IS OUR METRIC A FIXED POINT ON A FRACTAL?*

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(*Received October 7, 1985*)

The study of a discrete version of Laplace equation on a fractal is presented. While in the limit $n \to \infty$ the solutions of the equations are close to the usual ones in the continuum, a new essential feature appears, namely the presence of fixed points in the metric.

PACS numbers: 02.90.+p

Is our universe, the space-time we use to describe physical phenomena, a continuum? For centuries now we have taken to accept this continuous nature of the underlying framework of our world, although the objects we describe are, of course, of finite extension.

The question may take a special significance in a general relativistic or cosmological approach; what if the energy necessary to observe a small detail of space-time is comparable to that required to form a black hole? We[1] therefore thought it interesting to consider a simple case of discontinuous underlying structure. The structure we want to study is based on the construction of a self-similar fractal [1]. We first present this structure in the context of an underlying continuous space, although we will later have a more intrinsic look at it.

Consider a triangle (Fig. 1), which is subdivided into several subtriangles (by dividing each side into n equal parts). We discard all the points contained in, say, the downward pointing triangles, and perform a similar subdivision of the remaining ones. This procedure is then repeated indefinitely, and defines a (2-dimensional) n-fractal. The Hausdorff dimension is easily estimated to be: $\ln \frac{n(n+1)}{2} \Big/ \ln n$. The special case $n = 2$ corresponds to the "Sierpinsky gasket". This procedure can be used similarly in underlying d-dimensional

* Presented at the XXV Cracow School of Theoretical Physics, Zakopane, Poland, June 2–14, 1985.

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[1] This work was performed in collaboration with F. Englert, M. Rooman, Université Libre de Bruxelles, and Ph. Spindel, Université de l'Etat à Mons.

spaces, by substituting (hyper-) tetrahedra for the upward pointing triangles mentioned before (notice that the discarded parts are now more complicated polyhedra).

Having achieved this, we must stress two properties of our construction, namely the lack of translational invariance on one side, but, more interestingly for our purpose, the preservation of (discrete) dilatation invariance. Namely, if the division procedure is extended indefinitely (in both directions), the figure obtained by shrinking all lengths by $\frac{1}{n}$ is identical to the original one!

For convenience, our usual pictorial representation of this construction (Fig. 2) uses equilateral triangles; there is however nothing compulsory in this, and the underlying metric has this far not been specified.

We want to study some typical equation on this structure. One obvious choice is Laplace equation, which we write for a scalar field responding to a point-like test charge[2].

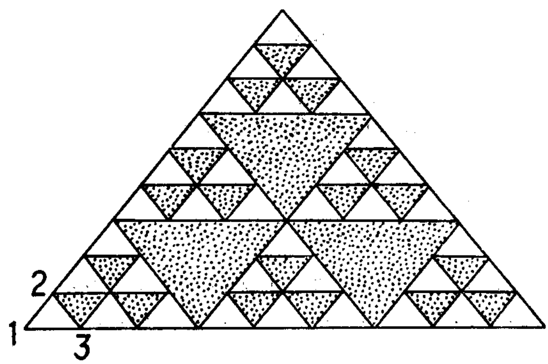

Fig. 1

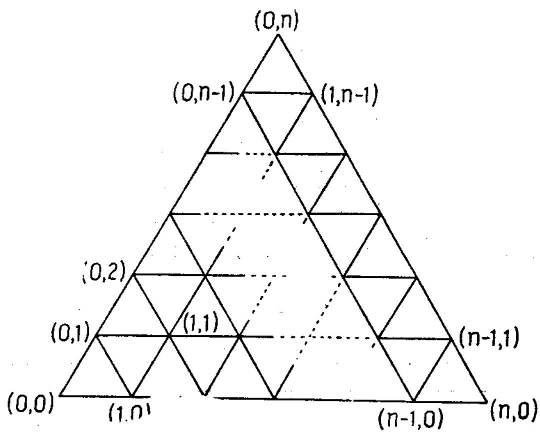

Fig. 2

[2] This notion will have to be adapted below.

