THE BACKGROUND FIELD METHOD IN THE AXIAL GAUGE

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(*Received July 11, 1985; final version received December 13, 1985*)

The background field method is applied to Yang-Mills theories (e.g. QCD) in the axial gauge. It provides a simple and elegant way to obtain relations between renormalization constants in QCD. Constraints on the form of gauge-fixing conditions in the background field method are derived.

PACS numbers: 11.15.Kc

1. *Introduction to the background field method*

The background field method was introduced by DeWitt [1]—extension of the formalism was made by t'Hooft [2], DeWitt [3], Boulware [4] and by Abbott [5].

Because in Ref. [6] one can find a beautiful introduction to the background field method, we shall be brief. However, our way will be slightly different, in order to obtain constraints on the form of gauge fixing term in the background field method.

We begin with the generating functional in Yang-Mills theories with the gauge fixing term depending on an additional auxiliary field A^a. For simplicity, we consider the theory without fermions.

$$Z_A[J] = N\int [dQ^a_\mu]\Delta(Q, A)\exp\left\{i\int d^4x\left[\mathscr{L}[Q]-\frac{1}{2\lambda}[G^a(Q, A)]^2+J^a_\mu Q^{a\mu}\right]\right\}. \quad (1)$$

Using $G^a(Q, A)$ (instead of usual $G^a(Q)$) does not affect gauge-invariant S matrix. ω^b is the parameter of the infinitesimal gauge transformation on Q. $\Delta(Q, A)$ is the Faddeev-Popov (F-P) determinant, which is defined (for $\lambda \to 0$) as:

$$\Delta^{-1}(Q, A) = \int Dg \prod_x \delta[G^a({}^gQ, {}^gA)], \quad (2)$$

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where g denotes full gauge transformation. F-P determinant is invariant under gauge transformations, as can be seen from

$$\Delta^{-1}({}^{g}Q, {}^{g}A) = \int Dh \prod_{x} \delta[G^{a}({}^{hg}Q, {}^{hg}A)]$$

$$= \int D(hg) \prod_{x} \delta[G^{a}({}^{hg}Q, {}^{hg}A)] = \Delta^{-1}(Q, A) \tag{3}$$

(we used group properties of gauge transformations and the invariance properties of the measure Dg). F-P determinant is usually calculated near the identity of the group as:

$$\Delta(Q, A) = \det \frac{\delta G^{a}(Q, A)}{\delta \omega^{b}}, \tag{4}$$

where ω^{b} is the parameter of a gauge transformation on Q and A.

After a change of integration variable $Q \to Q+A$, we can write ([5])

$$Z_{A}[J] \exp\{-i \int d^{4}x J \cdot A\} = \tilde{Z}[J, A]; \tag{5}$$

where

$$\tilde{Z}[J, A] = N \int [dQ] \Delta(Q+A, A)$$

$$\times \exp\left\{i \int d^{4}x \left[\mathscr{L}[Q+A] - \frac{1}{2\lambda}[G^{a}(Q+A, A)]^{2} + J \cdot Q\right]\right\}. \tag{6}$$

The great advantage of the background field method lies in the fact that there exists a class of functions G^{a} that retains explicit gauge invariance.

Let us make the following transformations in Eq. (6)

$$A_{\mu}^{a} \to A_{\mu}^{a'} = A_{\mu}^{a} - f^{abc} \omega^{b} A_{\mu}^{c} + \frac{1}{g} \partial_{\mu} \omega^{a}, \tag{7}$$

$$J_{\mu}^{a} \to J_{\mu}^{a(\mathrm{rot})} = J_{\mu}^{a} - f^{abc} \omega^{b} J_{\mu}^{c} \tag{8}$$

and make the change of integration variables

$$Q_{\mu}^{a} \to Q_{\mu}^{a(\mathrm{rot})} = Q_{\mu}^{a} - f^{abc} \omega^{b} Q_{\mu}^{c}. \tag{9}$$

$J \cdot Q$ in (6) is clearly invariant under (8) and (9). Adding (7) and (9) we find:

$$(A_{\mu}^{a} + Q_{\mu}^{a}) \to (A_{\mu}^{a} + Q_{\mu}^{a})' \tag{10}$$

(by prime we denote gauge transformation).

The Lagrangian (by construction) is invariant under gauge transformations of its argument.

