METHOD OF SEPARATING THE ANGULAR COORDINATES IN TWO-BODY WAVE EQUATIONS WITH SPIN*

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An explicit form of total angular momentum eigenfunctions is found for the physical systems described by one three-dimensional space coordinate and arbitrary spin degrees of freedom. The resulting formula is usefully parametrized by the multicomponent radial wave function. The dependence on the angular coordinates is given by action of generalized spherical harmonics. The formula gives a convenient method of separation of the angular coordinates in an arbitrary one- or two-body wave equation with spin. As an example, the method was applied to the relativistic wave equation for one Dirac and one Duffin-Kemmer particle, proposed recently by Królikowski. A corresponding set of radial equations is derived in the case of spherically symmetrical interaction potentials.

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1. Introduction

If we try to solve the wave equation for spherically symmetrical one- or two-particle system, it is necessary first to separate the angular coordinates. In the *spinless* case it can be easily done by looking for the solutions that are simultaneously eigenfunctions of the L^2 and L_3 operators. Since the angular part of those eigenfunctions is unambiguously determined by the spherical harmonics

$$\psi_{lm_l}(r, \theta, \varphi) = Y_{lm_l}(\theta, \varphi)R(r), \tag{1}$$

we are left with a much simpler equation for the radial function $R(r)$.

For particles *with spin* the situation is obviously more complicated because we must look for solutions that are eigenfunctions of J^2 and J_3 operators. The general form of such eigenfunctions can be written down using the well known formula for adding the angular momenta

$$\psi_{jm_j}(r, \theta, \varphi) = \sum_l \sum_s C_{ls}(r) \sum_{m_s} (l, s, m-m_s, m_s|j, m_j)\, \chi^s_{m_s} Y_{l,m-m_s}(\theta, \varphi), \tag{2}$$

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where $(j_1, j_2, m_1, m_2 | j, m)$ are the Clebsch-Gordan coefficients and $\chi^s_{m_s}$ are eigenfunctions of S^2 and S_3 operators. The wave function in the form (2) is parametrized by some arbitrary r-dependent functions $C_{ls}(r)$. Substituting (2) into the wave equation enables us to eliminate the spherical harmonics and obtain a system of coupled equations for the functions $C_{ls}(r)$. Practically, however, this procedure may be found inconvenient, because it needs multiple applications of reduction formulae for Clebsch-Gordan coefficients, spherical harmonics and spin matrices.

Another way of eliminating the angular coordinates was proposed many years ago [1] in the case of the Breit equation. The method is based on another formula for the eigenfunctions of J^2 and J_3,

$$\psi_{jm_j}(r, \theta, \varphi) = U^{-1} Z_j^{m_j}(\theta, \varphi) \tilde{\tilde{\psi}}(r), \tag{3}$$

where $\tilde{\tilde{\psi}}(r)$ is a multicomponent radial function, $Z_j^{m_j}(\theta, \varphi)$ describes a matrix operator dependent on angular coordinates, and U stands for a unitary transformation. The operator $Z_j^{m_j}$ plays the role of Y_{lm_l} in the formula (1) and, therefore, it will be called a generalized spherical harmonic. The form of operator $Z_j^{m_j}$ for two Dirac particles was explicitly found in Ref. [1]. In the present paper it is shown that the formula (3) can be generalized for an arbitrary physical system containing one or two particles. In particular, the explicit form of the corresponding operator $Z_j^{m_j}$ is derived.

Since in Eq. (3) the radial and angular coordinates are fully separated, it is convenient to use this formula to eliminate the angular coordinates from the wave equation. The general method is described in Sect. 2. In Sect. 3 some examples are given to illustrate general considerations. The first two examples, reproducing known results, show how the formula (3) works. In the third example a new result is described. Namely, our general method is applied to the equation for one Dirac and one Duffin-Kemmer particle, proposed recently by Królikowski [2]. The corresponding set of radial equations is derived. Results are summed up in Sect. 4.

