ON THE DYON-FERMION SYSTEM

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(*Received July 9, 1985; revised version received November 5, 1985*)

A dyon-fermion system is considered. We prove that there is no finite energy, spherically symmetric, time-independent solution of the Yang-Mills-Higgs-Dirac equations with non-vanishing fermionic currents.

PACS numbers: 03.50.Kk

1. Introduction

The classical Yang-Mills-Higgs SU(2) theory is known to possess time-independent, spherically symmetric solutions: the monopole and the dyons [1]. This model can be extended by adding a bispinor field. Suppose that a massless Dirac field in the fundamental representation of the SU(2) group is gauge covariantly coupled to the Higgs meson. The theory obtained in this way contains spherically symmetric, time-independent, finite energy solutions which describe fermion-dyon systems [2]. In such a field configuration all fermionic currents vanish so the fermion has no influence on the dyon (it will be convenient to understand by the fermionic current also the term $-\frac{g}{2}\bar{\psi}\psi$ appearing after variation of the action with respect to the scalar field).

The problem which may be of interest is whether the considered theory possesses solution with abovementioned properties and non-zero fermionic current. In this paper it is shown that the answer for this question is negative i.e. every spherically symmetric, time-independent, finite energy field satisfying equations of motion has all fermionic currents equal to zero.

2. The fermion-dyon system

The model which we shall consider is based on the Lagrangian [2]

$$\mathscr{L} = -\tfrac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a+\tfrac{1}{2}D^\mu\varphi_a D_\mu\varphi_a+\frac{i}{2}(\bar{\psi}_{AR}\gamma^\mu_{AB}D_\mu\psi_{BR} - D_\mu\bar{\psi}_{AR}\gamma^\mu_{AB}\psi_{BR})-\lambda(\varphi_a\varphi_a-b^2)^2-\frac{g}{2}\,\varphi_a\sigma^a_{RS}\bar{\psi}_{AB}\psi_{AS}, \tag{1}$$

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where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\varepsilon_{abc}A^b_\mu A^c_\nu,$$

$$D_\mu \varphi_a = \partial_\mu \varphi_a + e\varepsilon_{abc}A^b_\mu \varphi_c,$$

$$D_\mu \psi_{AR} = \partial_\mu \psi_{AR} - \frac{i}{2} eA^a_\mu \sigma^a_{RS}\psi_{AS},$$

and with $e, g, b, \lambda \geqslant 0$ being constants. A^a_μ are gauge potentials for SU(2) group, ψ_{AR} — bispinor isodoublet and φ_a — Higgs field in adjoint representation. The labels $a, b, c = 1, 2, 3$ and $R, S = 1, 2$ are SU(2) labels. We take Dirac matrices in the representation given by Jackiw and Rebbi [2]

$$\gamma^0 = -i\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \gamma^k = -i\begin{bmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{bmatrix}.$$

Using the Lagrangian (1) we can derive the equations of motion

$$D^\mu F^{a\nu}_\mu + \tfrac{1}{2} e\bar{\psi}_{AR}\gamma^\nu_{AB}\psi_{BS}\sigma^a_{RS} + e\varepsilon_{abc}\varphi_b D^\nu \varphi_c = 0, \tag{2a}$$

$$D^\mu D_\mu \varphi_a + \frac{g}{2}\sigma^a_{RS}\bar{\psi}_{AR}\psi_{AS} + \lambda(\varphi_b\varphi_b - b^2)\varphi_a = 0, \tag{2b}$$

$$i\gamma^\mu_{AB} D_\mu \psi_{BR} - \frac{g}{2}\sigma^a_{RS}\varphi_a\psi_{AS} = 0, \tag{2c}$$

and the energy density

$$T_{00} = \tfrac{1}{2}\{\sum_{a=1}^{3}(D_0\varphi_a)^2 + \sum_{j,a=1}^{3}[(D_j\varphi_a)^2 + (E^j_a)^2 + (B^j_a)^2]\}$$

$$+ \frac{i}{2}(\bar{\psi}_{AR}\gamma^0_{AB}D_0\psi_{BR} - D_0\bar{\psi}_{AR}\gamma^0_{AB}\psi_{BR}) + \frac{\lambda}{4}(\varphi_a\varphi_a - b^2)^2, \tag{3}$$

with $E^i_a = F^i_a$, $B^i_a = -\frac{1}{2}\varepsilon^{ijk}F^{jk}_a$.

