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INVESTIGATION OF THE WAVE EQUATION FOR ONE DIRAC AND ONE DUFFIN-KEMMER PARTICLE: A NEW FORM OF THE KLEIN PARADOX*

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The wave equation for one Dirac and one Duffin-Kemmer particle proposed recently by Królikowski is investigated. The radial equation derived in a previous paper is written down in the component form and reduced by eliminating auxiliary components of the wave function. Then, the limiting behaviour at $r \to 0$ is checked. In the case of the Duffin-Kemmer spin equal to 1 and the potential having the singularity r^{-a} ($a > 0$) it turns out that there is only one regular solution instead of three, two of them becoming oscillating solutions. It is shown that this phenomenon is a drastic form of the Klein paradox. A possibility is discussed how to apply the derived radial equations to quark-diquark systems, using the regular potential emerging from the finite size of diquarks.

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1. Introduction

The quark-diquark potential model of baryons needs a wave equation for the boson-fermion pair. Such an equation for a pair of a Dirac particle and a Duffin-Kemmer particle has been proposed recently by Królikowski [2]. In the center-of-mass frame this equation has the form

$$\{\beta^0[E-V-\vec{\alpha}\vec{p}-\beta(m+\tfrac{1}{2}S)]+\vec{\beta}\vec{p}-(M+\tfrac{1}{2}S)\}\psi = 0, \tag{1}$$

where $\gamma^\mu = (\beta, \beta\vec{\alpha})$ are Dirac matrices, $\beta^\mu = (\beta^0, \vec{\beta})$ denote Duffin-Kemmer matrices defined by the property

$$\beta^\mu\beta^\nu\beta^\varrho+\beta^\varrho\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\varrho+g^{\nu\varrho}\beta^\mu, \tag{2}$$

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$m(M)$ stands for mass of the Dirac (Duffin-Kemmer) particle, and the potentials V and S describe an interaction in the static approximation. For spherically symmetrical potentials: $V = V(r)$, $S = S(r)$ it is possible to eliminate the angular variables from Eq. (1). A general method of eliminating such variables from two-body wave equations is described in details in the previous paper [1]. Here we recall only the results concerning Eq. (1).

Let us look for solutions of Eq. (1) being eigenfunctions of J^2 and J_z operators with the eigenvalues $j(j+1)$ and m_j, respectively. The general form of such an eigenfunction is given by the formula [1]:

$$\psi_{jm_j}(r, \theta, \varphi) = U^{-1} Z_j^{m_j}(\theta, \varphi)\tilde{\tilde{\psi}}(r), \tag{3}$$

where U describes a unitary transformation

$$U = \exp(i\theta S_2)\exp(i\varphi S_3), \tag{4}$$

$Z_j^{m_j}$ has the meaning of a generalized spherical harmonic

$$\begin{aligned} Z_j^{m_j}(\theta, \varphi) = \sqrt{j+\tfrac{1}{2}}\, e^{i\varphi/2} \Bigg[& \sqrt{\frac{j+\frac{3}{2}}{j-\frac{1}{2}}}\, \Pi_{3/2} \sin\frac{\theta}{2} \cos^2\frac{\theta}{2}\, P_{j-3/2}^{(1,2)}(\cos\theta) \\ + & \quad \Pi_{1/2} \cos\frac{\theta}{2}\, P_{j-1/2}^{(0,1)}(\cos\theta) \\ + & \quad \Pi_{-1/2} \sin\frac{\theta}{2}\, P_{j-1/2}^{(1,0)}(\cos\theta) \\ - & \sqrt{\frac{j+\frac{3}{2}}{j-\frac{1}{2}}}\, \Pi_{-3/2} \sin^2\frac{\theta}{2} \cos\frac{\theta}{2}\, P_{j-3/2}^{(2,1)}(\cos\theta) \Bigg], \end{aligned} \tag{5}$$

and $\tilde{\tilde{\psi}}(r)$ stands for an arbitrary radial wave function. In the above formulae S_i denote the matrices of total spin, Π_{m_s} are the projectors on the subspaces with $S_z = m_s$ and $P_n^{(\alpha,\beta)}(x)$ are Jacobi polynomials. Function $\tilde{\tilde{\psi}}(r)$ has the same number of components as the full wave function $\psi(\vec{x})$. The only constraint for $\tilde{\tilde{\psi}}(r)$ comes in the case of $j = 1/2$ from the condition

$$(\Pi_{3/2}+\Pi_{-3/2})\tilde{\tilde{\psi}}(r) = 0. \tag{6}$$

Formula (3), when substituted into Eq. (1), gives the following radial equation [1]:

$$\Big\{\beta^0[(E-V)-\beta(m+\tfrac{1}{2}S)]-(M+\tfrac{1}{2}S)+[\beta^0\alpha_3-\beta^3]i\frac{d}{dr}$$

$$+\frac{1}{r}\sqrt{(j+\tfrac{3}{2})(j-\tfrac{1}{2})}\,[\beta^0(\Xi_3)^2\alpha_2-\tfrac{1}{2}\beta^2-\tfrac{1}{2}i\Sigma_3(1-2(\Xi_3)^2)\beta^1]$$

$$+\frac{1}{r}(j+\tfrac{1}{2})\,[-\beta^0(1-(\Xi_3)^2)\alpha_2+\tfrac{1}{2}\,\beta^2-\tfrac{1}{2}\,i\Sigma_3(1-2(\Xi_3)^2)\beta^1]$$

$$+\frac{1}{r}\,[\beta^0(\alpha_1 S_2-\alpha_2 S_1)-(\beta^1 S_2-\beta^2 S_1)]\Big\}\,\tilde{\tilde{\psi}}(r)=0, \tag{7}$$

where $\frac{1}{2}\Sigma_k = \frac{i}{4}\varepsilon_{ijk}\gamma^i\gamma^j$ and $\Xi_k = i\varepsilon_{ijk}\beta^i\beta^j$ are the matrices of Dirac and Duffin-Kemmer spin, respectively (total spin $\vec{S} = \frac{1}{2}\vec{\Sigma}+\vec{\Xi}$).

The algebra of Duffin-Kemmer matrices (2) has two irreducible representations: one 5-dimensional describing the spin 0 and another 10-dimensional corresponding to the spin 1. Thus, the radial equation (7) can be written as a system of 20 or 40 equations, respectively. Since some equations in these systems are algebraic and some components of the wave function are auxiliary, it is possible to decrease significantly the effective number of equations. This technically important problem will be discussed in two following sections.

To interpret the components of $\tilde{\tilde{\psi}}(r)$ it is useful to know transformed forms of some physical operators. For a given operator O let us define the operator $\tilde{\tilde{O}}$ by the formula

$$U^{-1}Z_j^{m_j}\tilde{\tilde{O}}\tilde{\tilde{\psi}}(r) \equiv OU^{-1}Z_j^{m_j}\tilde{\tilde{\psi}}(r). \tag{8}$$

Then one can find:

$$\tilde{\tilde{S}}^2 = S^2, \tag{9}$$

$$\tilde{\tilde{L}}^2 = j(j+1)+S^2-2(S_3)^2$$
$$+\sqrt{(j-\tfrac{1}{2})(j+\tfrac{3}{2})}\,[-\Sigma_1(\Xi_3)^2-\Xi_1+\Sigma_3(1-2(\Xi_3)^2)i\Xi_2]$$
$$+(j+\tfrac{1}{2})\,[\Sigma_1(1-(\Xi_3)^2)+\Xi_1+\Sigma_3(1-2(\Xi_3)^2)i\Xi_2], \tag{10}$$

$$\tilde{\tilde{P}} = (-1)^{j-\frac{1}{2}}\Sigma_1[2(\Xi_1)^2-1]\eta\beta[2(\beta^0)^2-1], \tag{11}$$

where P is the total parity operator and η denotes the phase factor in the intrinsic parity operator $\eta\beta[2(\beta^0)^2-1]$.

2. *The case of the Duffin-Kemmer spin equal to 0*

Let us use the specific representations of Dirac and Duffin-Kemmer matrices given in Table I. In the case of Duffin-Kemmer spin equal to 0 the wave function $\tilde{\tilde{\psi}}(r)$ is 20-component and has the form $\tilde{\tilde{\psi}}(r) = (\varphi(r),\ \psi_0(r),\ \psi_1(r),\ \psi_2(r),\ \psi_3(r))$, where $\varphi(r)$ and $\psi_\mu(r)$ are 4-component Dirac spinors. These 4-component spinors can be split into 2-component ones as follows: $\varphi = (\varphi^+, \sigma_3\varphi^-)$, $\psi_0 = (\psi_0^+, \sigma_3\psi_0^-)$, $\psi_1 = (\sigma_3\psi_1^-, \psi_1^+)$, $\psi_2 = (\psi_2^+, \sigma_3\psi_2^-)$, $\psi_3 = (\psi_3^+, \sigma_3\psi_3^-)$. So, if we substitute the wave function $\tilde{\tilde{\psi}}(r)$ expressed in terms of $\varphi^\varepsilon(r)$ and $\psi_\mu^\varepsilon(r)$ $(\varepsilon = \pm 1)$ into Eq. (7), we obtain two independent systems of equations with

$\varepsilon = \pm 1$ listed in Table II. The fact that the systems corresponding to $\varepsilon = +1$ and $\varepsilon = -1$ decouple from each other is connected with the correspondence of $\varepsilon = +1$ and $\varepsilon = -1$ to two different eigenvalues of total parity operator (11):

$$\tilde{\tilde{P}}\tilde{\tilde{\psi}}^{\varepsilon} = -(-1)^{j-\frac{1}{2}}\eta\varepsilon\tilde{\tilde{\psi}}^{\varepsilon}, \tag{12}$$

which implies that ε is an additional "good" quantum number.