2. *General discussion*

Let us consider the physical system described by one three-dimensional space coordinate $\boldsymbol{x}$, e.g. one particle in an external field or two particles in the centre-of-mass frame ($\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$). The orbital angular momentum of the system is given by the expression

$$\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p} = \boldsymbol{x} \times \left(-i \frac{\partial}{\partial \boldsymbol{x}}\right), \tag{4}$$

applicable both in non-relativistic and relativistic cases. The arbitrary spin angular momentum can be described by an appropriate matrix operator for which we have

$$[S_i, S_j] = i\varepsilon_{ijk} S_k, \quad [S_i, L_j] = 0, \quad i, j, k = 1, 2, 3. \tag{5}$$

The total angular momentum is a sum of orbital and spin angular momenta $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$. We look for the eigenfunctions of the J^2 and J_3 operators.

As the first step we introduce the spherical system of refeience: $x = (r, \theta, \varphi)$. Orbital momentum (4) expressed by the angles θ, φ takes its usual form

$$L_3 = -i\frac{\partial}{\partial\varphi},$$

$$L^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{6}$$

When the spherical space coordinates are introduced it is useful to rotate the internal spin coordinates by the following point-dependent unitary transformation:

$$\psi \to \tilde{\psi} = U\psi,$$

$$O \to \tilde{O} = UOU^{-1}, \tag{7}$$

$$U = \exp(i\theta S_2)\exp(i\varphi S_3), \tag{8}$$

wheie O represents an arbitrary operator. Acting on $\tilde{\psi}$ the operators $\tilde{O}$ maintain their primary physical meaning, e.g. $\tilde{S}_3$ is the operator of the z-component of spin. However, from the equation

$$\tilde{S}_r := \frac{x}{r}\cdot\tilde{S} = S_3 \tag{9}$$

we can see that the operator S_3 acting on $\tilde{\psi}$ gets a new meaning of the radial component of spin.

If the total spin has *half-integer* eigenvalues the transformation (7) is ambiguous: the substitution $\varphi \to \varphi+2\pi$ into expression (8) changes the sign of the operator U. So, we should consider the operator U and wave function $\tilde{\psi}$ as two-valued. This fact is closely related to the behaviour of systems with half-integer spin under spatial rotations.

Using the formulae given above it is possible to find the explicit form of transformed operators $\tilde{J}_3$ and $\tilde{J}^2$:

$$\tilde{J}_3 = L_3, \tag{10}$$

$$\tilde{J}^2 = L^2 + \frac{1}{\sin^2\theta}(S_3)^2 - 2\frac{\cos\theta}{\sin^2\theta}S_3L_3. \tag{11}$$

It is clear, that operator (9) commutes with (10) and (11), so we can look for common eigenfunctions of $\tilde{S}_r$, $\tilde{J}_3$ and $\tilde{J}^2$ with eigenvalues m_s, m_j and $j(j+1)$, respectively. Let us denote by $\chi^{(i)}_{m_s}$ the eigenvectors of (9) (index i labels different solutions with the same eigenvalue). Explicit forms of χ depend on the actual form of spin matrix S_3. The eigenvectors of (10) have the obvious form

$$\tilde{\psi}_{m_j}(\varphi) = \frac{1}{\sqrt{2\pi}}\exp(im_j\varphi), \tag{12}$$

where m_j is integer *or* half-integer (depending on the total spin). Substituting the form

$$\psi^{(i)}_{m_s m_j j}(\theta, \varphi) = \chi^{(i)}_{m_s}\tilde{\varphi}_{m_j}(\varphi)\tilde{\psi}_{m_s m_j j}(\theta) \tag{13}$$

for eigenfunctions of (11) we obtain the following equation for $\tilde{\psi}_{m_s m_j j}(\theta)$:

$$\left[-\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\frac{1}{\sin^2\theta}(m_j^2+m_s^2)-2\frac{\cos\theta}{\sin^2\theta}m_j m_s\right]\tilde{\psi}_{m_s m_j j}(\theta)$$
$$= j(j+1)\tilde{\psi}_{m_s m_j j}(\theta). \tag{14}$$

