PERIODIC SOLUTIONS OF YANG-MILLS-DIRAC THEORY

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New periodic in time and space independent solutions of Yang-Mills equations coupled to massless Dirac bispinors are presented.

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1. *Introduction*

The Yang-Mills-Dirac classical theory has been studied for several years but still the number of known solutions is rather limited [1]. In this paper we present new solutions of Yang-Mills equations coupled to massless Dirac equations. They are periodic in time and space independent.

The order of the work is the following: In Section 2 we present the SU(2) Yang-Mills-Dirac equations and recapitulate the Jackiw-Rebbi formalism [2]. Section 3 comprises solutions which were found previously [3]; they give an interesting example of dynamic bifurcation. New solutions are described in Section 4; for all of them the evolution of a Dirac bispinor is influenced by a Yang-Mills potential, but the feedback vanishes since the fermionic current is trivial. In the last Section we briefly comment on the possible utilization of the solutions.

2. *The equations of motion*

We will study a system of coupled SU(2) Yang-Mills-Dirac (YMD) equations. The SU(2) strength field tensor $F^a_{\mu\nu}$ can be expressed in terms of Yang-Mills (YM) potentials A^a_μ as follows

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon^{abc} A^b_\mu A^c_\nu.$$

Here ε^{abc} is the Levi-Čivita completely antisymmetric tensor, Latin indices $a, b, c\ldots = 1, 2, 3$ denote isospin directions while Latin $i, j, k = 1, 2, 3$ and Greek $\mu, \nu\ldots = 0, 1, 2, 3$ are space and space-time labels, respectively. The metric of Minkowski space is $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$.

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Dirac spinor consists of two 2×2 matrices

$$\psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix}, \tag{1}$$

whose first (spinor) index transforms under the Poincaré group while the second (isospinor) index transforms under the SU(2) gauge group.

The massless Dirac equations coupled to YM equations read

$$(\mathbf{1})^{AB}\partial_0(\psi^{\mp})^{BC} \mp (\sigma^i)^{AB}\partial_i(\psi^{\mp})^{BC}$$
$$+\left[(\mathbf{1})^{AB}A_0^a\left(\frac{\sigma^a}{2i}\right)^{CD} \mp (\sigma^i)^{AB}A_i^a\left(\frac{\sigma^a}{2i}\right)^{CD}\right](\psi^{\mp})^{BD} = 0, \tag{2}$$

$$\partial_\mu F^{a\mu\nu} + \varepsilon^{abc}A_\mu^b F^{c\mu\nu} = -\sum_{K=\pm} K(\psi^{K+})^{AB}(\sigma^\nu)^{AA'}\left(\frac{\sigma^a}{2}\right)^{BB'}(\psi^K)^{A'B'}. \tag{3}$$

Here $A, A', B, B' \ldots = 1, 2$, $+$ denotes the hermitian conjugation, σ's are Pauli matrices ($\sigma^0 = 1$) and Dirac matrices γ are given in the chiral representation,

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

We insert, following Jackiw and Rebbi [2]

$$\psi^{\mp} = \varphi^{\mp}\sigma^2 \tag{4}$$

into (2) and then use the identity

$$\sigma^2(\vec{\sigma})^T = -\vec{\sigma}\sigma^2;$$

this yields the following Dirac equations

$$\partial_0\varphi^{\mp} \mp \sigma^i\partial_i\varphi^{\mp} - A_0^a\varphi^{\mp}\frac{\sigma^a}{2i} \pm A_i^a\sigma^i\varphi^{\mp}\frac{\sigma^a}{2i} = 0. \tag{5}$$

The fermionic current is given now by

$$j^{a0} = \mathrm{tr}\left[((\varphi^+)^+\varphi^+ + (\varphi^-)^+\varphi^-)\frac{\sigma^a}{2}\right], \tag{5a}$$

$$j^{ai} = \mathrm{tr}\left[((\varphi^+)^+\sigma^i\varphi^+ - (\varphi^-)^+\sigma^i\varphi^-)\frac{\sigma^a}{2}\right]. \tag{5b}$$

From now on there is no need to distinguish between spinor and isospinor indices. The product appearing in formulae (4), (5) is the ordinary matrix product.

