ELECTROMAGNETIC FIELD AS A NON-LINEAR CONNECTION

By B. Broda*

Institute for Nuclear Studies, Warsaw

and M. Przanowski

Institute of Physics, Technical University, Łódź**

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The "unification" of the gravitation and the electromagnetism in a Randers' space with a non-linear connection is presented. The equations of motion and the field equations appear in the natural manner.

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The purpose of this note is to give some geometrical version of the "unification" of the gravitational and the electromagnetic interactions. By the word "geometrical" we mean that all gravitational and electromagnetic forces have an entirely geometrical origin in the Einsteinian spirit. The potentials must be components of a metric and the strengths must be components of a connection for N-dimensional manifold M. The motion of the test particle proceeds along the geodesic line in M and the action for fields has the Hilbertian-like form

$$S_f = \int \tilde{\mathscr{R}} |g|^{1/2} d^N x, \tag{1}$$

where $\tilde{\mathscr{R}}$ is a suitably defined curvature scalar, $g = \det ||g_{\alpha\beta}||$. Well known examples of such theories are Kaluza-Klein Theory [1–3] and Finslerian theories [4–11]. Kaluza-Klein Theory deals with a 5-dimensional Riemannian manifold, while the Finslerian theories use the direction-dependent fields. We propose an alternative approach. We start with the Randers' metric, which is a particular case of a Finsler metric. Next, we construct the suitable non-linear connection.

* Address: Instytut Fizyki, Uniwersytet Łódzki, Nowotki 149/153, 90-230 Łódź, Poland.

** Address: Instytut Fizyki, Politechnika Łódzka, Wólczańska 219, 93-005 Łódź, Poland.

The physical space is now a 4-dimensional Randers' space [12–14] rather than a Riemannian one; i.e., the physical space is a manifold with the metric

$$ds = \sqrt{g_{ij}(x)dx^i dx^j} + a_k(x)dx^k, \tag{2}$$

where $\det ||g_{ij}|| \neq 0$. (The Latin indices run through 1, 2, 3, 4.) This metric has been proved to be very useful in the unification schemes [15]. The metric given by (2) is defined by the pair of the tensor fields (g, a); g_{ij} influences the local inhomogeneity of the space, whereas a_i changes the local anisotropy. We identify g_{ij} with the potential of the gravitational field and $a_i = \frac{q}{m} A_i$, where A_i — four-potential of the electromagnetic field, q and m — charge and mass of the test particle. From formula (2) it follows that we deal with a non--standard notion of the test particle. Our test particle changes the local anisotropy of the space by the factor $\frac{q}{m}$. The action for the particle has the following well known form (we work with units such that $c = 1$)

$$S_m = -m \int ds = -m \int \left(\sqrt{g_{ij}\dot{x}^i\dot{x}^j} + \frac{q}{m} A_k \dot{x}^k \right) d\tau, \tag{3}$$

where $\dot{x}^i := \frac{dx^i}{d\tau}$, $d\tau := \sqrt{g_{ij}(x)dx^i dx^j}$. Minimal Action Principle gives now the geodesic equation, which is simultaneously the equation of motion for the test particle

$$\frac{d\dot{x}^i}{d\tau} + \Gamma^i{}_{jk}\dot{x}^j\dot{x}^k - f^i{}_j\dot{x}^j = 0, \tag{4}$$

where $\Gamma^i{}_{jk}$ — Christoffel's symbols, $f^i{}_j := \frac{q}{m} F^i{}_j$ — tensor of the electromagnetic field. Using the absolute derivative, we can rewrite formula (4) in the geometrical language

$$\frac{D\dot{x}^i}{d\tau} = \dot{x}^j \nabla_j \dot{x}^i = 0. \tag{5}$$

The natural choice of the covariant derivative, i.e.,

$$\nabla_j u^i := \partial_j u^i + \Gamma^i{}_{jk} u^k - f^i_j \tag{6}$$

leads to the concept of the non-linear connection defined by the pair (Γ, f). Next, we introduce the directional covariant derivative for the contravariant vector field u with respect to X