So, we see that

$$\tilde{Z}[J, A] = \tilde{Z}[J^{(\mathrm{rot})}, A'] \tag{11}$$

when the following condition is fulfilled ($G^a(Q+A, A) = \tilde{G}^a(Q, A)$):

$$[\tilde{G}^a(Q^{(rot)}, A')]^2 = [\tilde{G}^a(Q, A)]^2 \tag{12}$$

(as previously mentioned, the F-P determinant is invariant under gauge transformations on its arguments). If we denote by $\bar{G}(Q, A)$ an 8-vector (in QCD) with components $\tilde{G}^a(Q, A)$, then condition (12) will be fulfilled if and only if

$$\bar{G}(Q^{(rot)}, A') = e^{-i\vartheta^a T^a}\bar{G}(Q, A). \tag{13}$$

Note, that generally ϑ^a would not have to be equal to ω^a — G^a = const is an example — however, as will be shown, due to physical restrictions on the form of G^a, ϑ^a must be equal to ω^a. The prime denotes, as before, gauge transformations.

Examples of $\tilde{G}^a$ satisfying (12) are given in Section 2.

Now, we shall use results of Ref. [5]. We define the functional generating connected Green functions as:

$$\tilde{W}[J, A] = -i \ln \tilde{Z}[J, A] \tag{14}$$

and (by Legendre transformation) the effective action generating one-particle irreducible (1PI) Green functions as:

$$\tilde{\Gamma}[\tilde{Q}, A] = \tilde{W}[J, A] - \int d^4x J \cdot \tilde{Q} \tag{15}$$

with

$$\tilde{Q}^a_\mu = \frac{\delta \tilde{W}[J, A]}{\delta J^{a\mu}}.$$

From (11), (14) and (15) we can see, that

$$\tilde{\Gamma}[\tilde{Q}, A] = \tilde{\Gamma}[\tilde{Q}^{(rot)}, A'] \tag{16}$$

and hence

$$\tilde{\Gamma}[0, A] = \tilde{\Gamma}[0, A'] \tag{17}$$

which is one of the most important results of the background field method. The effective action (15) is invariant with respect to gauge transformations of A. It allows one to use "naive" Ward identities, simplifies search for possible counterterms etc. Green functions obtained from $\tilde{\Gamma}[0, A]$ are calculated in the following way: we put only A on external legs (because $\tilde{Q} = 0$), and Q only on internal legs (because only Q is integrated over). The only thing left to be shown now is the connection between the background field effective action and the effective action written in terms of the original fields Q and A.

For $Z_A[J]$, Eq. (1), with original field variables Q and A, we define (similarly to $\tilde{Z}[J, A]$):

$$W_A[J] = -i \ln Z_A[J] \tag{18}$$

and

$$\Gamma_A[\bar{Q}] = W_A[J] - \int d^4x J \cdot \bar{Q} \tag{19}$$

with

$$\bar{Q}^a_\mu = \frac{\delta W_A[J]}{\delta J^{a\mu}}.$$

Simple calculation shows ([5]) that

$$\tilde{\Gamma}[0, A] = \Gamma_A[\bar{Q}]_{\bar{Q}=A}. \tag{20}$$

This equation shows the equality of 1PI Green functions calculated in the background field method and in the theory described by the generating functional (1). The connection between the theory without the background field and the theory described by (1) is given in Appendix A (for the case of the axial gauge).

2. *Possible forms of gauge fixing terms*

Now we want to make use of Eq. (13) to obtain possible forms of gauge fixing terms in the background field method.

The distinction of two cases should be made: either we treat both, Q and A perturbatively or only Q is treated perturbatively.

In the first case, the gauge fixing term must have the term $K_\mu Q^{\mu a}$, where K_μ is an operator that does not contain A or Q. That term is useful (for pure Yang-Mills theory even necessary) to perform integration in Eq. (5) (or, equivalently, to obtain the propagator of field Q). In order to have the renormalizable theory, there must appear at most four point vertices — so, the only acceptable terms in $\tilde{G}^a$ are Q, A, Q^2, A^2, QA (in this case the dimension of the operator $[\tilde{G}^a]^2$ will not exceed four). If we require K_μ to contain no more than one operator ∂ (and to be local), we have two kinds of $\tilde{G}^a$ satisfying (13) (up to an overall rotation of $\tilde{G}$) and fulfilling all the above conditions:

a) the axial gauge

$$\tilde{G}^a(Q, A) = n^\mu Q^a_\mu, \tag{21}$$

n_μ must not depend on Q nor A.

b) the covariant gauge

$$\tilde{G}^a(Q, A) = s^{\mu\nu}(\partial_\mu Q^a_\nu - gf^{abc}Q^b_\mu A^c_\nu) \tag{22}$$

$s_{\mu\nu}$ — symmetric — depends neither on Q nor on A.

