(\hat{\vec{p}})Y_{\alpha L'}^{*M}(\hat{\vec{p}}'), \tag{13}$$

where α denotes the quantum numbers JTS, and $\bar{\lambda} = \hbar^2/m$ (m is the nucleon mass). The symbols $g_{\alpha L}(p)$ and $h_{\alpha L}(p)$ refer to the attractive and repulsive parts of the potential, respectively. The function $Y_{\alpha L}^{M}(\hat{\vec{p}})$ is a normalized eigenstate of total angular momentum J and its z component M; it is a combination of an orbital angular momentum state $Y_L^{M_L}(\hat{\vec{p}})$ and a total spin state $\chi_s^{M_s}$. The deuteron wave function expressed in terms of the vector spherical harmonics

$$|Y_{LSJ}^{M}\rangle = \sum_{M_1,M_2} |Y_L^{M_1}\rangle\,|SM_2\rangle\,\langle LSM_1M_2|JM\rangle, \tag{14}$$

where $|Y_L^{M_1}\rangle$ is an ordinary spherical harmonic, $|SM_2\rangle$ is a two-body spin vector and $\langle LSM_1M_2|JM\rangle$ is the appropriate Clebsch-Gordan coefficient. The deuteron wave function can be written in the form (assuming $J_z = M$)

$$|\Phi\rangle = |\phi_c\rangle\,|Y_{011}^M\rangle + |\phi_t\rangle\,|Y_{211}^M\rangle. \tag{15}$$

We introduce the tensor operator $\hat{S}_{12}$ which has the form (in momentum representation)

$$\hat{S}_{12} = 3(\vec{\sigma}_1\cdot\hat{\vec{P}})(\vec{\sigma}_2\cdot\hat{\vec{P}}) - \vec{\sigma}_1\cdot\vec{\sigma}_2. \tag{16}$$

$\vec{\sigma}_1$ and $\vec{\sigma}_2$ are the Pauli matrices and P a unit operator. Hence we have

$$\hat{S}_{12}|Y_{011}^M\rangle = \sqrt{8}\,|Y_{211}^M\rangle. \tag{17}$$

Then, by combining equation (17) with equation (15), we have

$$|\Phi\rangle = (1/\sqrt{4\pi})\,[|\phi_c\rangle + |\phi_t\rangle\,(1/\sqrt{8})\hat{S}_{12}]\,|\chi\rangle. \tag{18}$$

Let us introduce the definitions

$$|C\rangle = (-1/\sqrt{4\pi})\,(H_0+\varepsilon)\,|\phi_c\rangle, \tag{19}$$

$$|T\rangle = (-1/\sqrt{4\pi})\,(H_0+\varepsilon)\,|\phi_t\rangle, \tag{20}$$

where $|C\rangle$ and $|T\rangle$ are functions of the length of momentum vector in momentum representation and are chosen to give a reasonable deuteron wave function.

With these definitions, we arrive at the following form for $|g_{st}\rangle$ when $J_z = M$;

$$|g_{st}, M\rangle = [|C\rangle + |T\rangle\,(1/\sqrt{8})\hat{S}_{12}]\,|\chi\rangle. \tag{21}$$

Using the orthonormality of the vector spherical harmonics we can show that

$$D_{st}(Z) = (1/\lambda_{st}) + \langle C|G_0(Z)\,|C\rangle + \langle T|G_0(Z)\,|T\rangle. \tag{22}$$

The T matrix in the triplet channel is

$$t_{st}(Z) = -\sum_{M=-1}^{1} |g_{st}, M\rangle D_{st}^{-1}(Z)\,\langle g_{st}, M|. \tag{23}$$

The final form of the T matrix is then given as

$$T_{ij}(Z) = T_{ij}^{t}(Z)P_{ij}^{(-)}(T) + T_{ij}^{s}(Z)P_{ij}^{(+)}(T), \tag{24}$$

where

$$T_{ij}^{t}(Z) = -[|C_{ij}\rangle + (1/\sqrt{8})\,|T_{ij}\rangle\hat{S}_{12}]P_{ij}^{(+)}(s)D_{st}^{-1}(Z)$$
$$\times[\langle C_{ij}| + (1/\sqrt{8})\hat{S}_{12}\langle T_{ij}|], \tag{25}$$

