SOME σ-LIKE MODELS IN EVEN-DIMENSIONAL SPACES*

BY W. J. ZAKRZEWSKI

Department of Mathematical Sciences, University of Durham, Durham, U.K.

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We propose a class of grassmanian models in $2k$ dimensions which for $k = 1$ reduce to CP^N models and for $k = 2$ to composite SU(2) Yang-Mills models. We define and discuss the conditions of selfduality for these models and present corresponding one instanton field configurations. Next we consider a universally coupled Dirac field. We exploit the similarity of the matrices used in the construction of one-instanton field configuration to the convenient choice of Dirac gamma matrices to find zero modes of the associated Dirac background problem.

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1. Introduction

In recent years an increasing degree of attention has been paid to topological properties of field theoretical models. Field equations of these models are nonlinear and they often possess extended solutions in either Minkowski or in Euclidean space. In Minkowski space they describe real extended objects — like solitons of the Sine-Gordon or KdV equations, in Euclidean space they provide stationary points of functional integrals used in studying quantum properties of the theory. This latter case represents itself, for instance, in the existence of instanton solutions of nonabelian gauge theories.

In the functional integral formulation of a field theory we compute various integrals of the type

$$I = \int d[\phi] \exp\{-\int d^{2k}xL(\phi, \partial_\mu\phi, \ldots)\}O[\phi, \ldots]\}, \tag{1.1}$$

where $d[\phi]$ denotes functional integration measure and $O[\phi, \ldots]$ various expressions constructed out of ϕ and its derivatives [1]. The only practical way of calculating expressions like (1.1) in general is by expansion around stationary points of the action

$$S = \int d^{2k}xL(\phi, \partial_\mu\phi, \ldots) \tag{1.2}$$

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(i.e. solutions of the Euclidean "equations of motion"). The conventional purturbation theory corresponds to the $\phi = 0$ solution and the expansion around $\phi = 0$. As the action for this solution vanishes we find

$$I = g^{+l} \sum_{p=0}^{\infty} g^{P} a_{P}, \tag{1.3}$$

where g is the coupling constant. If there are further solutions they contribute to I as well. Their contribution is given by

$$I = \exp\left(-\frac{S_0}{g}\right) \sum_{l=0}^{\infty} g^{l+k} b_l, \tag{1.4}$$

where S_0 denotes the value of the classical action on such a solution. Are these further contributions important? Clearly each one of them is not — but their total contribution can become very important in some theories. In most theories we do not know and so further studies are needed in order to clarify this point.

As we have already said nonabelian gauge theories possess such solutions. To find them one has to solve

$$D_\mu F_{\mu\nu} = \partial_\mu F_{\mu\nu} - [A_\mu, F_{\mu\nu}] = 0, \tag{1.5}$$

where as usual

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{1.6}$$

These second order equations for A_μ are hard to solve. However, one can find first order equations which when solved give solutions to the full second order equations. These so called self duality equations

$$F_{\mu\nu'} = \pm {}^*F_{\mu\nu}, \tag{1.7}$$

where

$${}^*F_{\mu\nu} = \tfrac{1}{2}\, \varepsilon_{\mu\nu\alpha\beta} F_{\alpha\beta} \tag{1.8}$$

guarantee solutions of the full equations (1.5) due to the Bianchi identities

$$D_\mu^* F_{\mu\nu} = 0. \tag{1.9}$$

Similar properties are also true in other models and in different dimensions. The Yang--Mills self duality equations (1.7) were solved by Atiyah-Drinfeld-Hitchen and Manin using a construction (ADHM) [2] which involves an auxiliary σ-model like field ϕ. This V field is matrix valued, satisfies

$$V^+ V = 1 \tag{1.10}$$

and the Yang Mills A_μ field is given by

$$A_\mu = V^+ \partial_\mu V. \tag{1.11}$$

Can we exploit this composite nature of the A_μ field? Should we take seriously the more elementary V fields? It is interesting to recall that composite Yang-Mills fields arise naturally in some supergravity models [3]. These models are often considered as theories in higher dimensional spaces — and then dimensionally reduced to 4-dimensional space. Such a reduction often involves finding some classical solutions and then imposing constraints on the possible forms of quantum fluctuations about these solutions.