Putting in b): $s_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ ($n = (1, 0, 0, 0)$), we have "Coulomb-like" gauge; putting $s_{\mu\nu} = g_{\mu\nu}$ we recover the usual background field gauge condition.

Unlike the usual Yang-Mills theory, in the background field method only two choices of gauge are acceptable.

We see, that in both cases ϑ^a in Eq. (13) is equal to ω^a (parameter of rotation of Q) — the statement mentioned earlier. Any combination of a) and b) is also acceptable.

The second case (only Q is treated perturbatively) is possible in very simple cases only. In such cases there is much broader class of $\tilde{G}^a$ satisfying (13) and with no higher than second power of Q (now vertices connect only Q fields — requirement of renormalizability does not constrain the number of powers of the A fields in the Lagrangian — contrary to the previous case). For example, if we find $\tilde{G}^a(Q, A)$ satisfying (13), then $\tilde{G}^a([F^{a\mu\nu}(A)F^a_{\mu\nu}(A)]^l Q, A)$ with arbitrary l also satisfies (13). As previously, $\tilde{G}^a$ should contain term linear in Q (with a coefficient that may depend on A) to make possible integration in Eq. (6).

3. *Renormalization constants in the axial gauge*

We shall consider here the axial gauge in the background field method.

Feynman rules obtained from (6) for $\tilde{G}^a(Q, A) = n_\mu Q^{\mu\nu}$ are identical to those in the usual theory (without the background field) and remain the same under the interchange $Q \leftrightarrow A$; only the propagator of A is not defined, but A appears only as the external legs. This is true also in presence of fermions. Equations (5) and (6) in presence of fermions read:

$$Z_A[J, \bar{\eta}, \eta] \exp\{-i \int d^4x J \cdot A\} = \tilde{Z}[J, \bar{\eta}, \eta; A],$$

where

$$Z[J, \bar{\eta}, \eta; A] = N \int [dQ][d\bar{\psi}][d\psi]\Delta(Q+A, A)$$

$$\times \exp\left\{i \int d^4x \left[\mathscr{L}_{\text{YM}}[Q+A] + i\bar{\psi}_i(\delta_{ij}\not{\partial} - g(\not{Q}^a + \not{A}^a)T^a_{ij})\psi_j\right.\right.$$

$$\left.\left. - \frac{1}{2\lambda}[G^a(Q+A, A)]^2 + J \cdot Q + \bar{\eta}\psi + \bar{\psi}\eta\right]\right\}.$$

So, we see that coupling $\bar{\psi}Q\psi$ is the same as $\bar{\psi}A\psi$ one.

Following the standard arguments ([5]), we introduce renormalization constants:

$$A^0_\mu = Z_A^{1/2} A_\mu \mu^{-\varepsilon/2}, \tag{23}$$

$$g_0 = Z_g g \mu^\varepsilon. \tag{24}$$

Renormalization of the field Q is not needed (field Q appears only inside loops — we would have two $Z_Q^{1/2}$ from renormalization of the field at each vertex, and Z_Q^{-1} from renormalization of the propagator).

All divergences in $\tilde{\Gamma}(0, A)$ must take the form: infinite constant times $(F^a_{\mu\nu})^2$ (due to the preserved background gauge invariance). $F^a_{\mu\nu}$ is renormalized by

$$F^a_{\mu\nu} = Z_A^{1/2}\mu^{-\varepsilon/2}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g Z_g Z_A^{1/2} f^{abc} A^b_\mu A^c_\nu]. \tag{25}$$

Renormalized $F^a_{\mu\nu}$ will have group-covariant form only if $Z_g = Z_A^{-1/2}$, so (we define $\alpha := g^2/4\pi$, hence $Z_\alpha = Z_g^2$):

$$Z_\alpha = Z_A^{-1} \tag{26}$$

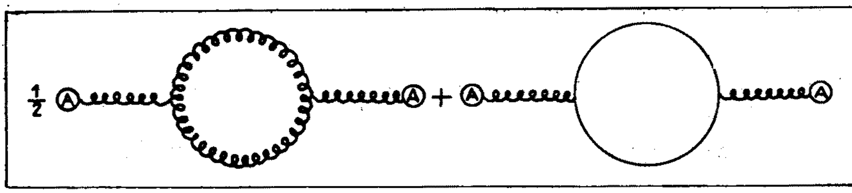

Fig. 1. Diagrams contributing to Z_A in order α

$Z_A - 1$ is calculated (to first order in α) from the diagrams shown in Fig. 1 (recall, that in the axial gauge, Faddeev-Popov ghosts do not couple to the physical fields). Feynman rules are the same for the background field vertices and the usual theory vertices (that is not true, for example, if we work in covariant gauges). We must then have:

$$Z_A = Z_3 \tag{27}$$

(definition of renormalization constants in QCD is given in Ref. [7]). From (26) we have

$$Z_\alpha = Z_3^{-1}. \tag{28}$$

We see that the background field method gives result (28) in a very straightforward way.