$$T_{ij}^{s}(Z) = -|S_{ij}\rangle P_{ij}^{(-)}(s)D_{st}^{-1}(Z)\,\langle S_{ij}| \tag{26}$$

and

$$D_{ss}(Z) = (1/\lambda_{ss}) + \langle S|G_0(Z)|S\rangle. \tag{27}$$

λ_{st} and λ_{ss} are the coupling constants with values which are chosen so that at the energy $Z = -\varepsilon$, $D_{st}(Z)$ and $D_{ss}(Z)$ vanish. $P^{(+)}(s)$, $P^{(-)}(s)$, $P^{(+)}(T)$ and $P^{(-)}(T)$ are the projection operators for spin-triplet, spin-singlet, isospin-triplet and isospin-singlet states respectively.

$|S\rangle$ is a function of the length of momentum vector in the case of the spin-singlet state. The wave function obtained in Faddeev scheme is the sum of the three components,

$$|\Psi\rangle = \sum_{\alpha=1}^{3} |\Psi_\alpha\rangle \tag{28}$$

which satisfy the coupled system of equations

$$|\Psi_\alpha\rangle = G_0(E)T_\alpha(E)\,[|\Psi_\beta\rangle + |\Psi_\gamma\rangle], \tag{29}$$

where $G_0(E)$ is the free resolvent of three-body system, E is the energy of three-body system, and T_α is the transition operator of three-body system.

In particular, we can write

$$|\Psi_1\rangle = (123)\,|\Psi_3\rangle, \tag{30}$$

$$|\Psi_2\rangle = (132)\,|\Psi_3\rangle. \tag{31}$$

Combining (30), (31) and (29), we have

$$|\Psi_3\rangle = G_0(E)T_3(E)\,[(123)+(132)]|\Psi_3\rangle. \tag{32}$$

Our transition operator T_3 is essentially a sum of projection operators, each of which is written in terms of two-body vector. Since the two-body states are anti-symmetric under exchange of particles 1 and 2, we have

$$T_3(E)\,(12) = -T_3(E), \tag{33}$$

where we have used the fact that (12) is Hermitian. Thus, in our case equation (32) becomes

$$|\Psi_3\rangle = 2G_0(E)T_3(E)\,(123)\,|\Psi_3\rangle. \tag{34}$$

It is known that the spin and isospin vectors which arise in the three-nucleon problem are basis vectors for the irreducible representations of S_3. Then, the Faddeev equations can be reduced to three coupled one-dimensional integral equations when the two-body transition operator is of the form given by equation (24). The transition operators T_3 which appear in the Faddeev equations [see Equations (29) and (34)] are closely related to the two-body T matrix as

$$\langle \vec{P}_k, \vec{q}_k|T_3(Z)\,|\vec{P}'_k, \vec{q}'_k\rangle = \delta(\vec{q}_k - \vec{q}'_k)\,\langle \vec{P}_k|T_{12}(Z - \tfrac{3}{4}\,q_k^2)\,|\vec{P}'_k\rangle, \tag{35}$$

where $\vec{P}$ is the relative momentum of particles 1 and 2, $\vec{q}$ is the momentum of particle 3 and T_{12} is the transition operator for particles 1 and 2. Hence, T_3 describes processes in which particles 1 and 2 scatter, while particle 3 goes straight through.

For the interaction given by equation (24), T_3 becomes

$$T_3(Z) = T^t(Z)P_\tau^{(-)}(12)+T^s(Z)P_\tau^{(+)}(12), \tag{36}$$

where

$$T^t(Z) = -[|C(12)\rangle+|T(12)\rangle\,(1/\sqrt{8})\hat{S}_{12}]P_\sigma^{(+)}(12)t_{st}(Z) \times[\langle C(12)|+\hat{S}_{12}(1/\sqrt{8})\,\langle T(12)|], \tag{37}$$

$$T^s(Z) = -|S(12)\rangle P_\sigma^{(-)}(12)t_{ss}(Z)\,\langle S(12)|, \tag{38}$$

$$\langle\vec{q}_k|t_{st}(Z)\,|\vec{q}_k'\rangle = \delta(\vec{q}_k-\vec{q}_k')D_{st}^{-1}(Z-\tfrac{3}{4}\,q_k^2), \tag{39}$$

$$\langle\vec{q}_k|t_{ss}(Z)\,|\vec{q}_k'\rangle = \delta(\vec{q}_k-\vec{q}_k')D_{ss}^{-1}(Z-\tfrac{3}{4}\,q_k^2). \tag{40}$$