The solution regular at the points $\theta = 0$ and $\theta = \pi$ exists only for $j \geqslant m \equiv \max(|m_j|, |m_s|)$ and has the form

$$\tilde{\psi}_{m_s m_j j}(\theta) = \left(\sin\frac{\theta}{2}\right)^{|m_s - m_j|}\left(\cos\frac{\theta}{2}\right)^{|m_s + m_j|} P^{(|m_s - m_j|, |m_s + m_j|)}_{j-m}(\cos\theta), \tag{15}$$

where $P^{(\alpha,\beta)}_n(x)$ is the Jacobi polynomial [3]. The general eigenfunction of $\tilde{J}^2$ and $\tilde{J}_3$ is a linear combination of the solutions (13) with arbitrary r-dependent coefficients

$$\tilde{\psi}_{jm_j}(r, \theta, \varphi) = \sum_{m_s}\sum_i C^{(i)}_{m_s}(r)\tilde{\psi}^{(i)}_{m_s m_j j}(\theta, \varphi). \tag{16}$$

The index m_s takes the values $|m_s| \leqslant j$, while the number of different values of the index i is equal to the dimension of the spin subspace corresponding to the specific m_s.

Eq. (16) can be rewritten in a more convenient form. Let us introduce the operator projecting onto the subspace corresponding to m_s:

$$\Pi_{m_s} = \sum_i \chi^{(i)}_{m_s}(\chi^{(i)}_{m_s})^+,$$

$$\Pi_{m_s}\chi_{m_s'} = \chi_{m_s}\delta_{m_s m_s'}. \tag{17}$$

Then formula (16) takes the form

$$\tilde{\psi}_{jm_j}(r, \theta, \varphi) = Z^{m_j}_j(\theta, \varphi)\tilde{\tilde{\psi}}(r), \tag{18}$$

where

$$Z^{m_j}_j(\theta, \varphi) = \sum_{m_s}\alpha_{m_s}\Pi_{m_s}\tilde{\psi}_{m_s m_j j}(\theta)\tilde{\varphi}_{m_j}(\varphi), \tag{19}$$

$$\tilde{\tilde{\psi}}(r) = \sum_{m_s}\sum_i \frac{1}{\alpha_{m_s}} C^{(i)}_{m_s}(r)\chi^{(i)}_{m_s}. \tag{20}$$

The functions $C^{(i)}_{m_s}$ can be arbitrary, but some limitation for $\tilde{\tilde{\psi}}(r)$ comes out from the fact that m_s cannot exceed the value of j:

$$\Pi_{m_s}\tilde{\tilde{\psi}}(r) \equiv 0 \quad \text{for} \quad |m_s| > j. \tag{21}$$

The limitation (21) is important only for the lowest values of j, so that the function $\tilde{\tilde{\psi}}(r)$ can be regarded as unrestricted if j is larger than the maximal value of the total spin.

The coefficients α_{m_s} in the formulae (19) and (20) can be arbitrary numbers different from zero. It is convenient to choose them in such a manner, that the operator $Z_j^{m_j}$ is "normalized", i.e.

$$\int r^2 dr d\Omega (\tilde{\psi}'_{jm}(r, \theta, \varphi))^+ \tilde{\psi}_{jm}(r, \theta, \varphi) = \int r^2 dr (\tilde{\tilde{\psi}}'(r))^+ \tilde{\tilde{\psi}}(r). \tag{22}$$

Using properties of Jacobi polynomials [3] one can get

$$|\alpha_{m_s}|^2 = (j+\tfrac{1}{2}) \frac{(j+m)!(j-m)!}{(j+m')!(j-m')!}, \tag{23}$$

where $m = \max(|m_s|, |m_j|)$, $m' = \min(|m_s|, |m_j|)$.