A simple lower-dimensional model with similar topological properties is the $\mathbf{CP}^{N-1}$ model in 2 dimensions [4]. In this case the basic field is the vector valued complex Z field which satisfies a constraint similar to (1.10) namely

$$Z^+Z = 1. \tag{1.12}$$

The action density is given by

$$(D_\mu Z)^+ D_\mu Z, \tag{1.13}$$

where D_μ is defined by

$$D_\mu \psi = \partial_\mu \psi - \psi(Z^+\partial_\mu Z). \tag{1.14}$$

As the field strength in (1.6) can be considered to be given by

$$F_{\mu\nu} = [D_\mu, D_\nu], \tag{1.15}$$

where D_μ is defined as in (1.14) with Z replaced by V we see striking similarities between the $\mathbf{CP}^{N-1}$ model in 2 dimensions and composite Yang-Mills fields in 4 dimensions. In fact we can generalize these similarities further and obtain a sequence of σ-like models in $2k$ dimensions, which for $k = 1$ and $k = 2$ reproduce the $\mathbf{CP}^1$ and composite SU(2) Yang Mills models. In the next section we present these models and discuss their properties. We show that we can define a topological charge and state appropriate selfduality relations.

Moreover, a simple generalization of a Bogomolny bound [5] allows us to show that solutions of the selfduality equations are local minima of the action. All this is discussed in the next section.

We then proceed to look for solutions to these selfduality equations. An ansatz is proposed which allows us to find solutions to these equations. We show that for $d = 2$ we recover one instanton solution of the $\mathbf{CP}^1$ model and for $d = 4$ — one instanton of the SU(2) Yang-Mills model. Next we consider the minimally coupled Dirac field. We show that the matrices used in the construction of our one instanton solution can be used to define appropriate Dirac matrices. This allows us to find zero modes solutions of the appropriate background problem [6].

Actually the problem of looking at Yang-Mills theories based on Lagrangian densities

$$L = \mathrm{tr}\,(\underbrace{F \wedge \dots F \wedge F}_{k\ \text{times}})^+(F \wedge \dots F \wedge F)$$

in $4k$ dimensions has a long history [7]. In these papers the selfduality equations are presented, shown to imply the full equations of motion, and some solutions of the equations of motion are given. Our solutions, when considered as expressions for the composite $SU(2^{2k-1})$ gauge fields coincide or are equivalent to some solutions given in reference [7].

2. Grassmann models in even dimensions

In analogy with nonabelian (grassmann) σ models in 2 dimensions our basic fields will be grassmann valued fields $Z(x)$ [8] defined over a $2k$-dimensional space ($x \in \mathbf{R}^{2k}$). A convenient representation is provided by $p \times N$ matrices,

$$Z_i^\alpha(x), \tag{2.1}$$

where $\alpha = 1 \dots N$ and $i = 1 \dots p$ denote the indices of the global and local symmetry respectively and we require that $N \geqslant 2p$.

On our fields Z we impose the constraint

$$Z^+Z = 1_p, \tag{2.2a}$$

which in our representation is given by

$$Z_\alpha^{+j} Z_i^\alpha = \delta_i^j. \tag{2.2b}$$

To construct the Lagrangian we define

$$D_\mu \psi = \partial_\mu \psi - \psi Z^+ \partial_\mu Z \tag{2.3}$$

for any grassmannian field ψ. Then

$$[D_\mu, D_\nu] = F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{2.4}$$

where

$$A_\mu = Z^+ \partial_\mu Z \tag{2.5}$$

is a composite Yang-Mills-like field. The curvature in (2.4) can be thought of as the coefficient of the form

$$D \wedge D\psi \tag{2.6}$$

and so it suggests that we consider also higher order forms

$$D^{[l]}\psi = \underbrace{D \wedge D \wedge \dots \wedge D}_{l \text{ times}}\psi. \tag{2.7}$$

Then for a Lagrangian density form in $2k$ dimension we can take

$$L d^{2k}x = \mathrm{tr}\, [D^{[k]}Z]^+ D^{[k]}Z, \tag{2.8}$$

where the fields Z are, as usual, subject to the constraint (2.2a).