This interaction conserves isospin but not the spin. There are two possible isospin states for a given J_z-component μ; $|0;\tfrac{1}{2},\mu\rangle\!\rangle$ and $|1;\tfrac{1}{2},\mu\rangle\!\rangle$. These two isospin states are due to the presence of the projection operators in equation (36). Expanding $|\Psi_3\rangle$ in terms of the isospin vectors as

$$|\Psi_3\rangle = |\phi\rangle\,|0;\tfrac{1}{2},\mu\rangle\!\rangle-|\phi'\rangle\,|1;\tfrac{1}{2},\mu\rangle\!\rangle, \tag{41}$$

and substituting the expansion (41) in equation (34), we find

$$|\phi\rangle = 2G_0(E)T^t(E)\,[(123)\,|\phi\rangle\,(-\tfrac{1}{2})-(123)\,|\phi'\rangle\,(-\tfrac{1}{2}\sqrt{3})], \tag{42}$$

$$|\phi'\rangle = 2G_0(E)T^s(E)\,[(123)\,|\phi\rangle\,(-\tfrac{1}{2}\sqrt{3})-(123)\,|\phi'\rangle\,(\tfrac{1}{2})], \tag{43}$$

where E is the three-body energy.

Consider the expansions of $|\phi\rangle$ and $|\phi'\rangle$ in terms of eigenfunctions of total angular momentum using $J-J$ coupling. This can be done by introducing the $J-J$ coupling. Since,

$$\vec{J}_{12} = \vec{L}_{12}+\vec{S}_{12}, \tag{44}$$

$$\vec{J}_3 = \vec{L}_3+\vec{S}_3, \tag{45}$$

$$\vec{J} = \vec{L}_{12}+\vec{J}_3. \tag{46}$$

As a result of the projection operator nature of $T^t(Z)$, we find that the expansion $|\phi\rangle$ contains only those states for which $J_{12} = 1$, $L_{12} = 0, 2$ and $S_{12} = 1$. There are five different possible states. These states are analysed as

L_{12}	S_{12}	J_{12}	L_3	S_3	J_3	State
0	1	1	0	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$\|\Omega_1\rangle$
2	1	1	0	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$\|\Omega_1'\rangle$
0	1	1	2	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$\|\Omega_2\rangle$
−2	1	1	2	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$\|\Omega_2'\rangle$
0	0	0	0	$\tfrac{1}{2}$	$\tfrac{1}{2}$	$\|\Omega_3\rangle$

Using the properties of the tensor operator $\hat{S}_{12}$, we can show that

$$|\Omega_1'\rangle = (1/\sqrt{8})\hat{S}_{12}|\Omega_1\rangle, \tag{47}$$

$$|\Omega_2'\rangle = (1/\sqrt{8})\hat{S}_{12}|\Omega_2\rangle, \tag{48}$$

$$\hat{S}_{12}|\Omega_3\rangle = 0; \tag{49}$$

then, we can write

$$|\phi\rangle = \sum_{i=1}^{2} [|\Sigma_i\rangle\, |\Omega_i\rangle + |\Sigma_i'\rangle\, |\Omega_i'\rangle], \tag{50}$$

$$|\phi'\rangle = |\Sigma_3\rangle\, |\Omega_3\rangle. \tag{51}$$

With the different relations for the $|\Sigma\rangle$ given as

$$|\Sigma_1\rangle = G_0(E)\, |C(12)\rangle\, |\chi_1(3)\rangle, \tag{52}$$

$$|\Sigma_1'\rangle = G_0(E)\, |T(12)\rangle\, |\chi_1(3)\rangle, \tag{53}$$

$$|\Sigma_2\rangle = G_0(E)\, |C(12)\rangle\, |\chi_2(3)\rangle, \tag{54}$$

$$|\Sigma_2'\rangle = G_0(E)\, |T(12)\rangle\, |\chi_2(3)\rangle, \tag{55}$$

$$|\Sigma_3\rangle = G_0(E)\, |S(12)\rangle\, |\chi_3(3)\rangle, \tag{56}$$

$$|\chi_1(3)\rangle = -8\pi\hat{t}_1(E)\, \langle\Omega_1|[\langle C(12)| + \langle T(12)|\, (1/\sqrt{8})\hat{S}_{12}] \\ [(123)\, |\phi\rangle\, (-\tfrac{1}{2}) - (123)\, |\phi'\rangle\, (-\tfrac{1}{2}\sqrt{3})], \tag{57}$$