The derived formulae give us a method for eliminating the angular coordinates from arbitrary equation with spin. Such a method consists of two steps:

1. performing the unitary transformation (7),
2. making the substitution (18).

If the equation is invariant under rotations, the operator $Z_j^{m_j}$ can be transferred to the left and removed. The resulting equation is the looked for radial equation.

3. Examples

3.1. Dirac particle, $j = 1/2$, $m_j = 1/2$

To give a simple example how the transformation (3) works we will find the form of the wave function for the Dirac particle in the state $j = 1/2$, $m_j = 1/2$. Introducing the spin matrices $S_k = \frac{1}{2}\Sigma_k = \frac{i}{8}\varepsilon_{ijk}[\gamma^i, \gamma^j]$ ($k\ i, j = 1, 2, 3$) we can write the projectors (17) as

$$\Pi_{+1/2} = \tfrac{1}{2}(1+\Sigma_3)$$

$$\Pi_{-1/2} = \tfrac{1}{2}(1-\Sigma_3). \tag{24}$$

From Eqs. (19), (15) and (12) we have

$$Z_{1/2}^{1/2}(\theta, \varphi) = \tfrac{1}{2}(1+\Sigma_3)\cos\frac{\theta}{2}\, e^{i\varphi/2} + \tfrac{1}{2}(1-\Sigma_3)\sin\frac{\theta}{2}\, e^{i\varphi/2} \tag{25}$$

and from Eq. (8):

$$U^{-1} = \exp(-\tfrac{i}{2}\Sigma_3\varphi)\exp(-\tfrac{i}{2}\Sigma_2\theta)$$

$$= \left(\cos\frac{\varphi}{2} - i\Sigma_3 \sin\frac{\varphi}{2}\right)\left(\cos\frac{\theta}{2} - i\Sigma_2 \sin\frac{\theta}{2}\right). \tag{26}$$

Using the Dirac representation of Dirac matrices and denoting the components of function $\tilde{\tilde{\psi}}(r)$ by f_1, f_2, g_1, g_2 we finally obtain

$$\psi_{\frac{1}{2},\frac{1}{2}}(r,\theta,\varphi) = U^{-1}Z_{1/2}^{1/2}(\theta,\varphi)\begin{bmatrix} f_1 \\ f_2 \\ g_1 \\ g_2 \end{bmatrix} = U^{-1}\begin{bmatrix} f_1 \cos\frac{\theta}{2}\, e^{i\varphi/2} \\ f_2 \sin\frac{\theta}{2}\, e^{i\varphi/2} \\ g_1 \cos\frac{\theta}{2}\, e^{i\varphi/2} \\ g_2 \sin\frac{\theta}{2}\, e^{i\varphi/2} \end{bmatrix} = \begin{bmatrix} f_- + f_+ \cos\theta \\ -f_+ \sin\theta\, e^{i\varphi} \\ g_- + g_+ \cos\theta \\ -g_+ \sin\theta\, e^{i\varphi} \end{bmatrix}, \tag{27}$$

where $f_\pm = \frac{1}{2}(f_1 \pm f_2)$, $g_\pm = \frac{1}{2}(g_1 \pm g_2)$. The result (27) is the standard textbook solution of the problem.