As D is a covariant derivative with respect to local $U(p)$ transformations

$$\psi(x) \to \psi'(x) = \psi(x) R(x) \tag{2.9}$$

where

$$R(x) \in U(p),$$

we see that the proposed model possesses a nonabelian local $U(p)$ symmetry. Moreover in the case $k = 1$ the model reduces to the familiar grassmann model in 2 dimensions

$$L = \mathrm{tr}\,(D_\mu Z)^+ D_\mu Z \tag{2.10}$$

and in the $k = 2$ case it leads to

$$L = \mathrm{tr}\,(F^+_{\mu\nu} Z^+ Z F_{\mu\nu}) = \mathrm{tr}\, F^+_{\mu\nu} F_{\mu\nu} \tag{2.11}$$

i.e. the familiar Yang-Mills Lagrangian density for the composite field A_μ (2.5). In 6 dimensions (i.e. for $k = 3$) we use $D^{[3]}Z$. The coefficients of this term correspond to antisymmetric combinations of $D_\alpha Z F_{\mu\nu}$ and so the Lagrangian density in this case is given by

$$L = \mathrm{tr}\,(A_{\mu\alpha\beta})^+ A_{\mu\alpha\beta}, \tag{2.12}$$

where

$$A_{\mu\alpha\beta} = D_{\{\alpha} Z F_{\mu\nu\}} = D_\alpha Z F_{\mu\nu} + D_\mu Z F_{\nu\alpha} + D_\nu Z F_{\alpha\mu}. \tag{2.13}$$

In 8 dimensions (i.e. for $K = 4$) we use $D^{[4]}Z$, whose coefficients correspond to antisymmetric combinations of $ZF_{\mu\nu}F_{\alpha\beta}$ and so are given by

$$ZA_{\alpha\beta\gamma\delta} = ZF_{\{\alpha\beta}F_{\gamma\delta\}} = Z[F_{\alpha\beta}F_{\gamma\delta} + F_{\gamma\alpha}F_{\beta\delta} + F_{\alpha\delta}F_{\beta\gamma} + F_{\beta\gamma}F_{\alpha\delta} + F_{\delta\beta}F_{\alpha\gamma} + F_{\gamma\delta}F_{\alpha\beta}] \tag{2.14}$$

and again the overall Z factor in (2.13) drops out of the Lagrangian density which is given in terms of $A_{\alpha\beta\gamma\delta}$ alone

$$L = \mathrm{tr}\,(A^+_{\alpha\beta\gamma\delta} Z^+ Z A_{\alpha\beta\gamma\delta}) = \mathrm{tr}\,(A^+_{\alpha\beta\gamma\delta} A_{\alpha\beta\gamma\delta}). \tag{2.15}$$

Returning back to the form language we see that due to the constraint (2.2a) the Lagrangian density form is given by

$$L d^8 x = \mathrm{tr}\,(F \wedge F)^+ (F \wedge F), \tag{2.16}$$

where $F = D \wedge D$.

This observation generalizes and so we see that in $d = 4K$ dimensions

$$L d^{4k} x = \mathrm{tr}\,(\underbrace{F \wedge F \wedge \ldots F}_{k \text{ times}})^+ (F \wedge F \wedge \ldots F) \tag{2.17}$$

while in $d = 4k+2$ dimensions

$$L d^{4k+2} x = \mathrm{tr}\,(DZ \wedge \underbrace{F \wedge \ldots \wedge F}_{k \text{ times}})^+ DZ \wedge \underbrace{F \wedge \ldots \wedge F}_{k \text{ times}}. \tag{2.18}$$

This demonstrates the alternating patterns of models mentioned in the introduction: $d = 4k$ dimensional theories involve only composite potentials and field strengths and the overall Z fields in $D^{[l]}Z$ can be dropped while in $d = 4k+2$ dimensions covariant derivatives of the grassmann fields appear as well.