From the assumption that fields are time-independent and spherically symmetric (in the extended meaning see (3)), we obtain the following ansatz

$$\varphi_a(x^0, \vec{x}) = \frac{1}{u} G(u)n_a, \quad A^a_0(x^0, \vec{x}) = \frac{1}{u} K(u)n_a,$$

$$A^a_j(x^0, \vec{x}) = \frac{1}{u}[(1 - H_1(u))\varepsilon_{ajm}n_m - H_2(u)(\delta_{aj} - n_a n_j) - T(u)n_a n_j],$$

$$\psi_{AR}(x^0, \vec{x}) = \frac{e^{\frac{1}{2}}}{u}\begin{pmatrix} X_1(u)\sigma^2_{CR} + X_2(u)n_a\sigma^a_{CS}\sigma^2_{SR} \\ Y_1(u)\sigma^2_{DR} + Y_2(u)n_a\sigma^a_{DS}\sigma^2_{SR} \end{pmatrix}$$

$$A = 1, 2, 3, 4, \quad C = 1, 2, \quad D = 3, 4, \quad u = e|\vec{x}|, \quad n_a = \frac{x_a}{|\vec{x}|}. \tag{4}$$

All fields which are regular at the origin must satisfy the conditions

$$G = O(u^2), \quad K = O(u^2), \quad T = O(u^2),$$
$$H_1 = 1+O(u^2), \quad H_2 = O(u^2),$$
$$X_1 = O(u), \quad X_2 = O(u^2), \quad Y_1 = O(u), \quad Y_2 = O(u^2). \tag{5}$$

Substituting (4) into the equations of motion (2) and the energy density formula (3) we obtain

$$u^2 \frac{d^2}{du^2} G = 2(H_1^2+H_2^2)G + \frac{\lambda}{e^2}(G^2-b^2)G - \frac{g}{e} uZ, \tag{6a}$$

$$u^2 \frac{d^2}{du^2} K = 2(H_1^2+H_2^2)K - uQ, \tag{6b}$$

$$u^2 \frac{d^2}{du^2} H_1 = -u \frac{d}{du}(TH_2) - uT \frac{d}{du} H_2 + TH_2$$
$$+(H_1^2+H_2^2+T^2+G^2-K^2-1)H_1 + uJ_1, \tag{6c}$$

$$u^2 \frac{d^2}{du^2} H_2 = u \frac{d}{du}(TH_1) + uT \frac{d}{du} H_1 - TH_1$$
$$+(H_1^2+H_2^2+T^2+G^2-K^2-1)H_2 + uJ_2, \tag{6d}$$

$$u\left(H_2 \frac{d}{du} H_1 - H_1 \frac{d}{du} H_2\right) + T(H_1^2+H_2^2) = \tfrac{1}{2} uJ, \tag{6e}$$

$$2u \frac{d}{du} X_1 = (iT-2iH_2)X_2 + \left(2H_1 - \frac{g}{e} G\right) X_1 - iKY_1, \tag{6f}$$

$$2u \frac{d}{du} X_2 = (iT+2iH_2)X_1 - \left(2H_1 + \frac{g}{e} G\right) X_2 - iKY_2, \tag{6g}$$

$$2u \frac{d}{du} Y_1 = (iT-2iH_2)Y_2 + \left(2H_1 + \frac{g}{e} G\right) Y_1 - iKX_1, \tag{6h}$$

$$2u \frac{d}{du} Y_2 = (iT+2iH_2)Y_1 - \left(2H_1 - \frac{g}{e} G\right) Y_2 - iKX_2, \tag{6i}$$

$$T_{00} = \frac{e^2}{u^2}\left\{\left(\frac{d}{du}K - \frac{K}{u}\right)^2 + \left(\frac{K}{u}H_1\right)^2 + \left(\frac{K}{u}H_2\right)^2\right.$$
$$+\left(\frac{d}{du}H_1 + \frac{T}{u}H_2\right)^2 + \left(\frac{d}{du}H_2 - \frac{T}{u}H_1\right)^2$$

$$+\frac{1}{2u^2}(H_1^2+H_2^2-1)^2+\frac{1}{2}\left(\frac{d}{du}G-\frac{G}{u}\right)^2$$

$$+\left(\frac{G}{u}H_1\right)^2+\left(\frac{G}{u}H_2\right)^2+\lambda\frac{u^2}{e^2}\left(\frac{G^2}{u^2}-b^2\right)^2+\frac{K}{u}Q\Big\}. \tag{7}$$