$$|\chi_2(3)\rangle = -8\pi\hat{t}_1(E)\, \langle\Omega_2|\, [\langle C(12)| + \langle T(12)|\, (1/\sqrt{8})\hat{S}_{12}] \\ [(123)\, |\phi\rangle\, (-\tfrac{1}{2}) - (123)\, |\phi'\rangle\, (-\tfrac{1}{2}\sqrt{3})], \tag{58}$$

$$|\chi_3(3)\rangle = -8\pi\hat{t}_2(E)\, \langle\Omega_3|\, \langle S(12)\, |[(123)\, |\phi\rangle\, (-\tfrac{1}{2}\sqrt{3}) - (123)\, |\phi\rangle\, (\tfrac{1}{2})]. \tag{59}$$

Combining equations (50) and (51) with equations (52–56) and using equations (47) and (48), we can write

$$|\phi\rangle = G_0(E)\, [|C(12)\rangle + |T(12)\rangle\, (1/\sqrt{8})\hat{S}_{12}]\, [|\chi_1(3)\rangle\, |\Omega_1\rangle + |\chi_2(3)\rangle\, |\Omega_2\rangle], \tag{60}$$

$$|\phi'\rangle = G^0(E)\, |S(12)\rangle\, |\chi_3(3)\rangle\, |\Omega_3\rangle. \tag{61}$$

These are the two space-spin parts of the component $|\Psi_3\rangle$. If one expands the angular momentum states in terms of the eigenstates of J_{12}^2, J_{12z}, J_3^2 and J_{3z}, the structure of $|\phi\rangle$ and $|\phi'\rangle$ becomes obvious. For example $|\phi\rangle$ consists of a sum of terms corresponding to the different values of J_{12z} and J_{3z}. Each term is of the form of a deuteron wave function (modified kinematically by the presence of the third particle) multiplied by a wave function which depends only on the coordinates of particle 3. We simply work out a part of the kernel which connects $|\chi_1\rangle$ to itself.

The part of the kernel is given by

$$\langle \Omega_1 | (1/\sqrt{8}) \hat{S}_{12} \langle T(12) | (123) G_0(E) | T(12) \rangle \hat{S}_{12} (1/\sqrt{8}) | \Omega_1 \rangle | \chi_1(3) \rangle, \tag{62}$$

where (123) operates to the right. Using equation (47), equation (62) can be given as

$$\langle \Omega_1' | \langle T(12) | (123) G_0(E) | T(12) \rangle | \Omega_1' \rangle | \chi_1(3) \rangle. \tag{63}$$

Expanding $|\Omega_1'\rangle$ in L–S representation for the quantum numbers involved and by considering that $|\Omega_1'\rangle$ contains only one state which has a total orbital angular momentum L of 2 and a total spin S of 3/2. The notation foı the eigenstates of L^2 and L_z is $|(L_{12}L_3)LM\rangle$. Similarly for the spin states by expanding $|\Omega_1'\rangle$ in terms of these eigenstates, we have

$$|\Omega_1'\rangle = \sum_{m,\mu} |(20)2m\rangle \, |(1\tfrac{1}{2})\tfrac{3}{2}\mu\rangle \, \langle 2\tfrac{3}{2} m\mu | \tfrac{1}{2} M\rangle. \tag{64}$$

Using the fact that L^2, L_z, S^2 and S_z commute with (123), and that the spin functions involved form a symmetric representation of S_3, we get by combining equation (64) and equation (63)

$$\langle (20)20 | \langle T(12) | (123) G_0(E) | T(12) \rangle | (20)20 \rangle | \chi_1(3) \rangle. \tag{65}$$

Since equation (65) is independent of L_z, it can be written in the form

$$5^{-1} \sum_{m=-2}^{2} \langle (20)2m | \langle T(12) | (123) G_0(E) | T(12) \rangle | (20)2m \rangle | \chi_1(3) \rangle. \tag{66}$$