3.2. Two Dirac particles, $m_j = 0$

As a second example let us consider a system of two Dirac particles. In the centre-of-mass frame the wave function is a 16-component function of relative coordinate $\boldsymbol{x}$. We look for the 16×16 matrix $Z_j^{m_j}$. If we restrict ourselves to the case $m_j = 0$, then the property $\tilde{\psi}_{m_s 0 j}(\theta) = \tilde{\psi}_{-m_s 0 j}(\theta)$ implies

$$Z_j^0(\theta,\varphi) = \sum_{m_s \geq 0} \alpha_{m_s}(\Pi_{m_s} + \Pi_{-m_s})\tilde{\psi}_{m_s 0 j}(\theta). \tag{28}$$

Projectors Π_{m_s} are given by the equations

$$\Pi_0 = \tfrac{1}{2}(1 - \Sigma_3^{(1)}\Sigma_3^{(2)}),$$
$$\Pi_1 + \Pi_{-1} = \tfrac{1}{2}(1 + \Sigma_3^{(1)}\Sigma_3^{(2)}), \tag{29}$$

where the upper indices refer to the first or second particle. From the relations between Jacobi and Legendre polynomials [3] we get

$$\tilde{\psi}_{00j} = P_j^{(0,0)}(\cos\theta) = P_j(\cos\theta),$$
$$\tilde{\psi}_{10j} = \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, P_{j-1}^{(1,1)}(\cos\theta) = \frac{1}{j+1}\, P_j^1(\cos\theta). \tag{30}$$

Thus finally

$$Z_j^0(\theta,\varphi) = \sqrt{j+\tfrac{1}{2}}\,[\tfrac{1}{2}(1 - \Sigma_3^{(1)}\Sigma_3^{(2)})P_j(\cos\theta)$$
$$+ \frac{1}{\sqrt{j(j+1)}}\,\tfrac{1}{2}(1 + \Sigma_3^{(1)}\Sigma_3^{(2)})P_j^1(\cos\theta)]. \tag{31}$$

Operator $Z_j^0(\theta,\varphi)$ in the form (31) was used in Ref. [1] to eliminate the angular coordinates from the Breit equation.

3.3. One Dirac and one Duffin-Kemmer particle, $m_j = 1/2$

Now let us consider a system of one Dirac and one Duffin-Kemmer particle in the centre-of-mass frame. The spin operator for the Duffin-Kemmer particle has the form

$$\Xi_k = \frac{i}{2}\varepsilon_{ijk}[\beta^i, \beta^j], \qquad i, j, k = 1, 2, 3, \tag{32}$$

where $\beta^\mu = (\beta^0, \beta^k)$ are Duffin-Kemmer matrices defined by the relation

$$\beta^\mu\beta^\nu\beta^\varrho + \beta^\varrho\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\varrho + g^{\nu\varrho}\beta^\mu. \tag{33}$$

Total spin is equal to the sum $S = \frac{1}{2}\Sigma + \Xi$. Here the projectors Π_{m_s} are the following:

$$\begin{aligned} \Pi_{3/2} &= \tfrac{1}{2}(1+\Sigma_3)\Gamma_1, & \Pi_{1/2} &= \tfrac{1}{2}(1+\Sigma_3)\Gamma_0 + \tfrac{1}{2}(1-\Sigma_3)\Gamma_1, \\ \Pi_{-3/2} &= \tfrac{1}{2}(1-\Sigma_3)\Gamma_{-1}, & \Pi_{-1/2} &= \tfrac{1}{2}(1-\Sigma_3)\Gamma_0 + \tfrac{1}{2}(1+\Sigma_3)\Gamma_{-1}, \end{aligned} \tag{34}$$

where Γ_n denote the projecting operators corresponding to the third component of Duffin--Kemmer spin

$$\begin{aligned} \Gamma_1 &= \Xi_3\tfrac{1}{2}(1+\Xi_3) = \tfrac{1}{2}[(\Xi_3)^2 + \Xi_3], \\ \Gamma_0 &= (1-\Xi_3)(1+\Xi_3) = 1-(\Xi_3)^2, \\ \Gamma_{-1} &= -\Xi_3\tfrac{1}{2}(1-\Xi_3) = \tfrac{1}{2}[(\Xi_3)^2 - \Xi_3]. \end{aligned} \tag{35}$$