We can proceed to define a topological charge and to write down the generalized self--duality equations. As the space is even-dimensional (say $d = 2k$) a dual to a k form is again a k form. This allows us to define a topological charge density form

$$Qd^{2k}x = \mathrm{tr}\,\{[D^{[k]}Z]^{+}{*D^{[k]}}Z\}, \tag{2.19}$$

where ${*D^{[k]}}Z$ denotes the dual of $D^{[k]}Z$ and so if the coefficients of $D^{[k]}Z$ are represented by $A_{\alpha_1\ldots\alpha_k}$ the coefficients of the dual form ${*D^{[k]}}Z$ are represented by

$$*A_{\alpha_1\ldots\alpha_k} = \frac{1}{k!}\,\varepsilon_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_k}A_{\beta_1\ldots\beta_k}, \tag{2.20}$$

where $\varepsilon_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_k}$ is the completely antisymmetric tensor in $2k$ dimensions. Notice that this definition of the dual form and of the topological charge density agrees with the familiar definitions in 2 and 4 dimensions. Moreover a few lines of algebra allow us to prove that Q is a total divergence.

$$Q = \partial^{\mu}T_{\mu}, \tag{2.21}$$

where

$$T_{\mu} = \mathrm{tr}\,\varepsilon_{\mu\alpha_1\ldots\alpha_{k-1}\nu\beta_1\ldots\beta_{k-1}}(Z^{+}\partial_{\nu}Z)(\partial_{\alpha_1}Z^{+}\partial_{\beta_1}Z)\ldots(\partial_{\alpha_{k-1}}Z^{+}\partial_{\beta_{k-1}}Z) \tag{2.22}$$

which in turn establishes the topological nature of Q.

The self duality (antiselfduality) relations are now given by

$$D^{[k]}Z = \pm i^{k}{*D^{[k]}}Z, \tag{2.23}$$

and again agree with the corresponding definitions in 2 and 4 dimensions.

As the form $D^{[k]}Z$ is completely antisymmetric so is its dual ${*D^{[k]}}Z$. This antisymmetry allows us to prove that

$$\mathrm{tr}\,[[{*D^{[k]}}Z]^{+}{*D^{[k]}}Z] = \mathrm{tr}\,[[D^{[k]}Z]^{+}D^{[k]}Z] \tag{2.24}$$

which in turn gives

$$\mathrm{tr}\,[[D^{[k]}Z]^{+}D^{[k]}Z] = \tfrac{1}{4}\,\mathrm{tr}\,[[D^{[k]}Z+i^{k}{*D^{[k]}}Z]^{+}[D^{[k]}Z+i^{k}{*D^{[k]}}Z]]$$
$$+\tfrac{1}{4}\,\mathrm{tr}\,[[D^{[k]}Z-i^{k}{*D^{[k]}}Z]^{+}[D^{[k]}Z-i^{k}{*D^{[k]}}Z]] \tag{2.25}$$

which shows that the selfdual and antiselfdual fields are local minima of the action [5]. Thus solutions to the selfduality equations [2.23] will automatically solve the full equations (Euler Lagrange equations for the variational problem based on the Lagrangian density (2.8)). In each case we always restrict ourselves only to fields which also satisfy the constraints (2.2).

