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ON SUMMATION OF PERTURBATION EXPANSIONS

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The problem of the restoration of physical quantities defined by divergent perturbation expansions is analysed. The Padé and Borel summability is proved for alternating perturbation expansions with factorially growing coefficients. The proof is based on the methods of the classical moments theory.

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1. Introduction

The idea that perturbation expansions (PE) in coupling constant g in quantum physics are divergent asymptotic series was suggested by Dyson [1] almost thirty five years ago. It is obvious that in such a case an ordinary summation is not a correct method of the restoration of the expanded function and some "generalized summation methods" (GSM) should be used. First attempts, based on the Padé method, made in the late 60's concerned model calculations [2] as well as theoretical investigations of the applicability of GSM (for example Padé method for anharmonic oscillator) [3]. They are necessary because in any particular case the applicability and uniqueness of GSM must be examined and that is the price which one should pay for giving up the condition of analyticity of the theory in the origin of the complex g plane. The Dyson idea was confirmed by the discovery of the factorial growth of the PE coefficients, showed as first numerically, for anharmonic oscillator [4], and next theoretically [5], for a very wide class of quantum mechanical and field theoretical models. This property drew theorists' attention to the Borel (in some cases connected with conformal mapping) method of summation of the divergent series and to its modification, the Padé-Borel method, which uses the Padé method in summation of the series in the integrand of the ordinary Borel integral. A significant progress achieved in computation of higher PE coefficients has made it possible to do explicit computations in many quantum mechanical and field theoretical models and to compare them, in the simplest cases, with exact results available by computer methods, for example, with the numerical solutions of the Schrödinger equation. A detailed discussion of the results

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obtained may be found in the literature, so here we only mention the best examined models. For example, in the quantum mechanics an analysis was made for the gx^{2n} anharmonic oscillator (by Borel method [6]), for screened Coulomb potential $V(r) = -r^{-1}f(\mu r)$ including the Yukawa and Hulthen potentials (by Padé method [7]) and for the funnel-like potential $V(r) = r^{-1}+gr^n$ (by Padé-Borel method [7]). In quantum field theory the applications of GSM have been based on the Borel method and have referred to the scalar φ^4 models, where critical exponents in the $\varphi^4_{(3)}$ model and asymptotics of the Gell-Mann-Low function in the $\varphi^4_{(4)}$ model were computed and successfully compared with experimental and general theoretical results [8]. As stated before, the use of GSM of the perturbation expansion in any particular case needs a proof of its applicability and uniqueness. The rigorous results are known in some cases like the already mentioned Padé summation of PE of energy levels for anharmonic oscillator or the Borel summation of the same quantities [3, 9] or the Borel summation of PE of Schwinger functions in φ^4 in two and three dimensional Euclidean space-time [10]. In what follows we present a proof of applicability and the uniqueness criteria of various GSM of the perturbation expansions in the case when they are alternating series and their coefficients have an asymptotically factorial shape.

2. *Perturbation series coefficients as Stieltjes moments*

Let us consider some quantum mechanical or field theoretical quantity $\Phi(g)$ (for example bound state energy, the generating functional of Green functions, the Gell--Mann-Low function and so on) which depends on real positive variable g interpreted as coupling constant. In the Euclidean version of the path integral formalism such a quantity is given by the non-gaussian functional integral

$$\Phi(g) = \int d\mu(\varphi)\tilde{\Phi}(\varphi)e^{-gS_I(\varphi,g)}, \tag{2.1}$$

where $\mu(\varphi)$ is the gaussian measure, $\tilde{\Phi}(\varphi)$ is a functional depending on fields φ which defines the meaning of Φ and $S_I(\varphi, g)$ is the full interaction action including the counterterms needed. For the majority of the non-trivial models the exact calculation of the integral (2.1) is impossible. The only quantities which may be calculated by existing computational technics are derivatives $\dfrac{\partial^n\Phi(g)}{\partial g^n}$ in the point $g = 0$. Let us define

$$f_n = \frac{1}{n!}\left.\frac{\partial^n\Phi(g)}{\partial g^n}\right|_{g=0} = \frac{1}{n!}\int d\mu(\varphi)\tilde{\Phi}(\varphi)P(n, S_I(\varphi)). \tag{2.2}$$