Let us consider an ultraviolet cut off, corresponding to some level of fractalisation p. We write at this level the Laplace equation as a finite difference equation for the fields evaluated at each vertex — this defines the equation for the vertices belonging to levels $p, p-1 \ldots p-k$, since all of these vertices are present at level p. In general, we can write this equation as: (if the test charge e is situated at vertex $i = 0$)[3]

$$\sum_j g_{ij}(\phi_j - \phi_i) = e\delta_{i0}, \tag{1}$$

where the sum over j extends over all nearest neighbors of vertex i. More complex charge distributions can be considered by using the superposition principle. The coefficients g^{ij} are obviously related in some way to the metric. This relation can be exhibited at the level of an individual $p-1$ triangle by imposing some values for the length of the p-triangle sides, and comparing (1) to the continuum limit when the sides of the triangle are simultaneously shrunk to zero.

Neglecting scalar factors (lattice spacing...), we have:

$$g^{ij} \sim l_{ik}^2 + l_{jk}^2 - l_{ij}^2 \tag{2}$$

where ijk is any p-triangle with summits i, j, k, and l_{ik}^2 is the length squared of the side ik.

As an example, for an equaliteral triangle in euclidean metric,

$$g^{12} = K(l_{13}^2 + l_{23}^2 - l_{12}^2) = Kl^2, \quad \text{and} \quad g^{12} = g^{13} = g^{23}. \tag{3}$$

Alternatively, in the case of a Minkowski triangle, with (12) and (23) light cone lines, we have:

$$\begin{aligned} g^{12} &= K(l_{13}^2 + l_{32}^2 - l_{12}^2) = Kl^2 = g^{23}, \\ g^{13} &= K(0 + 0 - l_{13}^2) = -Ki^2 = -g^{23}, \\ &\text{and} \quad g^{12} = g^{23} = -g^{13}. \end{aligned} \tag{4}$$

Up to a normalization, we may thus take (g^{23}, g^{13}, g^{12}) to be $(1, 1, 1)$ and $(1, -1, +1)$ respectively for "isotropic" Euclidean and Minkowski space.

Before proceeding further it may be appropriate to show some of the properties of the system of equations we are considering (this far a similar set of equations might have been written on a lattice).

Let us consider a single $p-1$ triangle, and all the p-triangles it includes. (See Fig. 2).

If we sum all the equations for the internal points of the triangle with the exclusion of the summits we get the equation:

$$(\phi^{1,0} + \phi^{0,1} - 2\phi^{0,0}) + (\phi^{n-1,0} + \phi^{n-1,1} - 2\phi^{n,0}) + [\phi^{0,n-1} + \phi^{1,n-1} - 2\phi^{0,n}) = \sum q_i \tag{5}$$

(we replaced the site index i by a pair of affine coordinates (i_1, i_2), we have also used the Euclidean metric to simplify the argument, which is however perfectly general).

[3] We have voluntarily omitted scale factors from the equation, as they are of little consequence for our main topic.

Equation (5) tells us that the total flux of ϕ leaving the triangle is equal to the sum of the charges inside.

Using the equation for $\phi^{0,0}$, we have further:

$$(\phi^{1,0}+\phi^{0,1}-2\phi^{0,0}) = -e, \tag{6}$$

which relates the flux to the charge at point (0, 0). We can use the same reasoning at point $(0, n)$ and $(n, 0)$ and replace effectively the outgoing flux by "mirror charges" as in usual electrostatic problems. We see that the only type of problems we are allowed to address are in fact *completely* neutral ones with the total "electric" charge in the triangle summing up to 0.