Let us recall that for k even $D^{[k]}Z$ and its dual $*D^{[k]}Z$ involve an overall factor Z. Thus for $k = 2p$ the selfduality (antiselfduality) relations (2.23) can be rewritten

$$\underbrace{F \wedge F \wedge \ldots \wedge F}_{p \text{ factors}} = \pm{*}(\underbrace{F \wedge F \wedge \ldots \wedge F}_{p \text{ factors}}), \tag{2.26}$$

while the corresponding relations for $k = 2p+1$ are given by

$$DZ \wedge \underbrace{F \wedge \ldots \wedge F}_{p \text{ factors}} = \pm i{*}(DZ \wedge \underbrace{F \wedge \ldots \wedge F}_{p \text{ factors}}). \tag{2.27}$$

As these relations involve the composite field strength form F but we treat them as equations for the grassmannian field Z we see that in order for the theory to have nontrivial topological properties the degrees of the local and global symmetry transformations that can be applied to the underlying grassmannian field Z (in 2.3) have to be sufficiently large. In the simplest case ($d = 2$ dimensions) there are no restrictions and so $p \geqslant 1$ and $N \geqslant 2$. Thus the simplest model ($p = 1$, $N = 2$) i.e. the $\mathbf{CP}^1$ model has already all required topological properties and though we can consider $p > 1$ and $N > 2p$ not much is gained by these generalizations. For $d = 4$ we should take $p \geqslant 2$ and so $N \geqslant 2p \geqslant 4$. The simplest of such models ($p = 2$, $N = 4$) is just the familiar HP^1 model, used in the ADHM construction of a SU(2) Yang--Mills one instanton solution [2]. Here not much (from the topological point of view) is gained from increasing p; the increase of N is required in order to find multi-instanton solutions. However, for the simplest nontrivial field configuration $p = 2$, $N = 4$ is required. In a similar way we expect to increase p and N in higher dimensions. For $d = 6$ we can take $p = 4$, $N = 8$ and in general for $d = 2k$ we will take $p = 2^{k-1}$, $N = 2^k$. Such a choice may not be strictly required — but as we will show in the next section, it is very convenient and in some ways very natural. It allows us to propose an ansatz which in turn allows us to find one instanton solutions of the selfduality equations (2.26)–(2.27) in arbitrary even dimensional space.

3. *One instanton solutions and their properties*

The discussion in the previous section centred on proposing self-duality relations (2.26)–(2.27). Here will find some solutions of these equations. In seeking such solutions we will require that their action is finite and nonzero (i.e. they correspond to the familiar instantons of 2 and 4 dimensional theories). This restriction is imposed for "physical reasons" — such solutions provide important contributions to the quantized version of the theory defined in terms of functional integrals in Euclidean space; they are genuinely nontrivial from the topological point of view. Relaxing this condition opens the door to the whole plenora of topologically trivial (by maybe physically important) solutions. For example if the grassmannian Z in (2.1) does not depend on one variable x^i of space-time it is automatically a solution of the relations (2.26) or (2.27).

To find solutions to our selfduality relations (2.23)–(2.26) we try the following ansatz: we take $d = 2k$, consider Z_i^α where $i = 1, \dots 2^{k-1}$, $\alpha = 1, \dots, 2^k$, set

$$Z_i^\alpha = \frac{f_i^\alpha}{\sqrt{1+x^2}}, \tag{3.1}$$

where $x^2 = x_1^2+x_2^2+ \dots +x_{2k}^2$, and choose

$$f_i^\alpha = \delta_i^{\alpha-k} \quad \text{for} \quad \alpha > k$$

$$f_i^\alpha = [\Gamma_\mu(k)X_\mu]_i^\alpha \quad \text{for} \quad \alpha \leqslant k. \tag{3.2}$$

The problem is reduced to finding a suitable set of $2k$ $2^{k-1}\times 2^{k-1}$ matrices Γ_μ. Before we construct them let us observe that the familiar one instanton solutions of the $\mathbf{CP}^1$ model and SU(2) Yang Mills theory (in the regular gauge) are reproduced by this ansatz. They correspond to, respectively, $\Gamma_1(1) = 1$, $\Gamma_2(1) = i$ and

$$\Gamma_1(2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1, \quad \Gamma_2(2) = i\begin{bmatrix} 0 & \Gamma_1^+(1) \\ \Gamma_1(1) & 0 \end{bmatrix} = i\sigma_1,$$

$$\Gamma_3(2) = i\begin{bmatrix} 0 & \Gamma_2^+(1) \\ \Gamma_1(1) & (0) \end{bmatrix} = i\sigma_2, \quad \Gamma_4(2) = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3. \tag{3.3}$$