In the standard perturbation method f_n are interpreted as coefficients of the Taylor--Maclaurin expansion of $\Phi(g)$ whose sum is identified with $\Phi(g)$. This is incorrect in the case when $\Phi(g)$ is not holomorphic in the neighbourhood of $g = 0$, that is, if $S_I(\varphi; g)$ is a singular perturbation. It seems to be the rule in quantum physics and means that the point $g = 0$ is essentially singular and the PE is divergent in the whole plane of complex g. Exact calculation of the integrals (2.2) is a serious technical problem solved explicitly only for the first few n's but approximations based on the saddle point method let state that for many typical models f_n have an asymptotic representation of the form [5]:

$$f_n \sim a^n \Gamma(\mu n+1) n^\beta \sum_{j/0}^{\infty} \alpha_j n^{-j}, \quad n \to \infty \tag{2.3}$$

with calculable parameters a, μ, β, α_j which, by applying Stirling formula, is equivalent to the representations

$$f_n \sim a^n \Gamma(\mu n+\beta+1) \sum_{j/0}^{\infty} \beta_j n^{-j}, \tag{2.4a}$$

$$f_n \sim a^n \sum_{j/0}^{\infty} \gamma_j \Gamma(\mu n+\beta+1-j), \tag{2.4b}$$

where $\alpha_0 = \beta_0 = \gamma_0 = 1$ and the coefficients β_j and γ_j are uniquely determined by α_j.

The Gaussian functional integral (2.2) is rigorously defined and calculable for natural n, but by analogy to the integrals of finite dimensionality it seems quite reasonable :o consider it as the definition of the function of complex variable z in the domain $|\arg z| < \frac{\pi}{2}$. In this sector of the complex plane the derivation of the asymptotic formulas (2.3, 2.4a and 2.4b) remains valid, so instead of (2.2) we may consider a complex function

$$f(z) = \frac{1}{\Gamma(z+1)} \int d\mu(\varphi) \tilde{\Phi}(\varphi) P(z; S_I(\varphi)) \tag{2.5}$$

characterized for $|z| \to \infty$, $|\arg z| < \frac{\pi}{2}$ by asymptotic expansion

$$f(z) \sim a^z \Gamma(\mu z+\beta+1) \sum_{j/0}^{\infty} \beta_j z^{-j} \tag{2.6}$$

which means that $f(z)$ belongs to the class of asymptotically equivalent functions [11] of the type

$$f(z) = a^z \Gamma(\mu z+\beta+1) F(z), \tag{2.7}$$

where $F(z)$ is any function with asymptotics defined in (2.6). The function $f(z)$ is analytic in its domain and uniquely determined by its values for positive integer z. Analyticity is a consequence of an analogy between (2.5) and finite dimensional integrals of such a type. Due to the analyticity of $f(z)$ the function $F(z)$ in (2.7) has also to be analytic for $|\arg z| < \frac{\pi}{2}$. Its analytical and asymptotical properties guarantee, according to Carlson's theorem [12], that $F(z)$ is uniquely determined by its values for natural z so is $f(z)$. The considered function (2.7) may be represented by Mellin's integral

$$f(z) = a^z \Gamma(\mu z+\beta+1) F(z) = \int_0^{\infty} dt\, t^{z-1} f^*(t), \tag{2.8}$$

where

$$f^*(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds t^{-s} a^s \Gamma(\mu s + \beta + 1) F(s) \tag{2.9}$$

because $f(z)$ is regular for $|\arg z| < \frac{\pi}{2}$ and owing to the asymptotics of $F(z)$ and Γ function we have