For convenience we have introduced the following notation

$$J_1 = i(X_1\bar{Y}_2-\bar{X}_1Y_2-X_2\bar{Y}_1+\bar{X}_2Y_1),$$

$$J_2 = -(X_1\bar{Y}_1+\bar{X}_1Y_1-X_2\bar{Y}_2-\bar{X}_2Y_2),$$

$$J = -(X_1\bar{Y}_1+\bar{X}_1Y_1+X_2\bar{Y}_2+\bar{X}_2Y_2),$$

$$Q = X_1\bar{X}_2+\bar{X}_1X_2+Y_1\bar{Y}_2+\bar{Y}_1Y_2,$$

$$Z = i(X_1\bar{Y}_2-\bar{X}_1Y_2+X_2\bar{Y}_1-\bar{X}_2Y_1).$$

In this paper we prove that for each finite energy solution of Eqs. (6) which satisfies conditions (5) the fermionic currents

$$j_a = -\frac{g}{2}\bar{\psi}_{AR}\psi_{AS}\sigma^a_{RS} = \frac{ge}{u^2}Zn_a, \tag{8a}$$

$$j_a^0 = -\frac{e}{2}\bar{\psi}_{AR}\gamma^0_{AB}\psi_{BS}\sigma^a_{RS} = \frac{e^2}{u^2}Qn_a, \tag{8b}$$

$$j_a^k = -\frac{e}{2}\bar{\psi}_{AR}\gamma^k_{AB}\psi_{BS}\sigma^a_{RS} = \frac{e^2}{u^2}\left[J_1\varepsilon_{aks}n_s+J_2(\delta_{ak}-n_an_k)+Jn_an_k\right] \tag{8c}$$

must vanish.

We will use in our considerations gauge transformations given by SU(2) matrices of the form

$$\omega_{AB}(x) = \left(\exp i\frac{\sigma_a}{2}n_a\vartheta(u)\right)_{AB}.$$

The class of spherically symmetric fields (4) is closed against these transformations. We will need transformations formulas describing changes of some functions appearing in (6)

$$H_1' = \cos\vartheta H_1+\sin\vartheta H_2, \quad H_2' = \cos\vartheta H_2-\sin\vartheta H_1,$$

$$J_1' = \cos\vartheta J_1+\sin\vartheta J_2, \quad J_2' = \cos\vartheta J_2-\sin\vartheta J_1,$$

$$J' = J, \quad T' = T-u\frac{d}{du}\vartheta. \tag{9}$$

Using equations (6f, g, h, i) it is easy to obtain

$$u\frac{d}{du}Q = -\frac{g}{e}GS, \tag{10a}$$

$$u\frac{d}{du}S = KZ-\frac{g}{e}GQ, \tag{10b}$$

$$u\frac{d}{du}Z = -KS, \tag{10c}$$

$$u\frac{d}{du}J_1 = -TJ_2-2H_2J, \tag{11a}$$

$$u\frac{d}{du}J_2 = TJ_1+2H_1J, \tag{11b}$$

$$u\frac{d}{du}J = 2H_1J_2-2H_2J_1, \tag{11c}$$

where $S = X_1\bar{X}_2+\bar{X}_1X_2-Y_1\bar{Y}_2-\bar{Y}_1Y_2$.

We shall consider (10) as a system of equations for Q, S, Z. From (5) we obtain $\lim\limits_{u\to 0^+}\frac{K(u)}{u} = 0$, $\lim\limits_{u\to 0^+}\frac{G(u)}{u} = 0$. Thus these equations are regular on the whole half-line $u \geqslant 0$. Using (5) we obtain the boundary condition $Q(0) = S(0) = Z(0) = 0$. This boundary condition uniquely determines the solution: $Q = S = Z = 0$ (the uniqueness of the solution may be proved by using the method of Picard, see [4]). Therefore we have $j_a = 0$, $j_a^0 = 0$. The equality $Q = 0$ implies that each term in the energy density formula is non-negative. Thus the integral $\int\limits_0^\infty\left(\frac{d}{du}G-\frac{G}{u}\right)^2 du$ has to be finite. This implies that $\lim\limits_{u\to\infty}\frac{G(u)}{u}$ exists. Since $\int\limits_0^\infty\left(\frac{G}{u}H_k\right)^2 du$ must also be finite, we obtain $\lim\limits_{u\to\infty}\frac{G(u)}{u} = 0$ or $\int\limits_0^\infty (H_k)^2du < \infty$, $k = 1, 2$ (the case $\lim\limits_{u\to\infty}\frac{G(u)}{u} = 0$ is possible only when $b = 0$ or $\lambda = 0$).