Hence it is natural to define $\Gamma_\mu(k)$ inductively

$$\Gamma_1(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\Gamma_l(k) = i\begin{bmatrix} 0 & \Gamma_{l-1}^+(k-1) \\ \Gamma_{l-1}(k-1) & 0 \end{bmatrix}, \quad l = 2, \dots 2k-1,$$

$$\Gamma_{2k}(2k) = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.4}$$

It is easy to prove that our matrices $\Gamma_\mu(k)$ satisfy

$$\Gamma^+{}_\mu(k)\Gamma_\nu(k)+\Gamma^+{}_\nu(k)\Gamma_\mu(k) = \Gamma_\mu(k)\Gamma^+{}_\nu(k)+\Gamma_\nu(k)\Gamma^+{}_\mu(k) = 2\delta_{\mu\nu}1. \tag{3.5}$$

This shows that our grassmannian Z is properly normalized (i.e. satisfies (2.2a)). Moreover a few lines of algebra show that

$$(D_\mu Z)_i^\alpha = \partial_\mu Z_i^\alpha - Z_j^\alpha Z_\beta^{+j}\partial_\mu Z_i^\beta = \frac{1}{\sqrt{1+x^2}}[(1-P)\Gamma_\mu]_i^\alpha, \tag{3.6}$$

where P is the projector

$$P_\alpha^r = Z_i^r Z_\alpha^{+i}, \tag{3.7}$$

and that

$$[F_{\mu\nu}]^i_j = [\partial_\mu Z^+(1-P)\partial_\nu Z]^i_j - (\mu \leftrightarrow \nu)$$

$$= \frac{1}{(1+x^2)^2}[\Gamma^+_\mu \Gamma_\nu]^j_i - (\mu \leftrightarrow \nu). \tag{3.8}$$

In addition, our choice of $\Gamma_\mu(k)$ matrices and the antisymmetry of the forms appearing in (2.23) and (2.25) reduces the condition of selfduality to

$$\Gamma_1 \Gamma_2^+ \dots \Gamma_k = \pm (i)^k \Gamma_{k+1} \Gamma^+_{k+2} \dots \Gamma_{2k}. \tag{3.9}$$

However, due to (3.5), this condition is equivalent to

$$\Gamma_1 \Gamma_2^+ \Gamma_3 \Gamma_4^+ \dots, \Gamma^+_{2k} = \pm (i)^k 1. \tag{3.10}$$

It is easy to show that this condition is satisfied by our $\Gamma_\mu(k)$. To do this we observe that as

$$\Gamma_1(k)\Gamma_2^+(k) \dots \Gamma^+_{2k}(k)$$

$$= i\begin{bmatrix} \Gamma_1^+(k-1)\Gamma_2(k-1)\dots\Gamma_{2k-2}(k-1) & 0 \\ 0 & -\Gamma_1(k-1)\Gamma_2^+(k-1)\dots\Gamma^+_{2k-2}(k-1) \end{bmatrix} \tag{3.11}$$

and

$$\Gamma_1^+(k-1)\Gamma_2(k-1)\dots\Gamma_{2k-2}(k-1) = -\Gamma_1(k-1)\Gamma_2^+(k-1)\dots\Gamma^+_{2k-2}(k-1) \tag{3.12}$$

we have

$$\Gamma_1(k)\Gamma_2^+(k)\dots\Gamma^+_{2k}(k) = i\Gamma_1(k-1)\Gamma_2^+(k-1)\dots\Gamma^+_{2k-2}(k-1) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.13}$$

and so repeating this procedure $k-1$ times we find

$$\Gamma_1(k)\Gamma_2^+(k)\dots\Gamma^+_{2k}(k) = -i^k 1. \tag{3.14}$$

What are the properties of our grassmannians? Clearly, as we noted in the previous section, they correspond to the minima of the action and so satisfy the corresponding equations of motion. What is the value of this action?