In particular, a test charge e at site (0, 0) has to be accompanied by mirror charges $q_{0,n}$ and $q_{n,0}$ at sites $(0, n)$ and $(n, 0)$ with

$$q_{0,n}+q_{n,0}+e = 0. \tag{7}$$

One typical choice would be $(q_{0,0};\, q_{n,0};\, q_{0,n}) = \left(e, -\frac{e}{2}, -\frac{e}{2}\right)$. As we will see later this is an essential difference with the continuum or even the lattice situation, where mirror charges associated with a single point source are usually isotropically spread, and for the most part altogether ignored.

Rescaling

We now turn away temporarily from the details of the solution of the "electrostatic problem". In order to consider what happens at larger scale let us imagine that after defining the rules as in Eq. (1) at level p, we want to eliminate the variables pertaining to level p, and write equivalent equations involving the summits of level $p-1$ alone. This can be done at the cost of solving the linear system of equations presented in (1), in terms of the values of $\phi^{(0,n)}$, $\phi^{(n,0)}$ and $\phi^{(0,0)}$, which can be identified (up to a scale factor) to $\Phi^{(0,1)}$, $\Phi^{(1,0)}$ and $\Phi^{(0,0)}$ — we use capital letters and indices Φ^{IJ} for the variables pertaining to fractal level $(p-1)$.

We will in general obtain a new system of equations which is equivalent to (1) for the vertices belonging to fractal level $p-1$; namely:

$$\sum G^{IJ}(\Phi^J-\Phi^I) = \tilde{e}\delta^{I0}, \tag{8}$$

where $\tilde{e}$ is the apparent charge seen at this level, in the metric corresponding to G_{IJ} (notice that $\tilde{e}$ and G_{IJ} do not have an independent meaning, since we do not postulate anything about the metric of the fractal structure as a whole). There is in other words no way to tell for sure the distance between two points without some measurement, and the only measuring rod which we have is provided by the "electrostatic" equation itself.

We have studied in some detail the processes leading from Eq. (1) to Eq. (8). The following discussion is a brief summary of our findings, but contains most of the important aspects.

For simplicity, we choose in all cases to norm both $g^{13} = G^{13} = 1$ (i.e. the g^{ij} coefficient corresponding to the horizontal link in Fig. 2). Furthermore, I will present here only the case where $g^{13} = g^{23}$; $g^{12} = a$, which, by symmetry implies $G^{13} = G^{23} = 1$; $G^{12} = A$. Full results will be presented in a detailed publication.

Our purpose now is to compute A as a function of a. This can be done very easily for the Sierpinsky gasket, $(n = 2)$

$$A = \frac{3a^2+6a+1}{4a+6}\,. \tag{9}$$

The non-linearity of this result shows us that, despite the scale invariance of the fractal, *a non-linear change in the metric occurs when we go to large distances.* A natural question arises, namely: what are the fixed points of the rescaling equation (9)? They can easily be seen from a study of Fig. 3, where we have plotted A as a function of a; fixed points occur for $A = a$.

The signature of the metric changes for the critical value $a = -1/2$, while $a = -1$ corresponds to Minkowski space with 2 sides of the triangle on the light cone.

We see directly that only 2 fixed points exist, corresponding to $a = \pm 1$. The Minkowski fixed point turns out to be unstable, while the Euclidean one is attractive.

We have of course generalized Eq. (9). The general situation where $g^{13}/g^{23} \neq 1$ will be discussed elsewhere; the conclusions above are however unaffected. We have also consid-

