CONSTRAINED LAGRANGIAN THEORY FOR MONOPOLES*

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The equations of motion of a non-abelian monopole are derived from an action principle using the definition of its charge as a constraint. These equations are the analogues of the Lorentz and Maxwell equations for a point charge in an electromagnetic field. Their derivation makes use exclusively of the intrinsic topology and dispenses with the introduction of an interaction term in the action. The resulting equations bear a formal resemblance to the Wong equations governing the motion of a classical point source in a Yang-Mills field.

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1. *Introduction*

In these lectures we wish to study the principles governing the motion of charged particles in a gauge field. The most well known example is that of a point (electric) charge in an electromagnetic field, which is governed by the Maxwell and Lorentz equations. These equations can be obtained from an action principle in two ways, as follows. The usual procedure is to write an action consisting of three terms:

$$\mathscr{A} = \mathscr{A}_0^{\mathrm{F}} + \mathscr{A}_0^{\mathrm{M}} + \mathscr{A}_{\mathrm{I}}, \tag{1.1}$$

where $\mathscr{A}_0^{\mathrm{F}}$ is the action of the free electromagnetic field, $\mathscr{A}_0^{\mathrm{M}}$ that of the free particle, and $\mathscr{A}_{\mathrm{I}}$ is the interaction term. The correct equations of motion are obtained if the interaction term corresponds to minimal coupling. The alternative is to use what can be termed a topological action principle, which we shall now investigate.

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It is generally accepted that the magnetic monopole is topological in origin, i.e. it exists as a result of some topological property of the field. Conversely, the existence of such a particle imposes topological constraints on the possible field configurations. Hence in an action principle, we expect the variation of both particle and field variables to be no longer free but constrained. In other words, even if we start with the *free* action:

$$\mathscr{A}_0 = \mathscr{A}_0^{\mathrm{F}} + \mathscr{A}_0^{\mathrm{M}} \tag{1.2}$$

this topological constraint relating particle variables to field variables will induce interactions between them. Indeed we find that the resulting Euler-Lagrange equations are the dual transforms of the Maxwell and Lorentz equations for an electric charge, which, by the dual invariance of Maxwell's theory, are exactly the equations governing the motion of a magnetic charge in an electromagnetic field. Now by the same dual invariance, electric charges can be considered as "electric monopoles" and the above topological action principle applies equally to give the expected Maxwell and Lorentz equations. In a sense this derivation is more intrinsic than the normal approach in that the topological constraint is nothing but the definition of the charge, and no extraneous interaction term need to be introduced. Moreover, the induced coupling turns out to be necessarily minimal.

Our aim in these lectures is to generalize these procedures to non-abelian gauge fields. There duality is no longer valid in general. The (electric) sources and (magnetic) monopoles in the theory carry different charges and hence their dynamics are presumably different. The dynamics of the sources are readily given by the usual action principle via the introduction of an interaction term representing minimal coupling. The monopoles in the theory are, however, not so well studied. Their interaction with the gauge field is relatively unknown. However, we know how to define the non-abelian magnetic charge, which is topological in origin. Hence our topological action principle should in principle give all their dynamics in such a way that the monopole-field coupling will come out as a consequence rather than as an initial input.

The idea to derive the dynamics of a monopole through the definition of its charge is inspired by a beautiful paper [1] by Wu and Yang on the classical monopole-charge--field system. However, they are able to make use of the dual invariance in electrodynamics to avoid the difficulty of patching. Here we wish to study non-abelian monopoles, and in non-abelian theory exact duality no longer holds. To implement such a programme we find it necessary to introduce field variables in terms of loops, rather than just points, in spacetime. The requisite analytic machinery for dealing with loop variables is described in the lecture by CHM (in the same volume) [2]. Using this, we are able to derive from the topological action principle a set of equations which are analogous to the Maxwell and Lorentz equations of classical electrodynamics. We find a somewhat unexpected *formal* resemblance of these equations to the Wong equations for a classical point source of a Yang-Mills field [3]. This suggests the emergence of a conserved "isospin" vector relating to the monopole and tempts us to speculate on the possibility that particles such as quarks, which are usually interpreted as sources, might in fact be monopoles in non-abelian theories.

2. *The magnetic monopole: a dual topological treatment*

We wish to spend some time studying the familiar abelian case, from several different angles, so that we learn exactly what to expect, and more importantly, what *not* to expect, when we come to make the generalization to the non-abelian case.

2.1. The usual action principle

Let us start with something very simple. Consider the interaction of an electric charge with the electromagnetic field. From the usual action

$$\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_\mathrm{I}, \tag{2.1}$$

where the free action

$$\mathscr{A}_0 = -\frac{1}{16\pi}\int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x) - m\int d\tau, \tag{2.2}$$

and the interaction term

$$\mathscr{A}_\mathrm{I} = -e\int a_\mu(Y(\tau))\frac{dY^\mu(\tau)}{d\tau}\,d\tau, \tag{2.3}$$

one obtains via the free variations with respect to $a_\mu(x)$ and $Y^\mu(\tau)$, the usual Maxwell and Lorentz equations:

$$\partial_\nu f^{\mu\nu}(x) = -4\pi e\int d\tau\,\frac{dY^\mu(\tau)}{d\tau}\,\delta^4(x-Y(\tau)) \qquad \text{(Me1) (2.4)}$$

$$m\frac{d^2Y^\mu(\tau)}{d\tau^2} = -ef^{\mu\nu}(Y(\tau))\frac{dY_\nu(\tau)}{d\tau}. \qquad \text{(Le) (2.5)}$$

Here the normal symbols are used:

m = mass of particle

e = electric charge

$Y^\mu(\tau)$ = worldline of particle

$a_\mu(x)$ = Maxwell potential

$$f_{\mu\nu}(x) = \partial_\nu a_\mu(x) - \partial_\mu a_\nu(x) = \text{Maxwell field.} \tag{2.6}$$

To (Me1) should be added the condition for the existence of the potential, which is equivalent to

$$\partial_\nu {}^*f^{\mu\nu}(x) = 0, \qquad \text{(Me2) (2.7)}$$

where the star denotes the dual (see the next sub-section).

All the above is invariant under the interchange of "electric" and "magnetic" (i.e. the duality operation), so that we readily obtain the corresponding equations of motion for a magnetic charge in an electromagnetic field:

$$\partial_\nu {}^*f^{\mu\nu}(x) = -4\pi g \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta^4(x - Y(\tau)), \qquad \text{(Mg1) (2.8)}$$

$$\partial_\nu f^{\mu\nu}(x) = 0, \qquad \text{(Mg2) (2.9)}$$

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = -g {}^*f^{\mu\nu}(Y(\tau)) \frac{dY_\nu(\tau)}{d\tau}, \qquad \text{(Lg) (2.10)}$$

where M and g replace m and e.

However, magnetic monopoles are known to be topological in origin (see sub-section 2.3), so that the constrained variation procedure as outlined in Section 1 applies. We shall see that this procedure gives exactly the same equations (Mg) and (Lg). Notice also that because of electric-magnetic duality, the constrained variation applies equally well to electric charges, which can be considered as electric monopoles of the dual field ${}^*f_{\mu\nu}(x)$.

2.2. Duality

Duality plays an important role in abelian theory, and the lack of duality plays perhaps an even more important role in non-abelian theory. Let us now make a few general remarks about duality.

For any rank two tensor (in dimension four) one can define its dual, which is again a rank two tensor. Thus

$${}^*f_{\mu\nu}(x) = \tfrac{1}{2}\, \varepsilon_{\mu\nu\varrho\sigma} f^{\varrho\sigma}(x), \qquad (2.11)$$

where $\varepsilon_{\mu\nu\varrho\sigma}$ is the totally skew symmetric tensor with $\varepsilon_{0123} = -1$, is the dual of the Maxwell field $f_{\mu\nu}(x)$. This definition is quite general and applies to the non-abelian case as well.