We will look for position-independent solutions of Eqs. (3), (5). Let us begin with the study of evolution of a Dirac field in a field of a prescribed YM time-dependent potential. We make the following Ansatz

$$A_0^a = 0, \quad A_3^a = A_i^3 = 0. \tag{6}$$

The matrices $\varphi^{\mp}$ could be expanded as follows

$$\varphi^{\mp} = \varphi_0^{\mp}\mathbf{1} + \varphi_k^{\mp}\sigma^k, \tag{7}$$

where $\varphi_0^{\mp}$, $\varphi_k^{\mp}$ are complex functions. Inserting (7) into (5), employing the identities

$$\begin{aligned} \sigma^i\sigma^k &= \delta^{ik} + i\varepsilon_{ikl}\sigma^l, \\ \sigma^i\sigma^k\sigma^a &= i\varepsilon_{ika} + \delta^{ik}\sigma^a - \varepsilon_{lik}\varepsilon_{lam}\sigma^m \end{aligned} \tag{8}$$

and using (6), one arrives at the following equations:

$$i\partial_0\varphi_0^{\mp} \pm \frac{A_a^a}{2}\varphi_0^{\mp} \pm i\varepsilon_{ika}A_i^a\varphi_k^{\mp} = 0, \tag{9a}$$

$$i\partial_0\varphi_k^{\mp} \pm i\varepsilon_{iak}\frac{A_i^a}{2}\varphi_0^{\mp} \pm \frac{A_a^k}{2}\varphi_a^{\mp} \mp \frac{A_a^a}{2}\varphi_k^{\mp} \pm \frac{A_k^a}{2}\varphi_a^{\mp} = 0. \tag{9b}$$

The spinor current j_μ^a is expressed below in terms of φ's

$$j_0^a = \sum_{K=\pm} [\overset{*}{\varphi}{}_0^K\varphi_a^K + \varphi_0^K\overset{*}{\varphi}{}_a^K + i\varepsilon_{kla}\overset{*}{\varphi}{}_k^K\varphi_l^K]$$

$$\begin{aligned} j^{aj} = \sum_{K=\pm} [&K\delta^{aj}|\varphi_0^K|^2 + i\varepsilon_{ajl}K(\overset{*}{\varphi}{}_0^K\varphi_l^K - \overset{*}{\varphi}{}_l^K\varphi_0^K) \\ &+ K(\overset{*}{\varphi}{}_a^K\varphi_j^K - \delta^{aj}\sum_{i/1}^{3}|\varphi_i^K|^2 + \varphi_a^K\overset{*}{\varphi}{}_j^K)]. \end{aligned}$$

3. *Solutions and bifurcation*

In this Section we put

$$A_1^1 = A_2^2 = A, \quad A_2^1 = A_1^2 = B \tag{10}$$

the remaining components of a potential were already assumed to vanish.

Equations (9a), (9b) reduce then to the system

$$i\partial_0\varphi_0^{\mp} = 0, \tag{11a}$$

$$i\partial_0\varphi_k^{\mp} \pm \frac{A_a^k + A_k^a}{2}\varphi_a^{\mp} = 0. \tag{11b}$$

In an earlier paper [3] we have found

$$\varphi_1^{\mp} = \varphi_2^{\mp} = 0, \quad \varphi_0^{-} = \overset{*}{\varphi}{}_0^{+} = \overset{*}{g}, \quad \varphi_3^{-} = \overset{*}{\varphi}{}_3^{+} = \chi, \quad \arg \varphi_3^{+} = \arg \varphi_0^{+} \pm \pi \tag{12}$$

φ's are complex constants. Here the asterisk denotes the complex conjugation and g, χ satisfy the identity

$$-2(\overset{*}{g}\chi + g\overset{*}{\chi}) = \alpha^3/\sqrt{2} \tag{13}$$

with a constant α appearing in formulae for YM potentials. The fermion current is abelian:

$$j_\mu^a = \delta^{\mu 0}\delta^{a3}\alpha^3/\sqrt{2}. \tag{14}$$

There are 2 classes of YM potentials which satisfy YM equations with the above current:

(i)
$$A = \frac{\alpha}{\sqrt{2}}\cos\left(\frac{t\alpha}{\sqrt{2}}\right), \quad B = \frac{\alpha}{\sqrt{2}}\sin\left(\frac{t\alpha}{\sqrt{2}}\right), \tag{15a}$$

(ii)
$$A = \frac{1}{\sqrt{2}}\left(-P\left(C+\frac{t}{2}, 32\varepsilon, 8\alpha^6\right)\right)^{1/2}\cos\theta(t),$$

$$B = \frac{1}{\sqrt{2}}\left(-P\left(C+\frac{t}{2}, 32\varepsilon, 8\alpha^6\right)\right)^{1/2}\sin\theta(t). \tag{15b}$$

Here P is the Weierstrass double periodic elliptic function, ε is the energy density of the solution and

$$\operatorname{Im} C = \frac{K\left(\sqrt{\dfrac{e_1-e_2}{e_1-e_3}}\right)}{\sqrt{e_1-e_3}},$$

with $e_1 > e_2 > e_3$ being the roots of $4z^3 - 32\varepsilon z - 8\alpha^6 = 0$.
The real part of C is the usual integration constant, while the function $\theta(t)$ is given below

$$\theta(t) = \frac{-\alpha^3}{\sqrt{2}}\int_0^t \frac{dt'}{P\left(C+\dfrac{t'}{2}, 32\varepsilon, 8\alpha^6\right)}. \tag{16}$$

A period ω of a solution expresses via the complete elliptic integral K,

$$\omega = 4\,\frac{K\left(\sqrt{\dfrac{e_2-e_3}{e_1-e_3}}\right)}{\sqrt{e_1-e_3}}. \tag{17}$$

Let us point out some remarkable features of the solutions. First, their full energy density is constant in time. Next, for a parameter α tending to zero solutions (12), (13), (15a) tend to zero while solutions (12), (13), (15b) remain nonzero and coalesce with the elliptic cosine solution of Baseyan et al. [4][1]. This gives us an explicit example of bifurcation. The second bifurcation is in a parameter ε; for ε tending to its minimal value $\varepsilon_{\min} = 3\alpha^4/8$ solutions (12), (13), (15b) tend to the sin-cosin solutions (12), (13), (15a). For greater detail and a theoretic bifurcation discussion see [5].

4. *Sourceless solutions*

In this Section we will show solutions that are characterized by a trivial fermionic current.

Let

$$\varphi_0^{\mp} = \varphi_3^{\mp} = 0, \quad A_2^1 = A_1^2 = 0,$$

$$A_1^1 = -A_2^2 = A = \frac{1}{\sqrt{2}}\left(-P\left(C+\frac{t}{2}, 32\varepsilon, 0\right)\right)^{1/2}. \tag{18}$$

Equations (9) reduce to the system

$$\begin{aligned} i\partial_0\varphi_1^{\mp} \pm A\varphi_1^{\mp} &= 0, \\ i\partial_0\varphi_2^{\mp} \mp A\varphi_2^{\mp} &= 0. \end{aligned} \tag{19}$$

Their solutions are given by

$$\begin{aligned} \varphi_1^{\mp} &= \exp\left[\pm i\int_{t_0}^{t} A(t')dt'\right]c_1^{\mp} \\ \varphi_2^{\mp} &= \exp\left[\mp i\int_{t_0}^{t} A(t')dt'\right]c_2^{\mp}, \end{aligned} \tag{20}$$

where c's are complex constants. This fermion field gives the following YM current

$$j_0^a = i\varepsilon_{12a}(\overset{*}{\varphi}{}_1^+\varphi_2^+ - \overset{*}{\varphi}{}_2^+\varphi_1^+ + \overset{*}{\varphi}{}_1^-\varphi_2^- - \overset{*}{\varphi}{}_2^-\varphi_1^-), \tag{21a}$$