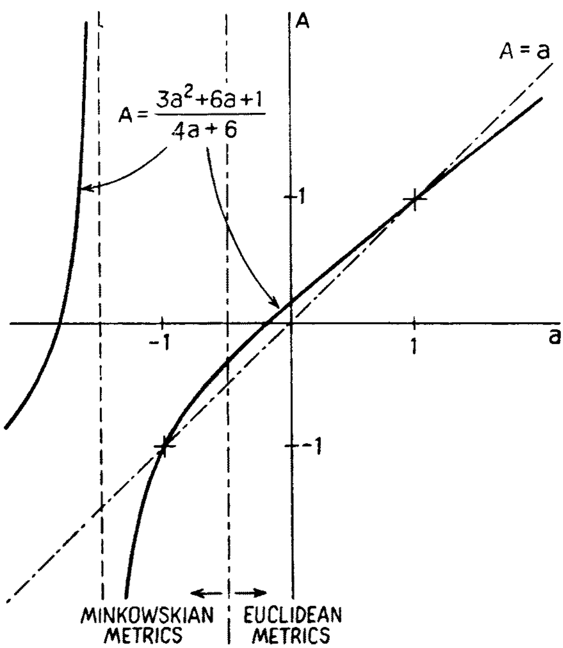

Fig. 3

ered more general n-fractals, with the following rescaling equations:

$$A = \frac{11a^4+80a^3+172a^2+132a+25}{2[9a^3+54a^2+95a+52]} \tag{10}$$

for $n = 3$ and:

$$A = \frac{25a^7+430a^6+2680a^5+8162a^4+13336a^3+11732a^2+5075a+790}{48a^6+712a^5+3880a^4+10312a^3+14368a^2+10090a+2820} \tag{11}$$

for $n = 4$. These expressions always have fixed points for $a = \pm 1$ (this can easily be demonstrated for arbitrary n-fractals) but with increasing n, extra (unstable) fixed points are introduced; for $n = 3$, we have two additional fixed points for $a = -2 \pm \sqrt{3/7}$ (Fig. 4), while for $n = 4$, there are 5 other fixed points for $a < -1$, therefore all minkowskian. All those extra fixed points have turned out to be unstable — with $a = 1$ the only attractive fixed point!

We have further considered n-fractal built upon larger dimensional underlying spaces — using hypertetrahedrons instead of triangles. For a d-dim 2-fractal, we get the general relation (assuming $(g^{ij}) = (1, \ldots, a)$):

$$A = \frac{(1+d)a^2+(4+d)a+1}{(2+d)a+(4+d)} \qquad (n = 2) \tag{12}$$

which exhibits characteristics similar to the relations above.

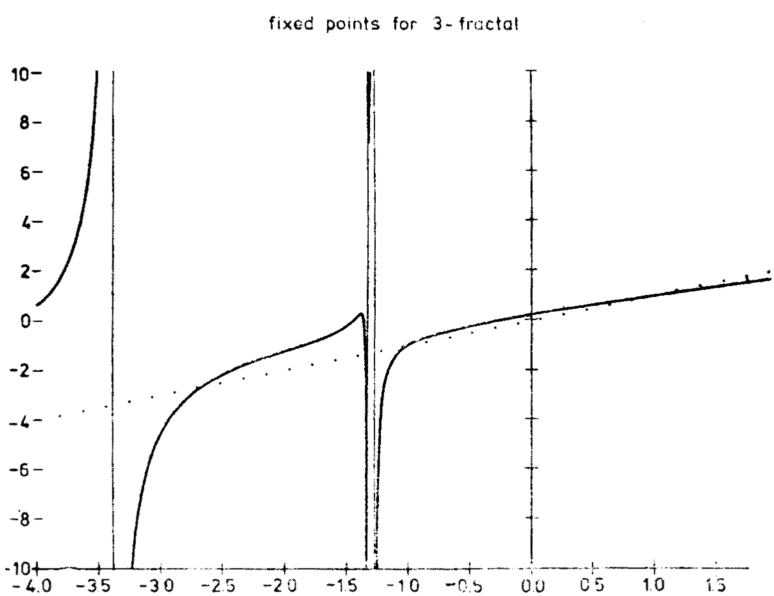

Fig. 4

While it may be disappointing that the large scale metric turns out to be euclidean, whatever the starting point (except for a small set of trajectories which land on the unstable points), the presence of this fixed point is already surprising.

It may be that our measuring tool is too crude at this stage and that, say, a similar equation for the vector potential might be more appropriate to Minkowski space.