A rank two tensor $F_{\mu\nu}(x)$ is defined to be a *gauge field* [5] if there exists a vector $A_\mu(x)$, called the potential, such that $F_{\mu\nu}(x)$ is its covariant curl, i.e.

$$F_{\mu\nu}(x) = \partial_\nu A_\mu(x) - \partial_\mu A_\nu(x) + ie[A_\mu(x), A_\nu(x)], \qquad (2.12)$$

where the last term is the commutator. In abelian theory (denoted by lower case $f_{\mu\nu}(x)$ and $a_\mu(x)$), the so-called Poincaré lemma is valid, which says that the existence of a potential $a_\mu(x)$ locally is equivalent to

$$\partial_\nu {}^*f^{\mu\nu}(x) = 0. \qquad (2.7)$$

This explains why, mathematically, that in Maxwell's theory, where $f_{\mu\nu}(x)$ is a gauge field everywhere, no point magnetic charges exist, and conversely, why if a magnetic monopole exists, that $f_{\mu\nu}(x)$ cannot be a gauge field everywhere [3].

Since ${}^{**}f = f$ and $f_{\mu\nu} f^{\mu\nu} = {}^*f_{\mu\nu} {}^*f^{\mu\nu}$, the action $\mathscr{A}$ is invariant under duality provided we also make the interchange $(m, e) \leftrightarrow (M, g)$. Hence one obtains equations (Mg) and (Lg) from (Me) and (Le) as stated in the last subsection. Notice also that the dual of (2.7) is $\partial_\nu f^{\mu\nu}(x) = 0$, which is the source-free condition. By the Poincaré lemma again, this means that $^*f_{\mu\nu}(x)$ is then a gauge field, i.e. there exists a potential $\bar{a}_\mu(x)$ for which $^*f_{\mu\nu}(x)$ is the curl.

The Poincaré lemma enables one to go freely, when the appropriate conditions are satisfied, from electric to magnetic and vice versa. This is an extremely useful device in many cases. The Poincaré lemma, however, does *not* apply in the non-abelian case. This means that although one can form the dual $^*F_{\mu\nu}(x)$, this dual, even in the source-free case, is not guaranteed to be a gauge field. Moreover, even if we can write down the equations of motion of non-abelian (electric) charges, we have no way to obtain from them directly the equations of motion for non-abelian (magnetic) monopoles, as we could in the abelian case (Section 2.1).

2.3. Topology of the magnetic monopole

In order to avoid confusion, we shall refer to the Dirac magnetic monopole as the magnetic monopole, and to any non-abelian generalizations with gauge group G as a G-monopole, or simply as a monopole.

Suppose we have a magnetic monopole. Then by the Poincaré lemma we know that we cannot find a well-defined singularity-free potential $a_\mu(x)$ in any region surrounding the magnetic monopole. This can also be seen in an elementary way using Stokes' theorem, because otherwise the magnetic flux would vanish. Hence if we insist on describing the physics in terms of a potential $a_\mu(x)$, we have either to introduce a string of singularities [6], or to use patching [7]. We shall use the latter.

At each time t consider a sphere surrounding the monopole, and divide it into the northern and southern hemispheres overlapping along the equator:

$$\mathcal{N}: 0 \leqslant \theta < \frac{\pi}{2} + \delta, \quad 0 \leqslant \varphi < 2\pi$$

$$\mathcal{S}: \frac{\pi}{2} - \delta < \theta \leqslant \pi, \quad 0 \leqslant \varphi < 2\pi, \tag{2.13}$$

for all $t, r > 0$, and $0 < \delta \leqslant \frac{\pi}{2}$. Then patching of $a_\mu(x)$ means that we have two functions $a_\mu^{\mathcal{N}}(x)$ and $a_\mu^{\mathcal{S}}(x)$, defined respectively in $\mathcal{N}$ and in $\mathcal{S}$, satisfying along the overlap the "patching condition",

$$a_\mu^{\mathcal{S}}(x) = a_\mu^{\mathcal{N}}(x) - \frac{i}{e} S(x)\partial_\mu S^{-1}(x), \tag{2.14}$$

where $S(x)$ is the "patching function". There are two things we want. Firstly, we want the total flux out of the sphere to be $4\pi g$, i.e.

$$\iint_{\mathcal{N}} f_{\mu\nu} + \iint_{\mathcal{S}} f_{\mu\nu} = 4\pi g, \tag{2.15}$$

which by Stokes' theorem is equivalent to:

$$\oint (a_\mu^{\mathcal{N}}(x) - a_\mu^{\mathcal{S}}(x))dx^\mu = 4\pi g. \qquad \text{(T) (2.16)}$$

Secondly, the function $S(x)$, which maps the equator into the electromagnetic group U(1), must be single-valued. $S(x)$ is in fact a map from a circle to a circle, so that for it to be well-defined it must have an integral winding number, i.e. as x goes once round the equator $S(x)$ must come back to its original value. This is best illustrated by an example. Consider the field due to a static monopole. In polar coordinates, we have

$$\begin{aligned} a_t^{\mathcal{N}} &= a_r^{\mathcal{N}} = a_\theta^{\mathcal{N}} = 0 \\ a_\varphi^{\mathcal{N}} &= \frac{g}{r \sin \theta}(1-\cos\theta) \\ a_t^{\mathcal{S}} &= a_r^{\mathcal{S}} = a_\theta^{\mathcal{S}} = 0 \\ a_\varphi^{\mathcal{S}} &= -\frac{g}{r\sin\theta}(1+\cos\theta). \end{aligned} \qquad (2.17)$$

Solving (2.14), we have in the overlap

$$S = e^{2ige\varphi}, \qquad (2.18)$$

from which we deduce that S is single-valued if and only if

$$2eg = \text{integer } n. \qquad (2.19)$$

This is precisely Dirac's quantization condition. The integer n here is the winding number of $S(x)$ mentioned above. If g_0 is the smallest magnetic charge, then

$$2eg_0 = 1, \qquad (2.20)$$

and the integer n can also be interpreted as the magnetic charge of the pole in units of g_0, in which case the RHS of (2.15) and (2.16) can be replaced by $4\pi n g_0$.[1]

The equation (2.16) is a condition imposed on the field configuration by the presence of a magnetic monopole, and is therefore the topological constraint (T) we are looking for. Wu and Yang [1] have shown that (T) is in fact equivalent to (Mg1). Hence in a constrained variation, one can use either (T) or (Mg1) as the topological constraint.

2.4. Constrained variation

As outlined in Section 1, our topological action principle now requires us to vary the free action

$$\mathcal{A}_0 = -\frac{1}{16\pi}\int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x) - M\int d\tau \qquad (2.21)$$

[1] A third meaning of n will be given in Section 3. For the sake of completeness we mention that n is also the Chern class of a certain fibre bundle describing the magnetic monopole, but we shall not go into this any further.

under the topological constraint (T) or (Mg1). From what we did in Section 2.1, the variation is apparently to be done against variations of the variables $a_\mu(x)$ and $Y^\mu(\tau)$. But here $a_\mu(x)$ is a patched quantity, and the patching condition (2.14) depends on $Y^\mu(\tau)$ in a complicated manner, as the division into the two hemispheres depends on where the monopole is. This looks rather hopeless at first sight.

This difficulty can actually be by-passed as follows. Let us recall that in free Maxwell theory one can equally obtain the equations of motion from the free action $\int f_{\mu\nu}(x)f^{\mu\nu}(x)d^4x$ by the variation of $f_{\mu\nu}(x)$ under the constraint $\partial_\nu {}^*f^{\mu\nu}(x) = 0$ instead of the free variation of $a_\mu(x)$. Furthermore this constraint is equivalent to the statement that $f_{\mu\nu}(x)$ is a gauge field (see Section 2.2).