$$|a^{x+iy}\Gamma(\mu x + \beta + 1 + i\mu y)F(x+iy)| \underset{|y|\to\infty}{\sim} \tag{2.10}$$

$$\sim \sqrt{2\pi}\, a^x |y|^{\mu x + \beta + \frac{1}{2}} e^{-\frac{1}{2}\pi|y|} \in L(dy, \mathbf{R}),$$

which is a necessary condition for (2.8) to hold according to general theorems in Mellin's transformation theory [13]. Asymptotic expansion of $f(z)$ (2.6) defines the asymptotic shape of $f^*(t)$, which belongs to the class of asymptotically equivalent functions

$$f^*(t) = \frac{1}{\mu}\left(\frac{t}{a}\right)^{\frac{\beta+1}{\mu}} e^{-\left(\frac{t}{a}\right)^{\frac{1}{\mu}}} h(t),$$

$$h(t) \underset{t\to\infty}{\sim} \sum_{j/0}^{\infty} \beta_j \left(\frac{t}{a}\right)^{-\frac{j}{\mu}}, \tag{2.11}$$

which follows from asymptotic properties of Mellin's transform [14] and means that $f^*(t)$ is positive definite in the neighbourhood of infinity. Moreover, $f^*(t)$ is non negative on the whole positive real semiaxis. Indeed, let us suppose that there exists a point $t_0 \in (0, \infty)$ such that

$$f^*(t_0) = \frac{a^s}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds t_0^{-s} \Gamma(\mu s + \beta + 1) F(s) < 0. \tag{2.12}$$

The integral (2.12) does not depend on the choice of the line of integration $\sigma + i\lambda$ because the integrand is analytic in the right half plane and tends to zero when imaginary part of its argument tends to infinity (2.10). Using this we may take σ big enough to dominate (2.12) by the leading term of the asymptotics of $F(s)$. Such a procedure gives

$$\frac{a^s}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} ds t_0^{-s} \Gamma(\mu s + \beta + 1) \tag{2.13}$$

$$= \frac{1}{\mu}\left(\frac{t_0}{a}\right)^{\frac{\beta+1}{\mu}} e^{-\left(\frac{t_0}{a}\right)^{\frac{1}{\mu}}} > 0$$

which contradicts supposition (2.12). Following our previous results we are able to state that $f(z)$ is represented by the Stieltjes integral, that is, that there exists the nondecreasing function $\sigma(u)$ defined on $[0, \infty)$ for which

$$f(z) = \int_0^\infty u^z d\sigma(u) \tag{2.14}$$

so we can formulate

Corollary 1: The sequence of perturbation expansion coefficients $\{f_n\}_{n=0}^\infty$, which are the values of $f(z)$ for z natural, defines the Stieltjes problem of moments [14].

The interpretation of the perturbation series coefficients as Stieltjes moments gives a possibility of the restoration of the expanded function even in the case when the expansion considered is divergent everywhere. It also gives some information about nature of the PE, namely about its asymptotic character. Restoration procedure, suggested by Corollary 1, may be based on either Padé [15] or Borel [16] generalized summation method and in the following considerations we shall analyze both of them in a more detailed way.

3. *Padé summation of the perturbation series*

According to Corollary 1 the perturbation expansion coefficients are the moments of the measure $\sigma(u)$ defined on $[0, \infty)$, that is

$$f_n = \int_0^\infty u^n d\sigma(u) \tag{3.1}$$

which means that the formal perturbation expansion of $\Phi(g)$

$$\Phi(g) = \sum_{n/0}^{\infty} f_n(-g)^n \tag{3.2}$$

is the so called Stieltjes series. The properties of the Padé approximants of the functions belonging to this class are well known and may be listed as follows [15]:

S1 — Any sequence of $[M+j, M]$ Padé approximants to a series of Stieltjes converges to an analytic function in the cut complex plane $-\infty \leqslant z \leqslant 0$ as M tends to infinity, $j \geqslant -1$. Moreover, if $\sum_{n/0}^{\infty} (f_n)^{\frac{-1}{2n+1}}$ diverges then all these sequences tend to a common limit.