$$\nabla_X u := \partial_X u + \Gamma_X u - f_X, \tag{7}$$

where $(\partial_X u)^i := X^j \partial_j u^i$, $(\Gamma_X)^i{}_j := X^k \Gamma^i{}_{kj}$, $(f_X)^i := X^j f^i{}_j$. The following identities hold (compare [16])

$$\nabla_{X+Y}u = \nabla_X u + \nabla_Y u, \tag{8a}$$

$$\nabla_{\lambda X}u = \lambda \nabla_X u, \tag{8b}$$

$$\nabla_X(\lambda u) = (\partial_X \lambda)u + \lambda \nabla_X u, \tag{8c}$$

where λ is a scalar. Of course, generally

$$\nabla_X(u+w) \neq \nabla_X u + \nabla_X w.$$

The non-linear connections have been considered in [17–22] but from a different point of view. Our connection is defined only in the tangent bundle and it is not induced by any connection in the bundle of linear frames. This is a G-connection in the following sense

$$T(h)\nabla_X u = \nabla'_X(T(h)u), \tag{9}$$

where $T(h)$ belongs to a linear representation ϱ of $G = GL(n, \mathbf{R})$

$$(h \in G, \quad \varrho : h \to T(h)).$$

In our case ϱ is the fundamental representation. From (9) it follows

$$T(h)\,(\partial_X u + \Gamma_X u - f_X) = (\partial_X + \Gamma'_X)\,(T(h)u) - f'_X, \tag{10}$$

where

$$\Gamma'_X = T(h)\Gamma_X T^{-1}(h) - (\partial_X T(h))T^{-1}(h), \tag{11a}$$

$$f'_X = T(h) f_X. \tag{11b}$$

Therefore, we have obtained the transformation rules for the connection.

It is easy to calculate

$$[\nabla_X, \nabla_Y]Z = \nabla_{[X,Y]}Z + R(X, Y)Z + P(X, Y) + f_{Y-X} \tag{12a}$$

and

$$\nabla_X Y - \nabla_Y X = [X, Y] + T(X, Y) + f_{Y-X}, \tag{12b}$$

where

$$R^i{}_{jkl} := \partial_k \Gamma^i{}_{lj} + \Gamma^i{}_{kr}\Gamma^r{}_{lj} - (k \leftrightarrow l) \tag{13a}$$

$$P^i{}_{jk} := -\partial_j f^i{}_k - \Gamma^i{}_{jl} f^l{}_k - (j \leftrightarrow k) \tag{13b}$$

$$T^i{}_{jk} := \Gamma^i{}_{jk} - \Gamma^i{}_{kj} \tag{13c}$$

We define the curvature as the pair (R, P) and torsion T ($= 0$ in our case).

Our theory possesses the following "inhomogeneous objects": metric (g, a), Eq. (2), connection (Γ, f), Eq. (6), curvature (R, P), Eq. (12a).

Finally, we construct the appropriate curvature scalar $\tilde{\mathscr{R}}$. Let us introduce

$$\tilde{\mathscr{R}}_\lambda := \mathscr{R}+\lambda\mathscr{P}, \tag{14}$$

where

$\mathscr{R} := g^{jk}R^i{}_{jki}$, $\mathscr{P} := P^i{}_{ij}a^j$, λ — constant; according to Eq. (1)

$$S_f = \int \tilde{\mathscr{R}}_\lambda |g|^{1/2}d^4x = \int \mathscr{R}|g|^{1/2}d^4x+\lambda \int \mathscr{P}|g|^{1/2}d^4x \tag{15}$$

and after some simple manipulations

$$S_f = \int \mathscr{R}|g|^{1/2}d^4x+\frac{\lambda}{2}\int f^i_j f_i^j|g|^{1/2}d^4x+\lambda\int \partial_j(a_i g^{ik}|g|^{1/2}f_k^j)d^4x. \tag{16}$$

In order to obtain the desired result we have to specify the constant λ and eliminate the total derivative. Concluding one can say that the approach proposed in our note is more natural than that using Finsler space theories. Finslerian connections lead to the curvature tensors which depend on the directional variable [17, 18]. That dependence has been eliminated by the averaging with respect to the directions [4, 5] or by removing the terms which depend on directions [9]; both techniques are very artificial.

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