Now (Mg1) implies $\partial_\nu {}^*f^{\mu\nu}(x) = 0$ outside the worldline $Y^\mu(\tau)$, and hence by the above argument the variation could actually be done against the variables $f_{\mu\nu}(x)$ rather than $a_\mu(x)$. The variation against $f_{\mu\nu}(x)$ is straightforward because it is not patched (in fact gauge invariant) and hence independent of the other variables $Y^\mu(\tau)$.

Using the method of Lagrange multipliers, we form now the auxiliary action

$$\mathscr{A}'_0 = \mathscr{A}_0 - \int d^4x \lambda_\mu(x) \left\{ \partial_\nu {}^*f^{\mu\nu}(x) + 4\pi g \int d\tau \frac{dY^\mu(\tau)}{d\tau} \delta^4(x - Y(\tau)) \right\}. \tag{2.22}$$

Variation against $f_{\mu\nu}(x)$ gives:

$$f^{\mu\nu}(x) = -4\pi\{\tfrac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}(\partial_\sigma\lambda_\varrho(x) - \partial_\varrho\lambda_\sigma(x))\}, \tag{2.23}$$

and variation against $Y^\mu(\tau)$ gives:

$$\mathrm{M}\frac{d^2Y_\mu(\tau)}{d\tau^2} = 4\pi g\{\partial_\nu\lambda_\mu(Y(\tau)) - \partial_\mu\lambda_\nu(Y(\tau))\}\frac{dY^\nu(\tau)}{d\tau}. \tag{2.24}$$

Equation (2.23) implies that:

$${}^*f_{\mu\nu}(x) = \frac{1}{2\pi}(\partial_\nu\lambda_\mu(x) - \partial_\mu\lambda_\nu(x)), \tag{2.25}$$

which is equivalent to saying that ${}^*f_{\mu\nu}(x)$ is gauge field *everywhere*, with the Lagrange multipliers fulfilling the role of the gauge potential. Equation (2.25) enables us to eliminate $\lambda_\mu(x)$ from our Euler-Lagrange equations (2.23) and (2.24), to obtain (Mg2) and (Lg):

$$\partial_\nu f^{\mu\nu}(x) = 0 \tag{2.9}$$

$$M\frac{d^2Y^\mu(\tau)}{d\tau^2} = -g{}^*f^{\mu\nu}(Y(\tau))\frac{dY_\nu(\tau)}{d\tau}. \tag{2.10}$$

These together with the constraint (Mg1) are exactly the same equations of motion as obtained in Section 2.1, and describe completely the interaction of a point magnetic monopole with its electromagnetic field. We have thus shown that in the case of a magnetic monopole, the constrained variation principle we propose is exactly equivalent to the usual action principle with minimal coupling, and we have truly obtained the dynamics of the system from purely topological considerations.

Now because of electric-magnetic duality, we can treat electrons as electric monopoles of the dual field $^*f_{\mu\nu}(x)$, which is now given in terms of a potential $\tilde{a}_\mu(x)$

$$^*f_{\mu\nu}(x) = \partial_\nu \tilde{a}_\mu(x) - \partial_\mu \tilde{a}_\nu(x), \tag{2.11}$$

except at the (electric) monopole position, where $\tilde{a}_\mu(x)$ is not defined. The above procedure goes through exactly, and the constrained variation will give us back (Me2) and (Le) together with (Mel) as the constraint.

Before we go on to our main objective — non-abelian theory — let us emphasize again the intrinsic character of this method:

(*i*) no interaction $\mathscr{A}_I$ is introduced,

(*ii*) the constraint *is* the definition of the charge, and

(*iii*) the induced coupling is automatically and necessarily minimal.

3. The non-abelian monopole charge

3.1. Definition

The magnetic charge in electromagnetic theory is both quantized and conserved. Indeed these two properties are usually considered to be the defining properties of the physical attribute "charge". If we want to generalize the magnetic charge to a non-abelian gauge theory, these two are then the properties we must endeavour to ensure. It turns out that the appropriate generalization has, besides these two properties, rather unfamiliar characteristics, which we shall now show.

The two equivalent definitions of (abelian) magnetic charge discussed in Section 2.3 cannot be used directly for the non-abelian case. We know of no straightforward and usable non-abelian analogue of Stokes' theorem. Equations (2.15) and (2.16) are no longer equivalent. Also when the gauge group G is not U(1), the patching function $S(x)$ will no longer map a circle to a circle, and the concept of winding number is less direct. However, there is yet a third (equivalent) definition for the magnetic charge which *can* be directly generalized, and after we have understood this new non-abelian definition, we shall also have realized how the two old definitions could have been generalized, though less straightforwardly. Consider then a closed loop[2] C parametrized by the function $\xi^\mu(s)$, $s = 0 \to 2\pi$. For technical reasons, we shall always consider based parametrized loops, i.e. functions from the closed interval $[0, 2\pi]$ to spacetime, passing through a fixed point P_0. The phase factor or Wilson loop corresponding to C is defined by:

$$\phi(C) = \exp ie \int_0^{2\pi} ds a_\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds} \, . \tag{3.1}$$

This is an element of U(1). Consider now a family of loops C_t which sweeps out a closed surface (Fig. 1). As t varies from 0 to 2π, $\phi(C_t)$ traces out a closed curve Γ in U(1), starting

[2] For a more detailed description of loops, please refer to the lecture [2] given by the first author.

and ending in the identity (Fig. 2). Γ is called the *total circuit*, and it has a winding number n. We see that [7]

$$2\pi n = \text{total change in phase of } \phi(C_t) \text{ as } t = 0 \to 2\pi. \tag{3.2}$$

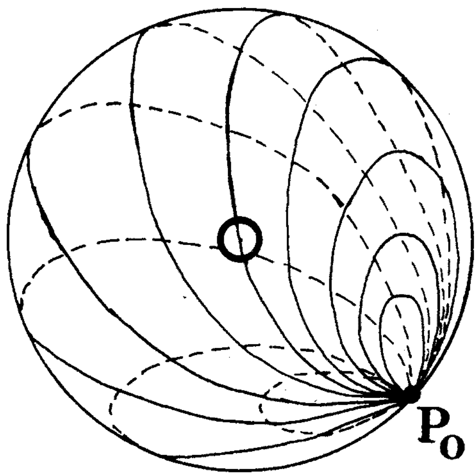

Fig. 1. A closed surface swept out by a family of loops

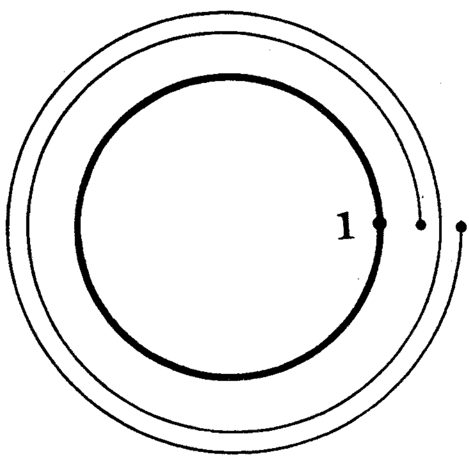

Fig. 2. A curve of winding number 2 in the group U(1)

Now by Stokes' theorem again, for any curve

$$e \oint_{C_t} a_\mu(x) dx^\mu = e \iint_{\text{surface}} f_{\mu\nu}(x) d\sigma^{\mu\nu}, \tag{3.3}$$

from which we deduce that:

$$\text{total change in phase} = (4\pi)^{-1} eg. \tag{3.4}$$

Hence

$$\text{winding number } n = 2eg, \tag{3.5}$$

as before.