The equality of the bare and renormalized Lagrangians gives the equation (Ref. [7] Eqs. 12.126b and 12.125):

$$Z_\alpha = Z_1^2 Z_3^{-3} = Z_4 Z_3^{-2} = Z_3^{-1} Z_{1F}^2 Z_2^{-2}. \tag{29}$$

So (from (28) and (29)),

$$Z_{1F} = Z_2 \tag{30}$$

and

$$Z_3 = Z_1 = Z_4. \tag{31}$$

We see that equations (30) and (31) make QCD in the axial gauge "QED-like". Calculation of Z_3 in the axial gauge is given, for example, in Ref. [8].

4. *Conclusions*

In the present work, the axial gauge in the background field method is considered. It is shown that the axial gauge is the only possible (physically acceptable) choice of a gauge in the background field method besides commonly used covariant gauge (as follows from considerations of Section 2). The background field method provides a simple and straightforward method of obtaining relations (28), (30) and (31), making the axial gauge so attractive. The connection between the usual theory and the theory with the background field is shown.

The author is thankful to Prof. S. Pokorski for many useful suggestions and discussions and to Dr C. Sachrajda for critical reading of the manuscript.

APPENDIX A

We shall now show that using changed gauge fixing term (in (1) we have $G^a = n_\mu(Q^{a\mu} - A^{a\mu})$ while in the usual theory we have $G^a = n_\mu Q^{a\mu}$) does not change any 1PI Green function (except 1-point Green function). Let us write (1) with $G^a = n_\mu(Q^{a\mu} - A^{a\mu})$ (we know that Faddeev-Popov determinant is independent of fields in the axial gauge)

$$Z_A[J] = N\int [dQ]\exp\left\{i\int d^4x\left[\mathscr{L}[Q] - \frac{1}{2\lambda}[n\cdot(Q^a - A^a)]^2 + J\cdot Q\right]\right\}$$

$$= \exp\left\{-\frac{i}{2\lambda}(n\cdot A^a)^2\right\}Z[\tilde{J}], \tag{A1}$$

where Z is the generating functional in the usual axial gauge and

$$\tilde{J}^a_\mu = J^a_\mu + \frac{1}{\lambda}n_\mu(n\cdot A^a). \tag{A2}$$

From (A1) we have

$$W_A[J] = W[\tilde{J}] - \frac{1}{2\lambda}\int d^4x(n\cdot A^a)^2. \tag{A3}$$

Differentiating both sides of (A3) over J, we find:

$$(\bar{Q}:=)\frac{\delta W_A[J]}{\delta J} = \frac{\delta W[\tilde{J}]}{\delta \tilde{J}}(=:\tilde{Q})$$

and

$$\Gamma_A[\bar{Q}] = W_A[J] - \int d^4x J\cdot\bar{Q} = \Gamma[\bar{Q}]$$

$$+\left\{\frac{1}{\lambda}\int d^4x(n\cdot\bar{Q}^a)(n\cdot A^a) - \frac{1}{2\lambda}\int d^4x(n\cdot A^a)^2\right\}.$$

So, we have

$$\frac{\delta\Gamma_A[\bar{Q}]}{\delta\bar{Q}^{\mu a}} = \frac{\delta\Gamma[\bar{Q}]}{\delta\bar{Q}^{\mu a}} + \frac{1}{\lambda}n_\mu(n\cdot A^a)$$

(which is equal to (A2)) and

$$\frac{\delta^n\Gamma_A[\bar{Q}]}{\delta\bar{Q}^{\mu_1 a_1}\dots\delta\bar{Q}^{\mu_n a_n}} = \frac{\delta^n\Gamma[\bar{Q}]}{\delta\bar{Q}^{\mu_1 a_1}\dots\delta\bar{Q}^{\mu_n a_n}} \quad \text{for} \quad n\geqslant 2.$$

So n-point 1PI Green functions (for $n \geqslant 2$ — for example propagator) are identical for both gauges.

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