Equation (66) can be given in the momentum representation as

$$-(4\pi)^{-2} \int P_2(\hat{P}_{12} \cdot \hat{P}_{23}) \frac{T(P_{12}) T(P_{23})}{P_{12}^2 + \frac{3}{4} P_3^2 - E} \chi(P_1) d\Omega_3 dP_{12}. \tag{67}$$

We have used the addition theorem for spherical harmonics. $\vec{P}_{ij}$ is the relative momentum of particle i and j. We make the transformation of variables

$$\vec{P}_{12} = \vec{k} + (\tfrac{1}{2})\vec{q}, \quad \vec{P}_{23} = -\vec{q} - (\tfrac{1}{2})\vec{k}. \tag{68}$$

In terms of k and q, equation (67) becomes

$$\chi_i(\vec{q}) = \frac{1}{4\pi} \int \frac{T(|\vec{k} + \frac{1}{2}\vec{q}|) T(|\vec{q} + \frac{1}{2}\vec{k}|)}{q^2 + \vec{q} \cdot \vec{k} + k^2 - E} \left[\frac{27}{32} \frac{q^2 k^2 \sin^2 \theta}{|\vec{q} + \frac{1}{2}\vec{k}|^2 \, |\vec{k} + \frac{1}{2}\vec{q}|^2} - 1 \right]$$

$$\times \chi_1(k) 2\pi k^2 dk \sin \theta d\theta; \tag{69}$$

θ is the angle between $\vec{k}$ and $\vec{q}$. The other kernels are worked out in a similar fashion. The final equations have the form

$$\chi_i(q) = F_i(q) \int d\vec{k} \frac{1}{q^2 + \vec{q} \cdot \vec{k} + k^2 - E} \sum_{j=1}^{3} G_{ij}(\vec{q}, \vec{k}) \chi_j(k), \tag{70}$$

where

$$d\vec{k} = 2\pi \sin\theta d\theta k^2 dk,$$

$$F_1(q) = F_2(q) = [2D_{st}(E - \tfrac{3}{4}q^2)]^{-1},$$

$$F_3(q) = [2D_{ss}(E - \tfrac{3}{4}q^2)]^{-1}; \tag{71}$$

G_{ij} $(i, j = 1, 3)$ are the kernels. The kernels satisfy the relation

$$G_{ij}(\vec{q}, \vec{k}) = G_{ji}(\vec{k}, \vec{q}). \tag{72}$$

We need only to represent six of the kernels as

$$G_{11}(\vec{q}, \vec{k}) = C(\vec{k} + \tfrac{1}{2}\vec{q})C(\vec{q} + \tfrac{1}{2}\vec{k}) + 2T(\vec{k} + \tfrac{1}{2}\vec{q})T(\vec{q} + \tfrac{1}{2}\vec{k}) \times \left[\tfrac{27}{32}\frac{q^2 k \sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2 |\vec{k} + \tfrac{1}{2}\vec{q}|^2} - 1\right], \tag{73}$$

$$G_{12}(\vec{q}, \vec{k}) = C(\vec{k} + \tfrac{1}{2}\vec{q})T(\vec{q} + \tfrac{1}{2}\vec{k})\left[1 - \tfrac{3}{2}\frac{q^2 \sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2}\right] + 2T(\vec{k} + \tfrac{1}{2}\vec{q})C(\vec{q} + \tfrac{1}{2}\vec{k})\left[\tfrac{3}{8}\frac{q^2 \sin^2\theta}{|\vec{k} + \tfrac{1}{2}\vec{q}|^2} - 1\right] + \sqrt{2}\,T(\vec{k} + \tfrac{1}{2}\vec{q})T(\vec{q} + \tfrac{1}{2}\vec{k})\left[1 - \tfrac{3}{4}\frac{q^2 \sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2} - \tfrac{3}{16}\frac{q^2 \sin^2\theta}{|\vec{k} + \tfrac{1}{2}\vec{q}|^2} - \tfrac{27}{64}\frac{q^2 k^2 \sin^2\theta}{|\vec{k} + \tfrac{1}{2}\vec{q}|^2\, |\vec{q} + \tfrac{1}{2}\vec{k}|^2}\right], \tag{74}$$