Quite generally in any manifold one can consider classes of closed curves Γ equivalent under continuous deformation, called (the first) homotopy classes. In particular we can consider classes of curves in any group G. These classes are obviously invariant under continuous changes, i.e. are topological invariants. Furthermore, they are always discrete.

Now for a non-abelian theory, if we replace (3.1) by

$$\Phi(C) = P_s \exp ie \int_0^{2\pi} ds A_\mu(\xi(s)) \frac{d\xi^\mu(s)}{ds}, \tag{3.6}$$

where P_s denotes path-ordering with respect to the loop parameter s, then for a family C_t as before, the total circuit is again a closed curve in the group manifold G. We now define [8, 7, 9] the *non-abelian monopole charge* as the homotopy class of the total circuit Γ. Although this may come as a surprise, on closer examination we see that the homotopy class is a topological invariant (hence is conserved), and is discrete (hence is quantized). Therefore it satisfies both criteria for the definition of a charge we set out at the beginning. Moreover, it reduces directly to the magnetic charge in the abelian case.[3]

Although it may not be apparent, these abstractly defined homotopy classes form an abelian group. Therefore, these classes can be added and have inverses. Thus it makes sense to consider systems of more than one monopole and also anti-monopoles, properties that any non-abelian generalization should ensure. Here the homotopy class containing the zero curve (i.e. just the identity in G) corresponds to the vacuum (i.e. no monopole).

The monopole charge defined above is rather abstract, and not easy to use. We shall give a more usable formula in Section 5, which will also serve as the topological constraint in our action principle, just as in the abelian case. Before doing so, we shall familiarize ourselves with the concept by looking at some specific examples. We shall consider only those cases that are of current interest in gauge theory.

3.2. Examples

1) The group SU(2) is topologically the same as the three-sphere S^3. All closed curves can be slipped off the sphere. Hence there is no monopole charge corresponding to SU(2). This is true of all the SU(N) groups, as well as the exceptional group E_8.

2) SO(3) is obtained from SU(2) by "forgetting the sign", or topologically, by identifying pairs of antipodal points on S^3. There are two classes of curves, those that can be slipped off and those that cannot (Fig. 3). We can, if we like, denote these two classes by a sign, just as in parity. The minus-charge is the simplest example of a non-abelian monopole charge. Two generalizations then obtain.

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[3] The non-abelian monopole charge we consider are primitive concepts in the gauge theory and have intrinsically non-abelian properties. It is therefore distinct from the soliton-monopoles considered by 't Hooft and Polyakov.

a) All SO(N) charges are similar, either plus or minus.
b) Just as SO(3) $\simeq$ SU(2)/Z_2, we can consider all SU(N)/Z_N, which has charges that behave like integers modulo N. In particular, SU(3)/Z_3 has charges that compose like triality.

We notice in the first two examples, that although SU(2) and SO(3) have the same Lie algebra, they have very different monopole charges. This shows that the global properties of the group are of paramount importance here.

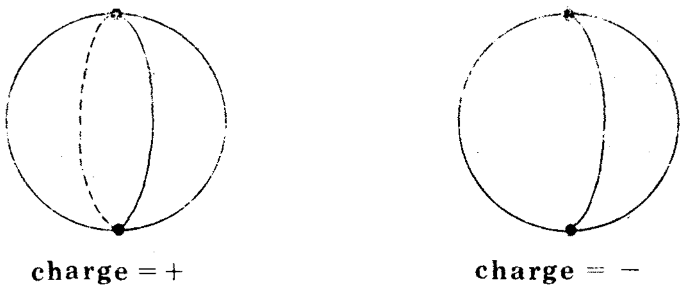


Fig. 3. Two classes of curves in SO(3) representing + and — charges

3) The electroweak group U(2) is locally isomorphic to the direct product SU(2) × U(1), and can be obtained from the latter by pairwise antipodal identification, as illustrated in Fig. 4, drawn with the usual periodic boundaries. The antipodal identification means that the part of the diagonal in the unshaded half is already a closed curve. The monopole

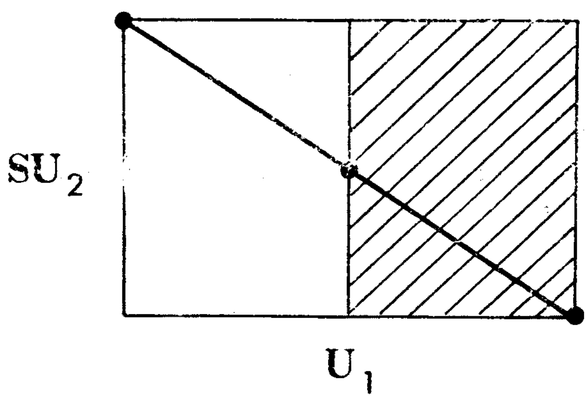


Fig. 4. Parametrization of U(2)

charge of such a theory can be described by two (related) classes, from the U(1) and SU(2) parts respectively [10]. The case is similar for a theory combining electromagnetism with chromodynamics, which has group U(3) (see Section 7).

4. Loop space variables

Our aim is to obtain equations of motion for a non-abelian monopole using a topological action principle. The difficulty attendant upon patching the gauge potential $A_\mu(x)$, mentioned at the beginning of Section 2.4 for the abelian theory, hits us in full force here. As in abelian theory, the solution lies in choosing a new, but equivalent, set of variables. We find that we need to go to the space of loops and formulate the problem there, as we shall now explain.

First we shall introduce the notations and list the main results [11]. The details are given in the lecture [2] by the first author, where references to other related work are also listed.

A loop always means a based parametrized loop, i.e. a map $\xi(x)$ from the interval $[0, 2\pi]$ to spacetime X passing through a fixed point P_0 such that $\xi(0) = \xi(2\pi) = P_0$. Let $\Omega^1 X$ denote the space of loops in X. One can also consider the space of loops in $\Omega^1 X$, i.e. $\Omega^2 X$, points of which represent closed surfaces in X.

Consider a gauge theory with gauge group G. Let $A_\mu(x)$ be the gauge potential. Then the phase factor $\Phi(C)$ is a map from $\Omega^1 X$ to G. Let x be a point on the loop C corresponding to the parameter s. One can then define the loop derivative $\dfrac{\delta}{\delta\xi^\mu(s)}$ of any quantity defined in $\Omega^1 X$ (e.g. $\Phi(C)$) by considering a delta-function type variation of $\xi^\mu(s)$ at the point s. The logarithmic derivative of $\Phi(C)$:

$$F_\mu(C|s) = \Phi^{-1}(C)\frac{\delta}{\delta\xi^\mu(s)}\Phi(C) \tag{4.1}$$

is Lie algebra-valued, defines a parallel displacement in $\Omega^1 X$, and is hence very similar to the gauge potential in X. One can also form the corresponding "field"

$$G_{\mu\nu}(C; s, s') = \frac{\delta}{\delta\xi^\nu(s')}F_\mu(C|s) - \frac{\delta}{\delta\xi^\mu(s)}F_\nu(C|s') + ie[F_\mu(C|s), F_\nu(C|s')]. \tag{4.2}$$

One can check immediately that $G_{\mu\nu}(C; s, s')$ vanishes identically except at the monopole position $\xi^\mu(s) = \xi^\mu(s') = Y^\mu(\tau)$ for some τ. Similarly, we can define the "phase factor" corresponding to $F_\mu(C|s)$:

$$\Theta(\Sigma) = P_t \exp ie \int_0^{2\pi}\int_0^{2\pi} ds\,dt\,F_\mu(C_t|s)\frac{\partial\xi_t^\mu(s)}{\partial t}, \tag{4.3}$$

TABLE I

Spacetime variables and loop variables

	Spacetime X	Loop space $\Omega^1 X$
Potential	$A_\mu(x)$	$F_\mu(C\|s)$
Field	$F_{\mu\nu}(x)$	$G_{\mu\nu}(C; s, s')$
Phase factor	$\Phi(C)$	$\Theta(\Sigma)$

where $\{C_t\}$ is a family of curves parametrizing the surface Σ (a point in $\Omega^2 X$). Table I gives a summary.