Phases of the coefficients α_{m_s} can be chosen as

$$\begin{aligned} \alpha_{1/2} &= \alpha_{-1/2} = \sqrt{j+\tfrac{1}{2}}, \\ \alpha_{3/2} &= -\alpha_{-3/2} = \sqrt{\frac{(j+\frac{1}{2})(j+\frac{3}{2})}{(j-\frac{1}{2})}}. \end{aligned} \tag{36}$$

Then the operator $Z_j^{1/2}$ takes the form

$$\begin{aligned} Z_j^{1/2}(\theta, \varphi) = \sqrt{j+\tfrac{1}{2}}\, e^{i\varphi/2}\Bigg[&\sqrt{\frac{j+\frac{3}{2}}{j-\frac{1}{2}}}\,\Pi_{3/2}\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}P^{(1,2)}_{j-3/2}(\cos\theta) \\ &+ \Pi_{1/2}\cos\frac{\theta}{2}P^{(0,1)}_{j-1/2}(\cos\theta) \\ &+ \Pi_{-1/2}\sin\frac{\theta}{2}P^{(1,0)}_{j-1/2}(\cos\theta) \\ &- \sqrt{\frac{j+\frac{3}{2}}{j-\frac{1}{2}}}\,\Pi_{-3/2}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}P^{(2,1)}_{j-3/2}(\cos\theta)\Bigg]. \end{aligned} \tag{37}$$

Recently, Królikowski [2] proposed a new equation describing in the static potential approximation a system consisting of one Dirac and one Duffin-Kemmer particle

$$\{\beta^0[E-V-\boldsymbol{\alpha p}-\beta(m+\tfrac{1}{2}S)]+\boldsymbol{\beta p}-(M+\tfrac{1}{2}S)\}\psi(x) = 0. \tag{38}$$

For some physical systems it is reasonable to assume that potentials V and S are spherically symmetrical: $V = V(r)$, $S = S(r)$. Let us apply our formulae to eliminating the angular coordinates from Eq. (38). Due to the rotational invariance the result of the angular variables separation cannot depend on m_j and, therefore, we can restrict ourselves to the case $m_j = \frac{1}{2}$.

The first step is to perform the unitary transformation (7). For $\boldsymbol{\alpha p}$ and $\boldsymbol{\beta p}$ we get

$$U(\boldsymbol{\alpha p})U^{-1} = -i\alpha_3\frac{\partial}{\partial r}-i\alpha_1\frac{1}{r}\frac{\partial}{\partial\theta}-i\alpha_2\frac{1}{\sin\theta}\frac{1}{r}\frac{\partial}{\partial\varphi}$$

$$-\frac{1}{r}(\alpha_1S_2-\alpha_2S_1)-\alpha_2S_3\frac{1}{r}\cot\theta,$$

$$U(\boldsymbol{\beta p})U^{-1} = -i\beta^3\frac{\partial}{\partial r}-i\beta^1\frac{1}{r}\frac{\partial}{\partial\theta}-i\beta^2\frac{1}{\sin\theta}\frac{1}{r}\frac{\partial}{\partial\varphi}$$

$$-\frac{1}{r}(\beta^1S_2-\beta^2S_1)-\beta^2S_3\frac{1}{r}\cot\theta. \tag{39}$$

The other terms in Eq. (38) are not changed and the transformed equation takes the form

$$\Big\{\beta^0[(E-V)-(m+\tfrac{1}{2}S)]-(M+\tfrac{1}{2}S)$$

$$-\beta^0\left[-i\alpha_3\frac{\partial}{\partial r}-\frac{1}{r}(\alpha_1S_2-\alpha_2S_1)-\frac{i}{r}K_1\right]$$

$$+\left[-i\beta^3\frac{\partial}{\partial r}-\frac{1}{r}(\beta^1S_2-\beta^2S_1)-\frac{i}{r}K_2\right]\Big\}\tilde{\psi}(r,\theta,\varphi) = 0, \tag{40}$$

where

$$K_1 = \alpha_1\frac{\partial}{\partial\theta}-i\alpha_2S_3\cot\theta+\alpha_2\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi},$$

$$K_2 = \beta^1\frac{\partial}{\partial\theta}-i\beta^2S_3\cot\theta+\beta^2\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}. \tag{41}$$