$$G_{13}(\vec{q}, \vec{k}) = 3C(\vec{k} + \tfrac{1}{2}\vec{q})S(\vec{q} + \tfrac{1}{2}\vec{k}), \tag{75}$$

$$G_{22}(\vec{q}, \vec{k}) = C(\vec{k} + \tfrac{1}{2}\vec{q})C(\vec{q} + \tfrac{1}{2}\vec{k})[1 - 3\cos^2\theta] + \sqrt{2}\,C(\vec{k} + \tfrac{1}{2}\vec{q})T(\vec{q} + \tfrac{1}{2}\vec{k})\left[\tfrac{1}{4}(3\cos^2\theta + 1) - \tfrac{3}{4}(q^2 + \tfrac{1}{4}k^2)\frac{\sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2}\right] + \sqrt{2}\,T(\vec{k} + \tfrac{1}{2}\vec{q})C(\vec{q} + \tfrac{1}{2}\vec{k})\left[\tfrac{1}{4}(3\cos^2\theta + 1) - \tfrac{3}{4}(k^2 + \tfrac{1}{4}q^2)\frac{\sin^2\theta}{|\vec{k} + \tfrac{1}{2}\vec{q}|^2}\right] + T(\vec{k} + \tfrac{1}{2}\vec{q})T(\vec{q} + \tfrac{1}{2}\vec{k})\left[\tfrac{3}{8}\sin^2\theta - \tfrac{3}{4}(q^2 - \tfrac{1}{8}k^2)\frac{\sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2} - \tfrac{3}{4}(k^2 - \tfrac{1}{8}q^2)\frac{\sin^2\theta}{|\vec{k} + \tfrac{1}{2}\vec{q}|^2} + \tfrac{27}{128}\frac{q^2 k^2 \sin^2\theta}{|\vec{q} + \tfrac{1}{2}\vec{k}|^2\, |\vec{k} + \tfrac{1}{2}\vec{q}|^2}\right], \tag{76}$$

$$G_{23}(\vec{q},\vec{k}) = 3S(\vec{q}+\tfrac{1}{2}\vec{k})T(\vec{k}+\tfrac{1}{2}\vec{q})\left[1-\tfrac{3}{2}\frac{k^2\sin^2\theta}{|\vec{k}+\tfrac{1}{2}\vec{q}|^2}\right], \tag{77}$$

$$G_{33}(\vec{q},\vec{k}) = S(\vec{q}+\tfrac{1}{2}\vec{k})S(\vec{k}+\tfrac{1}{2}\vec{q}). \tag{78}$$

At low energies the nuclear potentials and the interactions between nucleons in the triplet-triplet and singlet-singlet states are very small and may be neglected in comparison with those in the triplet-singlet and in singlet-triplet states. In our present calculations, we consider only the triplet-singlet and singlet-triplet states. The nucleon-nucleon interaction used in our work is of the form given by expression (13). We use different forms of parameters for the potential function of the Yamaguchi, Gaussian, Tabakin, Mongan and also of the Reid forms. The different parameters for these potentials are determined [27] by fitting the corresponding phase shifts. The effect of the Coulomb force in the case of proton-proton interaction is taken into account by calculating the pure Coulomb T matrix in a way described by Osman [29].

3. *Numerical calculations and results*

In the present work we use two-body interactions with potential functions of the Yamaguchi, Gaussian, Tabakin, Mongan and also of the Reid potentials. Each of these potentials includes both attraction and repulsion and also tensor forces. By fitting the different types of nucleon-nucleon interactions with the corresponding phase shifts, Osman [27] obtained the different values of the potential parameters. In the process of fitting, we use the phase-shifts presented in Refs. [30] [31] and [32] as stated in Ref. [27].

With these different two-body interactions, direct numerical calculations are carried out for the resulting integral equations. The Schmidt-Hilbert [33] theory of integral equations is applied. Since we use short-range nuclear potentials, the elements of the two-body T matrix rapidly decrease by increasing the number of partial wave expansions used. This dependence of the T matrix elements on the number of partial wave expansions off-the--energy-shell is given by Kowalski [34]. Then, in the equations obtained the summation over the number of partial wave expansions could be restricted to finite number of terms and then, the sets of the integral equations become finite. The integral equations are converted into matrix eigenvalue equation. In the present calculations, the eigenvalues are given as a function of the energy Z. The three-body integral equations are solved numerically by using a 45-point Gaussian integration. Then, the three-body bound state energies are those values of the energy Z for which a matrix eigenvalue takes the value one.