S2 — The $[M, M]$ and $[M, -1, M]$ approximants calculated for nonnegative real values of the argument form the best upper and lower bounds available using only a given number of coefficients. The use of higher M, that is of additional coefficients improves the bounds. The same holds for the first derivatives.

S3 — The function defined by the Stieltjes series admits integral representation $\Phi(g) = \int_0^\infty \frac{d\sigma(u)}{1+ug}$. The above properties mean that the considered field theoretical quantity $\Phi(g)$ (2.1) given by uncalculable functional integral, whose derivatives in $g = 0$ have the asymptotics (2.3), may be defined in terms of an ordinary integral with unknown Stieltjes measure

$$\Phi(g) = \int_0^\infty \frac{d\sigma(u)}{1+ug} \tag{3.3}$$

and bounded for physical values of the coupling constant by the Padé approximants to its perturbation expansion

$$[M-1, M]\{\sum_{n/0}^{\infty} f_n(-g)^n\} \leqslant \Phi(g) \leqslant [M, M]\{\sum_{n/0}^{\infty} f_n(-g)^n\} \tag{3.4}$$

with accuracy depending on the number of coefficients used in the calculation. In addition, if these coefficients do not grow too fast with the increasing n (the divergency condition in S1 is roughly equivalent to $f_n \lesssim (2n)!$), such a method determines $\Phi(g)$ in the unique way what follows immediately from S1 and (3.4).

Corollary 2: Perturbation series may be summed by the Padé method under some very general assumptions concerning the asymptotic behaviour of its coefficients, and there exists a class of models where this method gives a unique result.

4. *Borel summation of the perturbation series*

Borel's method is probably the most frequently used method of generalised summation, that is of the restoration of the function defined by divergent series. Nowadays we do not know what general conditions are necessary for this method to be used but one particular case is mathematically well understood. This is the case when an unknown function $\Phi(g)$ has formal series $\sum_{n/0}^{\infty} f_n(-z)^n$ as its strong asymptotic expansion of the order k and the Watson's theorem is valid [9, 17] — that is if we have

Theorem 1. The function $\Phi(z)$ has the series $\sum_{n/0}^{\infty} f_n(-z)^n$ as its strong asymptotic expansion of the order k if it is analytic in the sector $S = \left\{z; 0 < |z| < R, \arg z < \frac{k\pi}{2} + \varepsilon\right\}$ and there exist constants C, α and σ such that for all natural N and for all $z \in S$

(*i*) $$|\Phi(z) - \sum_{n/0}^{N-1} f_n(-z)^n| \leqslant C\sigma^N \Gamma(kN+\alpha+1)\,|z|^N.$$

Under these conditions $\Phi(z)$ may be uniquely summed by the Borel method, that is it may be represented by Laplace-like integral

(*ii*) $$\Phi(z) = \int_0^\infty dx e^{-x} x^\alpha B_\alpha(zx^k)$$

where

$$B_\alpha(zx^k) = \sum_{n/0}^{\infty} \frac{f_n(zx^k)^n}{\Gamma(kn+\alpha+1)}.$$

We see that conditions of this theorem, especially analyticity, are not easy to check but in the case when perturbation series coefficients are the Stieltjes moments we may use the result which is a small modification of the theorem of Hamburger and Nevanlinna [15].