As it may not be apparent from the above summary, loop space calculus is both complicated and delicate. One main reason is that loop space is infinite dimensional. The other is that for technical reasons we work with *parametrized* loops rather than actual geometric loops. The motivations for going into all these loop space gymnastics must be very strong. Indeed they are — or so we believe. Our task is twofold. We want to write down a topological constraint, and we want to find suitable variables in which to do so. We shall deal with the first question in the next section.

In abelian theory we knew already that $a_\mu(x)$ is a patched quantity in the presence of a monopole and we showed how by a judicious choice of variables ($f_{\mu\nu}(x)$ instead of $a_\mu(x)$) we were able to do the variation without undue difficulty. In non-abelian theory $F_{\mu\nu}(x)$ is not suitable for two reasons. Firstly, it is also patched. $A_\mu(x)$ and $F_{\mu\nu}(x)$ transform as follows:

$$A_\mu^{\mathscr{S}}(x) = S(x)A_\mu^{\mathscr{N}}(x)S^{-1}(x) - \frac{i}{e}S(x)\partial_\mu S^{-1}(x) \tag{4.4}$$

$$F_{\mu\nu}^{\mathscr{S}}(x) = S(x)F_{\mu\nu}^{\mathscr{N}}(x)S^{-1}(x). \tag{4.5}$$

Secondly, even in classical theory $F_{\mu\nu}(x)$ does not describe the theory completely, in the sense that two gauge inequivalent potentials may give rise to the same field.

From the definition of the non-abelian monopole charge one is naturally led to loop space variables. As $G_{\mu\nu}(C; s, s')$ vanishes identically except at the monopole position, the choice narrows down to $\Phi(C)$ and $F_\mu(C|s)$. Both are patch-independent, which is important from our point of view. Both are vastly degenerate and have to be severely constrained. Our choice of $F_\mu(C|s)$ is justified by the following theorem.

Theorem. The variables $F_\mu(C|s)$ satisfy

$$\text{(T):}\quad \Theta(\Sigma) = \text{monopole charge enclosed by } \Sigma \tag{4.6}$$

$$\text{(S):}\quad F_\mu(C|s)\frac{d\xi^\mu}{ds} = 0 \tag{4.7}$$

$$\text{(O):}\quad F_\mu(C_1|s) = F_\mu(C_2|s) \text{ if } \xi_1^\nu(s') = \xi_2^\nu(s'),\ s' \leqslant s, \tag{4.8}$$

if and only if there exists, except on the monopole worldline, a gauge potential $A_\mu(x)$ (unique up to gauge equivalence) such that $F_\mu(C|s)$ is defined by $A_\mu(x)$ through (4.1) and (3.6).

The theorem can be considered as the non-abelian analogue of the Poincaré lemma (Section 2.2) generalized to include monopoles, and establishes the entire equivalence between $F_\mu(C|s)$ under conditions (T), (S), (O) and the gauge potential $A_\mu(x)$ up to gauge transformations.

5. *The topological constraint*

It was promised at the end of Section 3.1 that a more computable form of the monopole charge be given. We now show how this can be obtained in terms of the variables $F_\mu(C|s)$ introduced above. Consider again the abelian case. The total circuit (Fig. 2) can be explicitly exhibited by drawing it out in the covering space of the circle which is the real line (Fig. 5a). The magnetic charge is nothing but the length of the total circuit measured out in R. Now SO(3) is the simplest case with a nontrivial non-abelian monopole charge, hence let us first concentrate on this case. So to "measure" the SO(3) monopole charge we consider the total circuit lifted to the covering group SU(2) (Fig. 5b), which

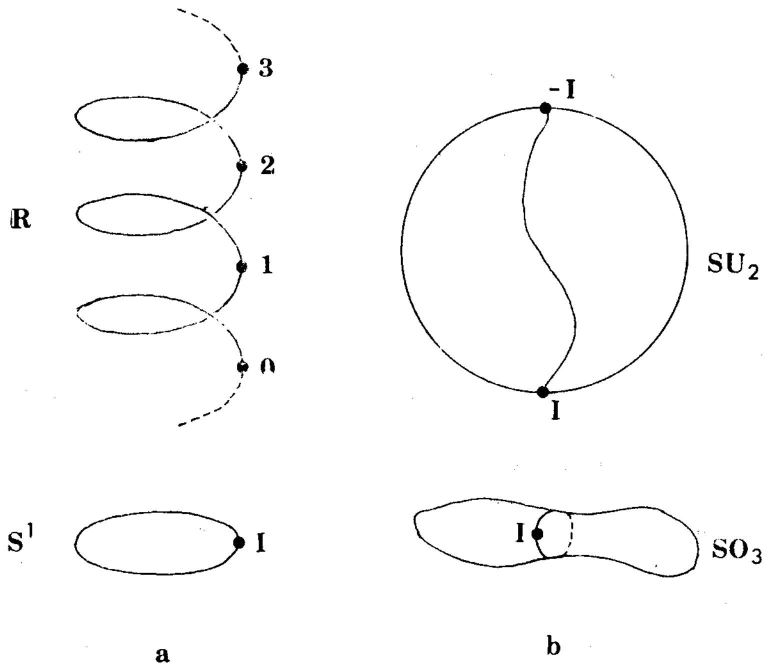


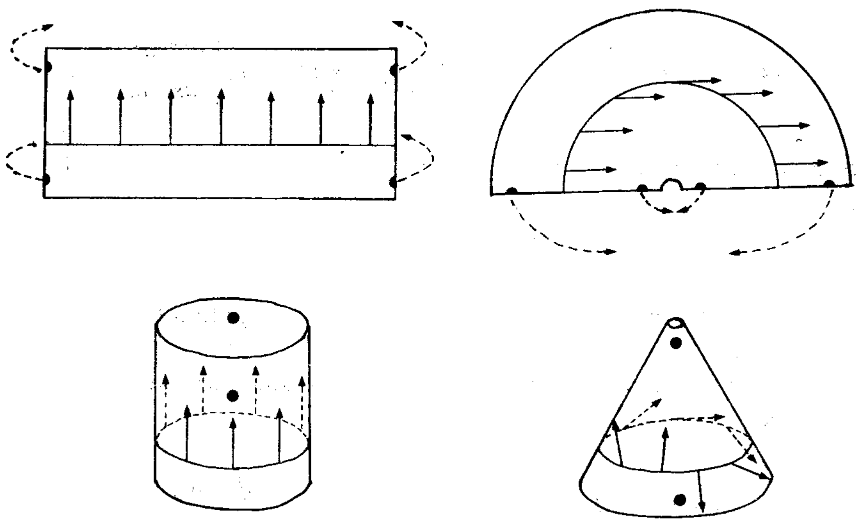
Fig. 5. Covering groups of (a) U(1) and (b) SO(3)

will be a path starting at I and ending at $-I$ for a monopole charge —. But the end-point is the same as the "phase factor" $\Theta(\Sigma)$ for the "potential" $F_\mu(C|s)$ in loop space. Hence we can now write the topological constraint (T) or (Mgl) as follows, where all group elements now belong to SU(2);

$$\Theta(\Sigma) = P_t \exp ie \int_0^{2\pi} dt \int_0^{2\pi} ds F_\mu(C_t|s) \frac{\partial \xi_t^\mu(s)}{\partial t}$$

$$= \begin{cases} I \text{ if } \Sigma \text{ does not enclose a monopole} \\ -I \text{ if } \Sigma \text{ encloses a monopole.} \end{cases} \tag{5.1}$$