The Euclidean fixed point

In this last section, we present the numerical solution of the equations for a test charge in the Euclidean metric. For 2-dimensions, we have represented (Fig. 5) the value of the potential along the side of the triangle (curve I) and along the mediator (curve M) for a

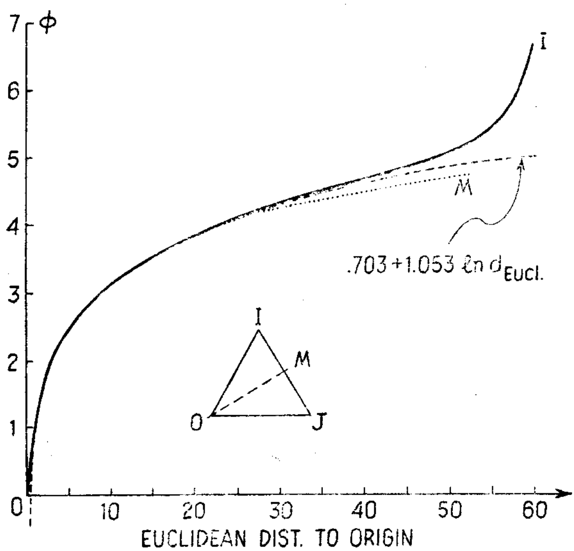

Fig. 5

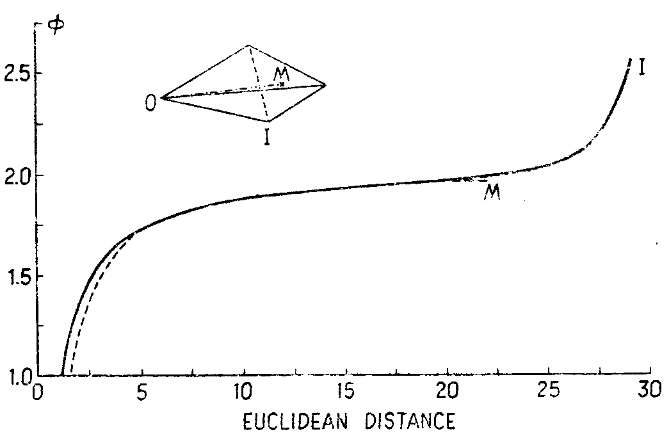

Fig. 6

$n = 60$ fractal; to this we superimpose a logarithmic fit inspired by the shape of the continuum potential; we see that the fit is excellent, except for points close to the boundary IJ, where the effect of the localized mirror charges is felt. We should keep in mind that the curve presented here is only a guide to the eye, and has no meaning for points which do not belong to the fractal. The correct picture would be given by a discontinuous set of points; furthermore scale invariance and flux conservation can be used to establish the shape of the potentials for vertices corresponding to subsequent levels of fractalisation. A similar picture is obtained in 3 dimensions, as shown by Fig. 6.

Conclusions and prospects

The study of a discrete version of Laplace equation on a fractal structure has brought some interesting results.

While in the limit $n \to \infty$ the solutions of the equations are close to the usual ones in the continuum, we have exhibited one essential new point, namely, the presence of fixed points in the metric. This shows that whatever first choice we make for the initial metric — and in particular its signature at a given scale, we are assured that the apparent metric at large distances will correspond to one of a discrete set of fixed points. In all the cases studied, the only stable fixed point yields an isotropic Euclidean metric. Further extensions should be envisaged, among which the case of vector fields ranks first.

This study was supported in part by the U.S. Department of Energy under contract #DE-ACO2-76ER01112. I wish to thank more specially Jos. Vermaseren (Nikhef), for assistance in some numerical calculations (especially Eq. (11)); the algebraic manipulation program Schoonschip was found helpful. I wish to thank the Organizing Committee of the Cracow School of Physics for its kind invitation.

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[1] P. Mandelbrot, *Fractals, Form, Chance and Dimension*, Freeman, San Francisco 1977.
[2] For more details, see F. Englert, J.-M. Frere, M. Rooman, Ph. Spindel, in preparation.