Theorem 2. Let nondecreasing function $\sigma(u)$, $u \in [0, \infty)$, have finite moments up to the order $2n$

(*i*) $$s_k = \int_0^\infty d\sigma(u)u^k, \quad k = 0, 1, 2, \ldots, 2n,$$

then there exists the function $\Sigma(z)$, analytic in the complex plane cut along real positive semiaxis, preserving the relation $\frac{\operatorname{Im} z\Sigma(z)}{\operatorname{Im} z} > 0$ for any $\operatorname{Im} z \neq 0$, and admitting the integral representation

(*ii*) $$\Sigma(z) = \int_0^\infty \frac{d\sigma(u)}{1+uz}$$

for which

(*iii*) $$\overline{\lim_{z\to 0}} \left| \frac{\Sigma(z) - \sum_{j/0}^{2n-1} s_j(-z)^j}{z^{2n}} \right| = s_{2n}$$

uniformly in the sector $S = \{z; 0 < |z| < R, \arg z \in (-\pi+\delta, \pi-\delta)\}$ with any $\delta > 0$ and (*iii*) means upper limit. The inverse theorem is true under condition (*iii*) valid even if $z = iy$, and when the conditions about analyticity are fulfilled.

The proofs of both theorems differ from the standard ones only in details and we omit them.

Using the fact that perturbation series coefficients f_n have representation (2.8), that is that they define the Stieltjes problem of moments and satisfy the conditions of theorem 2, we obtain for all N and all z belonging to

$$S = \{z; 0 < |z| < R, \arg z \in (-\pi+\delta, \pi-\delta), \delta > 0\}$$

$$|\Phi(z) - \sum_{n/0}^{N-1} f_n(-z)^n| \leqslant Ca^N\Gamma(\mu N+\beta+1)\,|z|^N, \tag{4.1}$$

where $C = \max_{z\in S} |F(z)|$ according to (2.8). Because of theorem 2 $\Phi(z)$ is analytic in S and all the conditions of Watson's theorem are satisfied for $\mu = 1$ and $\mu = 2$ so we have

Corollary 3. If perturbation series coefficients f_n do not grow too fast with the increasing n the series may be uniquely summed to an analytic function by the Borel method, that is

$$\Phi(g) = \sum_{n/0}^{\infty} f_n(-g)^n = \int_0^{\infty} dx e^{-x} x^{\beta} B_{\beta}(gx^k), \tag{4.2}$$

where

$$f_n = a^n \Gamma(kn+\beta+1)F(n) \tag{4.3}$$

and

$$B_{\beta}(z) = \sum_{n/0}^{\infty} \frac{f_n(-z)^n}{\Gamma(kn+\beta+1)}. \tag{4.4}$$

Comparing the above results with those obtained by the Padé method we see that in both methods the uniqueness condition is similar and, roughly speaking, has the form of the bound

$$f_n \leqslant (2n)! \tag{4.5}$$

In the case when (4.5) is not satisfied the Padé method seems to have some advantages because it gives the bounds (3.4) for $\Phi(g)$ for positive real values of g and this result does not depend on particular shape of f_n. In the Borel method we have to control the difference between $\Phi(g)$ and the truncated series (theorem 1(*i*)) which can be done because of theorem 2 but, furthermore, $\Phi(g)$ must have an analytic continuation in many sheeted Riemann surface according to the conditions of the Watson's theorem. The last is not guaranteed by theorem 2 and should be proved separately in a different way which has been done only in some special models mentioned in the introduction.

3. Conclusion

We have considered the Padé and Borel summation of the perturbation expansions with factorially growing coefficients. We showed that the set of perturbation expansion coefficients (which do not grow too fast) contains enough information to restore the expanded function in a unique way. In the models which do not fulfill these conditions we need additional information about the coupling constant dependence of the expanded quantities. The same happens when the perturbation expansion is nonalternating. The same is the rule for the models with degenerated vacuum, like double well potentials in quantum mechanics and nonabelian gauge theories. In these cases the proposed methods may be used only for unphysical values of the coupling constant and results obtained have to be continued to the physical values of this parameter. Such a continuation may be possible if the analytic properties of the function are known, but this kind of information is lost in the perturbation expansion because of its asymptotic character and may be achieved only by essentially nonperturbative methods of analysis of coupling constant dependence in these models.

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