This constraint holds in the infinite-dimensional loop space $\Omega^1 X$, which is difficult to visualize. Surprisingly enough, however, the two cases in (5.1) can be quite adequately illustrated in a simple two-dimensional model. Recall that $A_\mu(x)$ gives the change in phase from a point x to $x+dx^\mu$, and $F_\mu(C|s)$ gives the change in phase from the loop C to $C+\delta C$. Our example deals with the change in direction. It can in fact be thought of as an extremely simple model of two-dimensional general relativity. Fig. 6a shows a flat strip of paper made into a cylinder. The cylinder is flat, the curvature (corresponding to our field $G_{\mu\nu}(C; s, s')$) is zero, and the holonomy (corresponding to our phase factor $\Theta(\Sigma)$) of the curve shown is 1. This illustrates the case of no monopole. In contrast, Fig. 6b shows a flat semicircle with



a b

Fig. 6. (a) Globally flat space, curvature = 0, holonomy = 1; (b) Locally flat space, curvature = 0, holonomy = −1

its centre removed folded into a right cone without vertex. The cone is locally flat, its curvature is zero, but the holonomy of the curve shown is −1 (the arrows are rotated by π). This curve encircles the vertex which represents the monopole. Had we taken a curve on the cone which does not encircle the vertex, we would have found its holonomy to be 1, just as in the case of (5.1).

The case for other groups is similar. One can always write down the holonomy $\Theta(\Sigma)$ in terms of elements of the universal covering group of the given group. Some further examples are discussed in Section 7.

6. *The action principle*[4]

This section contains our main results, the derivation of which is neither short nor easy. We refer the reader to [4] for details. Here we shall only outline the steps, and for clarity work exclusively with SO(3)-monopoles.

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[4] Some of the results of this section were obtained after the lectures were given, but are included here for completeness.

First we have to re-write the free field action

$$\mathscr{A}_0^{\mathrm{F}} = -\frac{1}{16\pi}\int d^4x \,\mathrm{Tr}\,(F_{\mu\nu}(x)F^{\mu\nu}(x)) \tag{6.1}$$

in terms of our chosen loop variables $F_\mu(C|s)$. By direct computation, this is found to be

$$\mathscr{A}_0^{\mathrm{F}} = -(4\pi\bar{N})^{-1}\int \delta C \int_0^{2\pi} ds \,\mathrm{Tr}\,[F_\mu(C|s)F^\mu(C|s)]\left[\frac{d\xi^\mu}{ds}\frac{d\xi_\mu}{ds}\right]^{-1}, \tag{6.2}$$

where $\bar{N}$ is a normalization factor given by

$$\bar{N} = \int_0^{2\pi} ds \int \prod_{s'\neq s} d^4\xi(s'), \tag{6.3}$$

and where the integration over all loops is a functional integration because the loops have been purposely defined as functions.

The topological action principle for non-abelian monopoles can now be formulated: the free action

$$\mathscr{A}_0 = -(4\pi\bar{N})^{-1}\int \delta C \int_0^{2\pi} ds \,\mathrm{Tr}\,[(F_\mu(C|s)F^\mu(C|s)]\left(\frac{d\xi^\mu}{ds}\frac{d\xi_\mu}{ds}\right)^{-1} - M\int d\tau \tag{6.4}$$

is to be varied with respect to its variables $F_\mu(C|s)$ and $Y^\mu(\tau)$ under the constraints:

$$(\mathrm{T}): \quad \Theta(\Sigma) = \begin{cases} I \text{ if } \Sigma \text{ does not enclose } M \\ -I \text{ if } \Sigma \text{ encloses } M \end{cases} \tag{6.5}$$

$$(\mathrm{S}): \quad F_\mu(C|s)\frac{d\xi^\mu(s)}{ds} = 0 \tag{6.6}$$

$$(\mathrm{O}): \quad F_\mu(C_1|s) = F_\mu(C_2|s) \text{ if } \xi_1^\nu(s') = \xi_2^\nu(s') \text{ for } s' \lessgtr s. \tag{6.7}$$

This should be carefully compared with the action principle we considered before. Table II gives a summary. Notice that the constraint (T) is topological in all four cases. The relation between the two (T) constraints of the non-abelian theory exactly parallel the relation between those of the abelian theory. The constraint (S) in loop space formulation corresponds to the skew-symmetry of $F_{\mu\nu}$ in spacetime formulation. The constraint (O) for the non-abelian theories obtains as a result of path-ordering and hence has no counterpart in abelian theory (cf. theorem of Section 4).

We shall now concentrate on action principle IV. The constrained variation of the action with respect to the field variables $F_\mu(C|s)$ leading to the Yang-Mills equations is already done in our other lecture [2]. Here we shall concern ourselves with the variation of the monopole worldline.

TABLE II

Summary of the action principles

	Abelian theory	Non-abelian theory
Pure gauge theory	(I) $\mathscr{A}_0^F = -\frac{1}{16\pi}\int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x)$ Constraints (T) $\partial_\nu^* f^{\mu\nu}(x) = 0$ (S) $f^{\mu\nu}$ skew-symmetric	(III) $\mathscr{A}_0^F = -(4\pi\bar{N})^{-1}\int \delta C \int_0^{2\pi} ds$ $\mathrm{Tr}\,[F_\mu(C\|s)F^\mu(C\|s)]\left[\frac{d\xi^{\mu}}{ds}' \frac{d\xi_\mu}{ds}\right]^{-1}$ Constraints (T) $\Theta(\Sigma) = 1$ (S) $F_\mu(C\|s)\frac{d\xi^\mu(s)}{ds} = 0$ (O) $F_\mu(C_1\|s) = F_\mu(C_2\|s)$ if $\xi_1^\mu(s') = \xi_2^\nu(s')$ for $s' \leqslant s$.
Monopole theory	(II) $\mathscr{A}_0 = \mathscr{A}_0^F - M\int d\tau$ Constraints (T) $\partial_\nu^* f^{\mu\nu}(x) = -4\pi g \int d\tau \frac{dY^\mu}{d\tau}$ $\delta^4(x - Y(\tau))$ (S) $f^{\mu\nu}$ skew-symmetric	(IV) $\mathscr{A}_0 = \mathscr{A}_0^F - M\int d\tau$ Constraints (T) $\Theta(\Sigma) = \begin{cases} I \text{ if } \Sigma \text{ does not enclose } M \\ -I \text{ if } \Sigma \text{ encloses } M \end{cases}$ (S) $F_\mu(C\|s)\frac{d\xi^\mu(s)}{ds} = 0$ (O) $F_\mu(C_1\|s) = F_\mu(C_2\|s)$ if $\xi_1^\nu(s') = \xi_2^\nu(s')$ for $s' \leqslant s$.