$$\begin{aligned} j_i^a = {} & (|\varphi_1^+|^2 - |\varphi_1^-|^2)(2\delta^{1a}\delta^{1i} - \delta^{ia}) + (|\varphi_2^+|^2 - |\varphi_2^-|^2)(2\delta^{2a}\delta^{2i} - \delta^{ai}) \\ & + (\overset{*}{\varphi}{}_1^+\varphi_2^+ - \overset{*}{\varphi}{}_1^-\varphi_2^- + \varphi_1^+\overset{*}{\varphi}{}_2^+ - \varphi_1^-\overset{*}{\varphi}{}_2^-)(\delta^{1i}\delta^{2a} + \delta^{2i}\delta^{1a}) \end{aligned} \tag{21b}$$

which vanishes if

$$c_1^{\mp} = c_2^{\pm}, \quad |c_1^-| = |c_1^+|. \tag{22}$$

Let us put

$$\begin{aligned} A_1^1 = A_2^2 = A &= \left(-P\left(C+\frac{t}{2}, 32\varepsilon, 0\right)\right)^{1/2}, \\ A_2^1 = A_1^2 &= 0. \end{aligned} \tag{23}$$

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[1] This solution has been earlier found by R. Treat, *Nuovo Cimento* A6, 121 (1971). The author is grateful to A. Actor and J. M. Cervero for pointing out this fact.

If in addition $\varphi_k = 0$, $k = 1, 2, 3$, then from equations (9) we get

$$\varphi_0^{\mp} = \exp\left[\pm i \int_{t_0}^{t} A(t')dt'\right]c^{\mp}; \tag{24}$$

since the current j_μ^a has to vanish, we have to impose

$$c^- = c^+. \tag{25}$$

Further solutions corresponding to Ansatz (23) are listed below:

a) $\varphi_0^{\mp} = \varphi_2^{\mp} = \varphi_3^{\mp} = 0, \quad \varphi_1^{\mp} = c_1$

b) $\varphi_0^{\mp} = \varphi_1^{\mp} = \varphi_3^{\mp} = 0, \quad \varphi_2^{\mp} = c_2$

c) $\varphi_0^{\mp} = \varphi_1^{\mp} = \varphi_2^{\mp} = 0, \quad \varphi_3^{\mp} = \exp\left[\mp i \int_{t_0}^{t} A(t')dt'\right]c_3$

d) $\varphi_0^{\mp} = \varphi_3^{\mp} = 0, \quad \varphi_1^{\mp} = \varphi_2^{\mp}c = \text{const};$

for all of them the fermionic current vanishes identically. Potentials (18), (23) (which coincide with Baseyan et al. elliptic cosine solutions [4]) solve sourceless YM equations (see Section V in [5]). The YM potential influences the evolution of the Dirac field, but the feedback exactly vanishes, since $j^{a\mu} = 0$ if (22) or (23) hold. Thus we found new self-consistent solutions of YMD equations. Let us point out that the full energy density of all solutions presented in this Section is constant, although they are time dependent.

5. *Concluding remarks*

Our main intention is to give an argument that the above solutions are of interest. To begin with, we remind that in quantum field theory the fundamental role is played by the generating functional

$$J(f) = Z^{-1} \int \exp\left[i\phi(f) + \frac{i}{\hbar} S(\phi)\right] D\phi, \tag{26}$$

where $S(\phi)$ is an action functional and $Z = \int \exp\left(\frac{i}{\hbar} S(\phi)\right) D\phi$; $D\phi$ denotes a functional integration measure. The standard procedure to calculate J is to expand the action S around a classical solution ϕ_0 (if $\phi_0 = 0$ then the corresponding QFT is said to be perturbative while for $\phi_0 \neq 0$ the obtained QFT is called nonperturbative):

$$S(\varphi_0 + \delta\phi) = S(\phi_0) + \tfrac{1}{2} S''(\phi_0)\delta\phi^2; \tag{27}$$

the term linear in $\delta\phi$ is absent because ϕ_0 satisfies the classical equations of motion. Integral (27) is then of Fresnel type and can be formally calculated. The effective Green's functions might be obtained by a functional differentiation of the generating functional with respect to the source term f.