We lead in the present work with the ^{3}H and ^{3}He nuclei, which are treated as a three--body problem. Each of the ^{3}H and ^{3}He nuclei are considered as a three-nucleon bound state. With the Faddeev-Lovelace formalism, the binding energies of these nuclei are numerically calculated. The present equations give only the nuclear binding energies of these nuclei. Osman [29] had calculated Coulomb energies which we added to the calculated nuclear energies in the case of proton-proton interaction in ^{3}He nucleus. By this addition we get the actual three-body ground-state energies.

In the present work we calculate the binding energies for the ^{3}H and ^{3}He nuclei in three different cases. In the first case, the two body interactions are purely attractive. In the second case, the two-body interactions contain both attraction and repulsion. And in the third case, the two-body interactions contain both attraction and repulsion and the tensor forces also. All the results obtained for the theoretical calculations of the three-body binding energies for the ^{3}H and ^{3}He nuclei, using the different two-body interactions in the three different cases mentioned, are listed in Table I. The experimental values [35–37]

TABLE I

Calculated binding energies (MeV)

Potential	Nucleon-nucleon force	^{3}H	^{3}He
Yamaguchi	Central (att.)	9.855	9.011
Gaussian	Central (att.)	9.605	8.868
	Central (att. + rep.)	8.754	8.088
	(Central + Tensor) (att. + rep.)	8.208	7.678
Tabakin	Central (att.)	9.888	9.389
	Central (att. + rep.)	8.916	8.405
	(Central + Tensor) (att. + rep.)	8.524	7.991
Mongan	Central (att.)	9.964	9.189
	Central (att. + rep.)	8.936	8.196
	(Central + Tensor) (att. + rep.)	8.451	7.789
Reid	Central (att.)	10.371	9.488
	Central (att. + rep.)	9.207	8.573
	(Central + Tensor) (att. + rep.)	8.501	7.888
Experimental	binding energy	8.480	7.720

are also introduced in the same table. From the Table I, we can see that our theoretically calculated values are in reasonable agreement with the experimental values taking into account both the effect of short-range repulsion and the effect of the tensor forces. Also, by comparing the values calculated for the same nucleus and the same two-body interaction, for the case with and without repulsion forces respectively, we can get the effect of short--range repulsion on the binding energies. It can be easily done by comparing the results obtained for the first case with those obtained for the second one. This comparison gives the short-range repulsion effect. Comparing the values calculated for the same nucleus and the same two-body interaction, for the case not including tensor forces, with those for the case including tensor forces, we get the effect of tensor forces. So, the effect of including tensor forces is obtained by comparing results obtained for the second case with those obtained for the third case. The effects of including the short-range repulsion as well as effects of including tensor forces are presented in Table II.

TABLE II

The effect of including short-range repulsion (Rep.) and the effect of including tensor forces (Tensor) (Percent)

Potential	^3H		^3He	
	Rep.	Tensor	Rep.	Tensor
Yamaguchi	—	—	—	—
Gaussian	9.259	6.430	9.176	5.221
Tabakin	10.335	4.490	11.055	5.057
Mongan	10.863	5.584	11.429	5.102
Reid	11.893	7.986	10.129	8.324

4. Discussion and conclusions

We use the Faddeev-Lovelace formalism in the present three-body calculations. It is known that the three-body Faddeev equations include the two-body T matrix with connected variables of the momentum and the energy. At low energies, we obtain rapid convergence for the series of the separable expansion. But if the absolute value of the energy increases, the momentum also increases, and then more terms of the separable expansion are needed to obtain a given accuracy. In our calculations, the three-body integral equations are solved numerically with a good accuracy since we obtain a good convergence by using the present separable expansion. From Table I, we can see the good agreement between our theoretical values of the binding energies with the experimentally observed values. We introduce the effects of both short-range repulsion and tensor forces in Table II. From the results given in Table II, we can articulate two conclusions. First, introducing short--range repulsion in the two-body interactions affects the three-body binding energies in 9.176% to 11.893% for the nuclei and two-body interactions considered. Second, the inclusion of the tensor forces in the two-body interactions improves the three-body binding energies of the nuclei considered in our work in 4.490% to 8.324%.

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