Consider a variation $\Delta Y(\tau)$ at a given point along the monopole worldline. Denoting the right-hand side of the constraint (T) in (6.5) as ζ_Σ, we see that if the monopole does not cross the surface represented by Σ, then

$$\zeta_\Sigma^{-1}\Delta\zeta_\Sigma = 0, \tag{6.8}$$

and if the monopole crosses the surface (Fig. 7a),

$$\zeta_\Sigma^{-1}\Delta\zeta_\Sigma = i\pi\kappa, \tag{6.9}$$

where the quantity κ satisfies only the group relation:

$$\exp i\pi\kappa = -I. \tag{6.10}$$

To understand the significance of κ, we observe that the same change in phase can be obtained by moving the surface an amount $-\Delta Y(\tau)$ while keeping the monopole worldline fixed (Fig. 7b). The change in phase in this case can be achieved by composing infinitesimal variations in loop space, and arises from having a non-vanishing loop space curvature $G_{\mu\nu}(C; s, s')$ due to the monopole. Hence we can write

$$\zeta_\Sigma^{-1}\Delta\zeta_\Sigma = ie\int d\tau ds ds' \Theta_\Sigma^{-1}(t,0)G_{\mu\nu}(C_t; s, s')\Theta_\Sigma(t,0)\frac{\partial\xi_t^\nu(s')}{\partial t}\Delta Y^\mu(t), \tag{6.11}$$

which equals $i\pi\kappa$ by (6.9). This together with an algebraic form of (T) derived in Ref. [2], which is (6.17) further on, constrains the two variations $\Delta F_\mu(C|s)$ and $\Delta Y(\tau)$ according to (6.5), as is required.

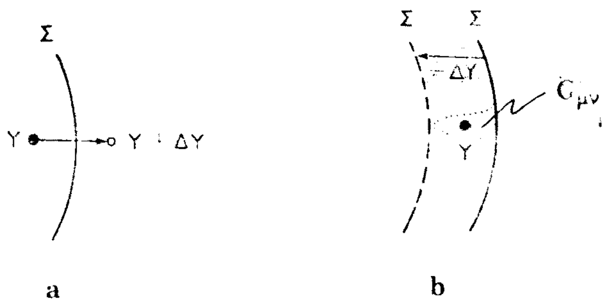

Fig. 7. Equivalent variations are obtained by varying (a) the monopole worldline and (b) the surface Σ

To obtain the Euler-Lagrange equations, we introduce the algebra-valued Lagrange multipliers $\lambda(C|s)$ and Λ_Σ respectively for (S) and (T). To account for (O) we introduce the truncated δ-function

$$\delta(C'-C)|_s = \prod_{s'<s}\delta^4(\xi'(s')-\xi(s')), \tag{6.12}$$

and define a normalization factor

$$\varrho(s) = \int\prod_{s'>s} d^4\xi'(s'). \tag{6.13}$$

We now require that the first variation of the action

$$\begin{aligned}\Delta\mathscr{A} = {}& \Delta\mathscr{A}_0 + ie\int\delta\Sigma\int dt ds \,\mathrm{Tr}\,[\Lambda_\Sigma\Theta_\Sigma^{-1}(t,0)\Delta F_\mu(C_t|s)\Theta_\Sigma(t,0)\frac{\partial\xi_t^\mu(s)}{\partial t}\\ &-i\pi\int\delta\Sigma\int dt ds d\tau\,\mathrm{Tr}\,[\Lambda_\Sigma\Theta_\Sigma^{-1}(t,0)\kappa(C_t|s)\Theta_\Sigma(t,0)]\\ &\times\varepsilon_{\mu\nu\varrho\sigma}\frac{\partial\xi_t^\nu(s)}{\partial t}\frac{\partial\xi_t^\varrho(s)}{\partial s}\frac{dY^\sigma(\tau)}{d\tau}\Delta Y^\mu(\tau)\delta^4(\xi_t(s)-Y(\tau))\\ &+\int\delta C\int ds\,\mathrm{Tr}\,[\lambda(C|s)\Delta F_\mu(C|s)]\frac{d\xi^\mu(s)}{ds}\end{aligned} \tag{6.14}$$

should vanish for all variations $\Delta F_\mu(C|s)$ and $\Delta Y^\mu(\tau)$. From this we obtain the Euler--Lagrange equations:

$$(2\pi\bar{N})^{-1}F^\mu(C|s)\varrho(s)\left[\frac{d\xi^\alpha(s)}{ds}\frac{d\xi_\alpha(s)}{ds}\right]^{-1}$$

$$= ie\int\delta\Sigma\int dt\Theta_\Sigma(t,0)\Lambda_\Sigma\Theta_\Sigma^{-1}(t,0)\frac{\partial\xi_t^\mu(s)}{\partial t}\delta(C_t-C)|_s+\lambda(C|s)\frac{d\xi^\mu(s)}{ds}\varrho(s), \tag{6.15}$$

$$M\frac{d^2Y_\mu(\tau)}{d\tau^2} = i\pi\int\delta\Sigma\int dtds\,\mathrm{Tr}\,[\Theta_\Sigma(t,0)\Lambda_\Sigma\Theta_\Sigma^{-1}(t,0)\kappa(C_t|s)]$$

$$\times\varepsilon_{\mu\nu\varrho\sigma}\frac{\partial\xi_t^\nu(s)}{\partial t}\frac{\partial\xi_t^\varrho(s)}{\partial s}\frac{dY^\sigma(\tau)}{d\tau}\delta^4(\xi_t(s)-Y(\tau)). \tag{6.16}$$

The Lagrange multipliers $\lambda(C|s)$ and Λ_Σ can be eliminated, after some effort, from these equations using the constraints. Finally we arrive at the equations of motion:

(Mg 1)
$$G_{\mu\nu}(C;s',s) = \frac{\pi}{e}\int d\tau\kappa(C|s)\varepsilon_{\mu\nu\varrho\sigma}\frac{d\xi^\varrho(s)}{ds}\frac{dY^\sigma(\tau)}{d\tau}$$

$$\times\delta^4(\xi(s)-Y(\tau))\delta(s-s') \tag{6.17}$$

(Mg 2)
$$\frac{\delta}{\delta\xi^\mu(s)}F^\mu(C|s) = 0 \tag{6.18}$$

(Lg)
$$M\frac{d^2Y_\mu(\tau)}{d\tau^2} = (2\bar{N}e)^{-1}\int ds\int\delta C\varepsilon_{\mu\nu\varrho\sigma}\,\mathrm{Tr}\,[\kappa(C|s)F^\nu(C|s)]$$

$$\times\frac{d\xi^\varrho(s)}{ds}\frac{dY^\sigma(\tau)}{d\tau}\left[\frac{d\xi^\alpha(s)}{ds}\frac{d\xi_\alpha(s)}{ds}\right]^{-1}\delta^4(\xi(s)-Y(\tau)). \tag{6.19}$$

These equations are analogous to the Maxwell and Lorentz equations of abelian theory (2.8)–(2.10).

The equations of motion for a non-abelian monopole, (6.17)–(6.19), which we have just derived, are in terms of the rather unfamiliar loop variables. They will presumably take some time to understand, and more to solve. We have, however, noted an intriguing relation between equations (6.17)–(6.19) and the Wong equations for a "classical" point source of a Yang-Mills field [3]. This may help towards understanding the former and arises as follows. At every spacetime point x where the potential $A_\mu(x)$ is well-defined, one can express loop variables in terms of spacetime variables:

$$\frac{\delta}{\delta\xi^\mu(s)}F^\mu(C|s) = \Phi_C^{-1}(s,0)D_\mu F^{\mu\nu}(\xi(s))\Phi_C(s,0)\frac{d\xi_\nu(s)}{ds}, \tag{6.20}$$

$$G_{\mu\nu}(C;s,s') = -\Phi_C^{-1}(s,0)\varepsilon_{\mu\nu\varrho\sigma}D_\alpha^* F^{\varrho\alpha}(\xi(s))\Phi_C(s,0)\frac{d\xi^\sigma(s)}{ds}\delta(s-s'). \tag{6.21}$$

These relations fail precisely on the monopole worldline, since $A_\mu(x)$, $F_{\mu\nu}(x)$ and $^*F_{\mu\nu}(x)$ are patched around the monopole and do not have a unique limit at the monopole position.