Now we shall show that $J_1 = J_2 = J = 0$. We start with some remarks concerning the case $b = 0$, $\lim\limits_{u\to\infty}\frac{G(u)}{u} = 0$. Setting $b = 0$, $Z = 0$ in Eq. (6a) we obtain

$$u^2\frac{d^2}{du^2}G = (H_1^2+H_2^2)G+\lambda G^3$$

and we see that $\frac{d}{du}G$ is non-increasing (non-decreasing) if $G \leqslant 0$ ($G \geqslant 0$). On the other hand L'Hospital's rule gives us $\lim_{u\to\infty} \frac{d}{du}G = 0$. Since $G(0) = 0$, the function G vanishes identically. Applying arguments based on scale-invariance one can show that the Yang--Mills potential is pure gauge [5]. Thus we may assume that $\lim_{u\to\infty} \frac{G(u)}{u} \neq 0$. From (10) it follows that

$$\frac{d}{du}(J_1^2+J_2^2-J^2) = 0.$$

The boundary condition (5) yields:

$$J_1^2+J_2^2 = J^2. \tag{12}$$

In the gauge in which $J_1 = 0$ (see (8)) we obtain

$$J_2 = \pm J. \tag{13}$$

Inserting (13) into Eq. (11c) we find that

$$u\frac{d}{du}J = \pm 2H_1 J,$$

$$J(u) = J(u_0)\exp \pm 2\int_{u_0}^{u} H_1(w)\frac{dw}{w}.$$

Assuming the energy is finite we have $\int_{0u}^{\infty}(H_1(w))^2 dw < \infty$, so $\int_{u_0}^{\infty}|H_1(w)|\frac{dw}{w} < \infty$ and

$$\lim_{u\to\infty} \exp\pm 2\int_{u_0}^{u} H_1(w)\frac{dw}{w} \neq 0. \tag{14}$$

On the other hand, if we choose the gauge in such a way that $T = 0$ then Eq. (6e) becomes

$$J = 2\left(H_2\frac{d}{du}H_1 - H_1\frac{d}{du}H_2\right). \tag{15}$$

But in this gauge the integrals $\int_0^{\infty}(H_k)^2 dw$, $\int_0^{\infty}\left(\frac{d}{dw}H_k\right)^2 dw$ are finite for all finite energy fields and from (15) we obtain

$$\int_0^{\infty}|J(u)|du < \infty. \tag{16}$$

This condition has the same form in each gauge. Comparing (13) and (16) we see that $J = 0$. From (11) we immediately have $J_1 = J_2 = 0$.

3. Conclusions

In this paper we have considered the dyon-fermion system with spherical symmetry. We have proved that for each time-independent, finite energy solution of the Yang-Mills-Higgs-Dirac equations the fermion does not act on the dyon. One can expect that the same problem, but with the fermionic field being time-dependent through the factor $\exp -i\omega x^0$ (i.e. being of the form $\psi_{AR}(x) = \exp -i\omega x^0 \tilde{\psi}_{AR}(x)$, where $\tilde{\psi}_{AR}(x)$ is given by (4)), will be more interesting in this respect. The case of massless Yang-Mills-Dirac system (the Higgs field being absent) was studied by Magg [5]. He proved that finite energy, static, spherically symmetric solutions can only be trivial ones. In the case when the Higgs field is present it can be shown that if such a solution with non-zero fermionic field and positive ω exists, then at least one of the expressions (7) does not vanish.

REFERENCES

[1] G. 't Hooft, *Nucl. Phys.* **B79**, 276 (1974); B. Julia, A. Zee, *Phys. Rev.* **D11**, 2227 (1975); M. K. Prasad, C. M. Sommerfield, *Phys. Rev. Lett.* **35**, 760 (1975); E. B. Bogomolny, *Sov. J. Nucl. Phys.* **24**, 449 (1976).
[2] R. Jackiw, C. Rebbi, *Phys. Rev.* **D13**, 3398 (1976).
[3] P. Forgacs, N. S. Manton, *Commun. Math. Phys.* **72**, 15 (1980).
[4] H. T. Davis, *Introduction to Nonlinear Differential and Integral Equations*, Dover Publications Inc., New York 1962.
[5] M. Magg, *J. Math. Phys.* **25**, 1531 (1984).