The second step is to substitute Eq. (18) into Eq. (40) and to transfer to the left the operator $Z_j^{1/2}$ given by Eq. (37). Using properties of Dirac and Duffin-Kemmer matrices

(Eq. (32)) and Jacobi polynomials [3] one can obtain

$$K_1 Z_j^{1/2} = Z_j^{1/2}[-\sqrt{(j+\tfrac{3}{2})(j-\tfrac{1}{2})}\,(\Xi_3)^2+(j+\tfrac{1}{2})\,(1-(\Xi_3)^2)]i\alpha_2,$$

$$K_2 Z_j^{1/2} = Z_j^{1/2}\{\sqrt{(j+\tfrac{3}{2})(j-\tfrac{1}{2})}\,[-\tfrac{1}{2}\,i\beta^2+\tfrac{1}{2}\,\Sigma_3(1-2(\Xi_3)^2)\beta^1]$$
$$+(j+\tfrac{1}{2})\,[\tfrac{1}{2}\,i\beta^2+\tfrac{1}{2}\,\Sigma_3(1-2(\Xi_3)^2)\beta^1]\}. \tag{42}$$

All other terms in Eq. (40) commute with the operator $Z_j^{1/2}$. Eventually, we get the following radial equation corresponding to Eq. (38):

$$\Big\{\beta^0[(E-V)-\beta(m+\tfrac{1}{2}\,S)]-(M+\tfrac{1}{2}\,S)+[\beta^0\alpha_3-\beta^3]i\,\frac{d}{dr}$$
$$+\frac{1}{r}\sqrt{(j+\tfrac{3}{2})(j-\tfrac{1}{2})}\,[\beta^0(\Xi_3)^2\alpha_2-\tfrac{1}{2}\,\beta^2-\tfrac{1}{2}\,i\Sigma_3(1-2(\Xi_3)^2)\beta^1]$$
$$+\frac{1}{r}(j+\tfrac{1}{2})\,[-\beta^0(1-(\Xi_3)^2)\alpha_2+\tfrac{1}{2}\,\beta^2-\tfrac{1}{2}\,i\Sigma_3(1-2(\Xi_3)^2)\beta^1]$$
$$+\frac{1}{r}\,[\beta^0(\alpha_1 S_2-\alpha_2 S_1)-(\beta^1 S_2-\beta^2 S_1)]\Big\}\,\tilde{\tilde{\psi}}(r) = \dot{0}. \tag{43}$$

Inserting the explicit forms of Dirac and Duffin-Kemmer matrices into Eq. (43) gives us the corresponding set of radial equations [5].

One can easily check that for $j = 1/2$ Eq. (43) is correct, but the function $\tilde{\tilde{\psi}}(r)$ must satisfy the additional condition

$$(\Pi_{3/2}+\Pi_{-3/2})\tilde{\tilde{\psi}}(r) = 0, \tag{44}$$

which diminishes the number of independent components. There are no similar limitations if j is larger than 1/2.

4. *Final remarks*

A general method of constructing the J^2 and J_3 eigen functions was described. This method, applied to a system of one Dirac and one Duffin-Kemmer particle, allowed one to obtain the radial equation (43) equivalent to the Królikowski equation (38). The Duffin-Kemmer formalism is appropriate to describe the spin-0 or spin-1 particle, so the derived equation offers various physical applications. One such example is a quark-diquark model of baryons [4]. Nevertheless, a careful investigation of properties of Eq. (43) showed, that for many physically interesting potentials the Królikowski equation suffers from the Klein paradox at $r \to 0$ [5]. Modifications removing this Klein paradox are discussed in Ref. [6].

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