If ϕ_0 has a symmetry lower than the full symmetry of classical equations of motion, then the Green's functions are meaningless [6]. This is related to the well known problem of zero mode solutions (ZMS) which cause great technical difficulties in practical calculation [7]. Let us explain briefly how the problems with ZMS arise. To calculate the functional integral

$$J(f) = Z^{-1} \int \exp\left[i\phi(f) + \frac{i}{\hbar} S(\phi_0) + \tfrac{1}{2} S''(\phi_0)\delta\phi^2 \right] D\phi \tag{28}$$

one can do the following:
(*i*) find the eigenfunctions of

$$S''(\phi_0)\delta\varphi_\lambda = \lambda\delta\varphi_\lambda, \tag{29}$$

where $\delta\varphi_\lambda$ should belong to a certain functional space (in our case, it should be square integrable)
(*ii*) expand quantum fluctuations in the basis of $\delta\varphi_\lambda$,

$$\delta\varphi = \sum c_\lambda \delta\varphi_\lambda \tag{30}$$

(*iii*) specify the functional measure of integration $D\phi$ as

$$D\phi = \prod_\lambda dc_\lambda. \tag{31}$$

The troubles with ZMS are due to the fact that $S''(\phi_0)\delta\varphi_0 = 0$, that is in certain directions (in a function space) the second variation of the action vanishes and the integration over dc_0 produces infinity. The solutions which we have found have a symmetry lower than the full 15-parameter G group of space-time transformations[2] (Poincaré group and conformal transformations) which is the symmetry of Yang-Mills-(massless) Dirac equations.

Let G_0 be a symmetry group of a solution ϕ_0, and define G_t by $G = G_0 \cup G_t$. To each generator $x_a \in \mathbf{G}_t(\mathbf{G}_t$ is the algebra of $G_t)$ corresponds a ZMS : $X_a\phi_0$ (X_a is a suitable representation of x_a; for gauge potentials this is simply a Lie derivative $\pounds_{x_a}$). The situation is obscured by gauge degrees of freedom — some ZMS can be compensated by a gauge rotation — but it is clear that all ZMS[3] are not square integrable. Thus they can be ignored.

The first term in the expansion of S in (27) is infinite for the solutions which we found in this paper. But if one quantizes the theory around exactly one classical solution, then the formula (26) is still well defined, due to the factor Z^{-1}.

Thus we think that from the technical point of view our solutions are good candidates for a background of QFT. Yet another problem is whether or not the solutions are physically relevant.

The author thanks the referee for pointing out an error in the last section of the paper.

[2] I do not discuss zero modes related to the gauge symmetry. To remove them, one has to impose a background gauge condition and to repeat the Faddeev-Popov trick (see, e.g., D. Amati, A. Rouet, *Nuovo Cimento* **A50**, 265 (1979)).

[3] I mean those ZMS that are related to the symmetry breaking.

REFERENCES

[1] J. Doneaux, Y. Saint-Aubin, L. Vinet, *Phys. Rev.* **D25**, 484 (1982); K. Meetz, *Z. Phys.* **C6**, 41 (1980); M. Basler, *Z. Phys.* **C20**, 57 (1983); R. Akhouri, W. I. Weisberger, *Nucl. Phys.* **B174**, 225 (1980); see also Ref. [5] below. Almost all of the solutions are listed in a review by E. Malec, TPJU preprint 15/16 (1984).

[2] R. Jackiw, C. Rebbi, *Phys. Rev.* **D13**, 3398 (1976).

[3] E. Malec, *Selfconsistent solutions of YMD eqautions*, report no 1251/P1, IFJ, Cracow 1984.

[4] A. A. Baseyan, S. G. Matinyan, G. K. Savvidi, *Pisma Zh. Eksp. Teor. Fiz.* **29**, 641 (1979).

[5] U. Kursawe, E. Malec, *J. Math. Phys.* **26**, 2643 (1985).

[6] R. Rączka, L. Roszkowski, IC/83/190 preprint.

[7] R. Jackiw, *Rev. Mod. Phys.* **49**, 681 (1977); R. Rajaraman, *Phys. Rep.* **21**, 228 (1975).