Let us look first at the case $d = 4k$ dimensions. Then, using the result (3.8) we see that all x dependence of the action density resides in the factor $(1+x^2)^{-2}$ associated with each field strength tensor F. So the action is proportional to

$$S \sim \int d^{4k}x \frac{1}{(1+x^2)^{4k}} = S^{4k-1} \int_0^\infty dx \frac{x^{4k-1}}{(1+x^2)^{4k}} = \pi^{2k} \frac{\Gamma(2k)}{\Gamma(4k)} \tag{3.15}$$

as S^{4k-1}, the volume of the $4k-1$ sphere in $4k$ dimensions is given by

$$S^{4k-1} = \pi^{2k} \frac{2}{\Gamma(2k)}. \tag{3.16}$$

In $d = 4k+2$ dimensions in addition to the x dependence in the field strength factors Γ, x appears also in the DZ term. However as

$$(DZ \wedge F \wedge \dots \wedge F)^+ = F^+ \wedge F^+ \wedge \dots \wedge (DZ)^+ \tag{3.17}$$

we see that the DZ factors contribute

$$\frac{1}{(1+x^2)}\Gamma_\nu^+(1-P)^2\Gamma_\mu = \frac{1}{(1+x^2)^2}\Gamma_\nu^+\Gamma_\mu \tag{3.18}$$

and so the action is proportional to

$$S \sim \int d^{4k+2}x \frac{1}{(1+x^2)^{4k+2}} = \pi^{2k+1}\frac{\Gamma(2k+1)}{\Gamma(4k+2)}. \tag{3.19}$$

The coefficient of proportionality in (3.16) and (3.19) is quite difficult to determine. It comes from the trace of strings of Γ_μ and Γ_ν^+. As our Γ_μ matrices satisfy (3.5) its calculation is straightforward though tedious. It is easy, however, to check that this coefficient is non-zero and so that our field configurations represent genuine nontrivial stationary points of the action.

Our field configurations are clearly not very general. First of all, using translational invariance we can shift vector x_μ in $x_\mu\Gamma_\mu$ by an arbitrary constant $x_\mu \to x_\mu - a_\mu$; moreover $\alpha > k$ components of f_i^α in (3.2) do not have to be given by a Krönecker delta function (but then the normalization in (3.1) has to be appropriately modified). Further generalizations can be performed exploiting conformal invariance of our action (2.8). We could also exploit the global $U(N)$ and local $U(p)$ symmetry of the action based on our grassmannian (2.1).

Even though we discussed the general case we presented our ansatz for a $d = 2k$ dimensional model for $p = 2^{k-1}$ and $N = 2^k$. Clearly we can increase these values. In particular, for $k = 1$, the increase in N takes us from CP^1 to CP^{N-1}. The increase in p leads to a nonabelian grassmannian. With our ansatz the increase in N is particularly easy if N increases in multiples of 2^{k-1}. Then for f_i^α in (3.2) corresponding to additional values of α we can take $\lambda(x-a)_\mu\Gamma_\mu$ — and, of course, we have to modify appropriately the normalization factor in (3.1). The increase in p (together with N) can be achieved for example by embeddings.

So far we have discussed only one instanton field configurations. Moreover; these field configurations were such that the composite fields A_μ (of (2.5)) were in the so called — regular gauge. Can we also find multi-instanton configurations [2]? In $d = 2$ this is simple but for $d \geq 4$ the situation is more complicated. In the Yang-Mills field case we have to go to the singular gauge and as ADHM construction showed it is convenient (and in their construction essential) to increase N linearly with q (instanton number). In fact we would take $N = 2q+2$ $(d = 4)$.

The necessity of going to the singular gauge complicates the calculations; we have so far been unable to determine whether we can conveniently adapt ADHM construction to generate multi-instanton configurations for $d > 4$.

4. The associated background Dirac problem

Now we turn our attention to the problem of a fermion in the fixed background of the instanton composite Yang-Mills-like field A_μ [6]. Such a fermion satisfies

$$\gamma_\mu(\partial_\mu\psi - \psi A_\mu) = 0, \tag{4.1}$$

where

$$A_\mu = Z^+\partial_\mu Z \tag{4.2}$$

and the fermion field ψ transforms under the local $U(2^{k-1})$ transformations in the same way the grassmann field Z.