Now suppose we choose to ignore this very essential obstruction and insist on using the relations (6.20) and (6.21) on the monopole worldline. Then the equations of motion (6.17)–(6.19) take the following form:

$$D_\nu^* F^{\mu\nu}(x) = \frac{\pi}{e} \int d\tau K(\tau) \frac{dY^\mu(\tau)}{d\tau} \delta^4(x - Y(\tau)), \tag{6.22}$$

$$D_\nu F^{\mu\nu}(x) = 0 \tag{6.23}$$

$$M \frac{d^2 Y^\mu(\tau)}{d\tau^2} = (4e)^{-1} \operatorname{Tr} [K(\tau)^* F^{\mu\nu}(Y(\tau))] \frac{dY_\nu(\tau)}{d\tau}, \tag{6.24}$$

where $K(\tau)$ replaces $\kappa(C|s)$ and satisfies again (6.8). These equations look like the dual transform of the Wong equations under the replacements $F_{\mu\nu}(x) \to {}^*F_{\mu\nu}(x)$, $I(\tau) \to K(\tau)$ and $e \to g = \dfrac{1}{4e}$, except that curiously the covariant derivative is still defined in terms of $A_\mu(x)$ rather than some potential of the dual field $^*F_{\mu\nu}(x)$.

We emphasize again that we cannot as yet give a meaning to equations (6.22)–(6.24), except in the abelian case in which they reduce to (2.8)–(2.10). However, the formal duality noted in the last paragraph suggests that the exact dual invariance of electromagnetism which is lost in the non-abelian case might be reflected in the relation between the dynamics of monopoles and sources of a non-abelian gauge field. If one could thus assign a meaning to (6.22)–(6.24), then the monopoles which are defined by their topological charge (a minus--charge in the case of SO(3)) would seem to acquire another conserved attribute, namely an isospin-like $K(\tau)$, as a result of their dynamics. In that case, might it not be that quarks, which are usually thought of as sources, are in fact the monopoles of colour gauge field, and that their dynamics are to be governed by (6.17)–(6.19)? We find this an altogether attractive possibility, as then both the charges and their dynamics would be necessary consequences of the topology, and neither need be introduced by hand in the theory.

7. *Further applications*

Although SO(3) monopoles are representative of the essential non-abelian features there are many other cases of theoretical interest. Given any Lie group G, the collection of its first homotopy classes is itself a group, called the fundamental group of G, usually denoted $\pi_1(G)$. Of present interest to us are groups whose fundamental group is cyclic. This includes all the classical and exceptional groups. (Some interesting examples were considered in detail in Section 3.2). Two very different cases then arise. Either $\pi_1(G)$ is finite or it is infinite. In the first case it is isomorphic to the integers modulo p, where p is the number of elements of $\pi_1(G)$. Such monopoles behave very similarly to the SO(3) monopoles we have considered. In the second case it is isomorphic to the integers. Such monopoles exhibit both abelian and SO(3)-like properties. [10] as we shall now show.

The most widely accepted theory of chromodynamics has quarks with fractional charges. This means that the corresponding gauge group (for colour and electromagnetism combined) is U(3) and not SU(3) × U(1). Now U(3) can be obtained from SU(3) × U(1) by making the following triple identification:

$$\left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}, 1\right) \equiv \left(\begin{pmatrix}\omega^2 & 0 & 0\\ 0 & \omega^2 & 0\\ 0 & 0 & \omega^2\end{pmatrix}, \omega\right) \equiv \left(\begin{pmatrix}\omega & 0 & 0\\ 0 & \omega & 0\\ 0 & 0 & \omega\end{pmatrix}, \omega^2\right), \tag{7.1}$$

where ω is any cube root of unity other than 1 (Fig. 8, cf. Fig. 4). For parametrization purposes, the shaded two-thirds can be ignored. The groups U(3) and SU(3) × U(1) are locally isomorphic and hence have the same Lie algebra. The potential therefore splits

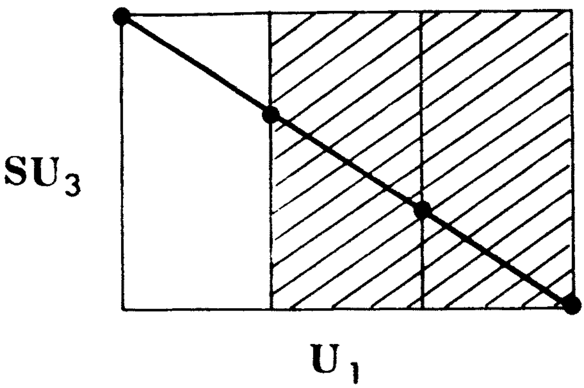

Fig. 8. Parametrization of U(3)

into two parts, the U(1) part $a_\mu(x)$ and the SU(3) part $A_\mu(x)$. To set up the topological action principle we have first to name the variables. For expediency, we shall use $a_\mu(x)$ for the abelian part, although we could of course use loop variables for this part also. The loop variables $F_\mu(C|s)$ then refer to the SU(3) part.

The constraints are also of two parts:

$$U(1) \quad : \partial_\nu {}^*f^{\mu\nu}(x) = -\frac{2n\pi}{e}\int_0^{2\pi} d\tau \, \frac{dY^\mu(\tau)}{d\tau}\, \delta^4(x - Y(\tau)) \qquad (\mathrm{T}_1)\ (7.2)$$

$$\mathrm{SU(3)}: \begin{cases} \Theta(\Sigma) = \begin{cases} (\omega^{2n}) & \text{if } \Sigma \text{ encloses the monopole} \\ 1 & \text{if } \Sigma \text{ does not enclose the monopole} \end{cases} \\ \text{plus constraints (S) and (O),} \end{cases} \qquad (\mathrm{T}_3)\ (7.3)$$

where the group elements are all in SU(3) notation. Notice that the same integer n occurs in (T_1) and (T_3). When n is a multiple of 3, the monopole is a "free" (i.e. colourless) magnetic monopole. Hence we see that

$$\text{unit "free" magnetic charge in U(3)} = 3 \times \text{unit magnetic charge in U(1)} = \frac{3}{2e}. \tag{7.4}$$

The action can now be written:

$$\mathscr{A}_0 = -\frac{1}{16\pi}\int d^4x f_{\mu\nu}(x) f^{\mu\nu}(x) - \frac{1}{4\pi\bar{N}}\int \delta C \int ds \operatorname{Tr}\left[F_\mu(C|s)F^\mu(C|s)\right] - M\int d\tau. \quad (7.5)$$

Variation with respect to the variables $a_\mu(x)$, $F_\mu(C|s)$ and $Y^\mu(\tau)$ give two sets of Maxwell--Yang-Mills equations:

$$\partial_\nu f^{\mu\nu}(x) = 0 \quad (7.6)$$

$$\frac{\delta F_\mu(C|s)}{\delta \xi^\mu(s)} = 0, \quad (7.7)$$

plus the constraints (T_1) and (T_3) (or its equivalent differential form (6.17)), all coupled via the Lorentz equation:

$$M\frac{d^2 Y_\mu}{d\tau^2} = -\frac{n}{2e}\,{}^*f_{\mu\nu}\frac{dY^\nu(\tau)}{d\tau}$$

$$+(2\bar{N}e')^{-1}\int ds \int \delta C \varepsilon_{\mu\nu\varrho\sigma} \operatorname{Tr}\left[\kappa(C|s)F^\nu(C|s)\right]$$

$$\times \frac{d\xi^\varrho(s)}{ds}\frac{dY^\sigma(\tau)}{d\tau}\left[\frac{d\xi^\alpha(s)}{ds}\frac{d\xi_\alpha(s)}{ds}\right]^{-1}\delta^4(\xi(s)-Y(\tau)), \quad (7.8)$$

where $\kappa = 0$ for $n = 3m$, and $e^\kappa = (\omega^{2n})$ otherwise.

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