For the $2k$ Dirac gamma matrices γ_μ we can take

$$\gamma_\mu(k) = \begin{bmatrix} 0 & \Gamma_\mu^+(k) \\ \Gamma_\mu(k) & 0 \end{bmatrix}, \quad \mu = 1, \dots 2k. \tag{4.3}$$

Notice that due to (3.5) this is a possible choice as these matrices satisfy as required, the Clifford algebra condition of the Dirac gamma matrices

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}1. \tag{4.4}$$

Moreover, this is a very convenient choice as it will allow us to solve the background equations (4.1).

To do this we define a generalization of the familiar γ_5 of 4 dimension

$$\gamma_h = (-i)^k \sum_{i=1}^{2k} \gamma_i = (-i)^k \begin{pmatrix} \Gamma_1^+\Gamma_2 \dots \Gamma_{2k} & 0 \\ 0 & \Gamma_1\Gamma_2^+ \dots \Gamma_{2k}^+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.5}$$

Its eigenstates play the role of chiral eigenstates in 4 dimensions.

Next we let

$$\psi = \begin{pmatrix} \psi_u \\ \psi_D \end{pmatrix}$$

such that ψ_u and ψ_D are $2^{k-1} \times 2^{k-1}$ matrices in spin and the local symmetry group. The equations of ψ_u and ψ_D decouple:

$$\Gamma_\mu^+(\partial_\mu\psi_D - \psi_D Z^+\partial_\mu Z) = 0, \tag{4.6a}$$

$$\Gamma_\mu^+(\partial_\mu\psi_u - \psi_u Z^+\partial_\mu Z) = 0. \tag{4.6b}$$

Of course, in our case

$$Z^+\partial_\mu Z = \frac{(\Gamma^+x)\Gamma_\mu - x_\mu 1}{1+x^2}. \tag{4.7}$$

We can now essentially repeat the discussion of B. Grossman [6] and show that there exists a normalizable solution to Eq. (4.6b).

To do this we put

$$\psi_u = (1+x^2)^a \psi_0, \tag{4.8}$$

where ψ_0 is a constant spinor.

Then, as

$$\Gamma_\mu(x\Gamma^+)\Gamma_\mu = -(2k-2)(\Gamma x) \tag{4.9}$$

we find that (4.6b) is satisfied if $a = 1/2-k$. This solution is normalizable (if $k > 1$) as

$$\int d^{2k}x|\psi|^2 \sim \int d^{2k}x \frac{1}{(1+x^2)^{2k-1}} < \infty \tag{4.10}$$

and for $k = 2$ coincides with the solution of Grossman. For $k = 1$ we have to proceed differently. Its solution, however, can also be found with ease.

One can also find solutions to Eq. (2.6a). They are given by

$$\psi = \psi_0(\Gamma \cdot x)\frac{(1+x^2)^{k-\frac{1}{2}}}{x^{2k}}, \tag{4.11}$$

but, they are not normalizable. Let us finish with some remarks about further work and some open problems.

Clearly an important problem is that of finding further solutions to the selfduality relations (2.26) and (2.27). Some modification of the ADHM construction [3] is required. The tantalizing connection between Dirac gamma matrices and Γ_μ matrices used in the construction of one instanton solutions calls for further applications. It is easy to check that some straightforward generalizations do not go through. For example the beautiful result [9] for the scalar Green function

$$D^2G = -1, \quad G = \frac{Z^+(x)Z(y)}{4\pi^2(x-y)^2} \tag{4.12}$$

does not generalize to

$$G \propto \frac{Z^+(x)Z(y)}{(x-y)^{2(k-1)}}. \tag{4.13}$$

At the same time the more physically relevant questions such as "what is the relevance of all this to supergravity, dimensional reduction, etc...?" also remain unanswered.

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