TABLE I

Used representations of Dirac and Duffin-Kemmer matrices

Representation of Dirac matrices:

$$\alpha_1 = -\sigma_2 \otimes \sigma_2 = \left[\begin{array}{c|c} & i\sigma_2 \\ \hline -i\sigma_2 & \end{array}\right], \quad \alpha_2 = 1 \otimes \sigma_1 = \left[\begin{array}{c|c} \sigma_1 & \\ \hline & \sigma_1 \end{array}\right], \quad \alpha_3 = \sigma_3 \otimes \sigma_2 = \left[\begin{array}{c|c} \sigma_2 & \\ \hline & -\sigma_2 \end{array}\right],$$

$$\beta = 1 \otimes \sigma_3 = \left[\begin{array}{c|c} \sigma_3 & \\ \hline & \sigma_3 \end{array}\right].$$

5-dimensional representation of Duffin-Kemmer matrices (corresponding to the spin 0):

$$\beta^0 = \left[\begin{array}{c|cccc} & i & 0 & 0 & 0 \\ \hline -i & & & & \\ 0 & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right], \quad \beta^1 = \left[\begin{array}{c|cccc} & 0 & i & 0 & 0 \\ \hline 0 & & & & \\ i & & & & \\ 0 & & & & \\ 0 & & & & \end{array}\right], \quad \beta^2 = \left[\begin{array}{c|cccc} & 0 & 0 & i & 0 \\ \hline 0 & & & & \\ 0 & & & & \\ i & & & & \\ 0 & & & & \end{array}\right], \quad \beta^3 = \left[\begin{array}{c|cccc} & 0 & 0 & 0 & i \\ \hline 0 & & & & \\ 0 & & & & \\ 0 & & & & \\ i & & & & \end{array}\right].$$

10-dimensional representation of Duffin-Kemmer matrices (corresponding to the spin 1):

$$\beta^0 = \left[\begin{array}{c|ccc|ccc|ccc} 0 & & & & & & & & & \\ \hline & & & & & & & i & 0 & 0 \\ & & & & & & & 0 & i & 0 \\ & & & & & & & 0 & 0 & i \\ \hline & & & & 0 & 0 & 0 & & & \\ & & & & 0 & 0 & 0 & & & \\ & & & & 0 & 0 & 0 & & & \\ \hline & -i & 0 & 0 & & & & & & \\ & 0 & -i & 0 & & & & & & \\ & 0 & 0 & -i & & & & & & \end{array}\right], \quad \beta^1 = \left[\begin{array}{c|ccc|ccc|ccc} & & & & & & & i & 0 & 0 \\ \hline & & & & 0 & 0 & 0 & & & \\ & & & & 0 & 0 & -i & & & \\ & & & & 0 & -i & 0 & & & \\ \hline & 0 & 0 & 0 & & & & & & \\ & 0 & 0 & i & & & & & & \\ & 0 & -i & 0 & & & & & & \\ \hline i & & & & & & & & & \\ 0 & & & & & & & & & \\ 0 & & & & & & & & & \end{array}\right],$$

$$\beta^2 = \left[\begin{array}{c|ccc|ccc|ccc} & & & & & & & 0 & i & 0 \\ \hline & & & & 0 & 0 & i & & & \\ & & & & 0 & 0 & 0 & & & \\ & & & & -i & 0 & 0 & & & \\ \hline & 0 & 0 & -i & & & & & & \\ & 0 & 0 & 0 & & & & & & \\ & i & 0 & 0 & & & & & & \\ \hline 0 & & & & & & & & & \\ i & & & & & & & & & \\ 0 & & & & & & & & & \end{array}\right], \quad \beta^3 = \left[\begin{array}{c|ccc|ccc|ccc} & & & & & & & 0 & 0 & i \\ \hline & & & & 0 & -i & 0 & & & \\ & & & & i & 0 & 0 & & & \\ & & & & 0 & 0 & 0 & & & \\ \hline & 0 & i & 0 & & & & & & \\ & -i & 0 & 0 & & & & & & \\ & 0 & 0 & 0 & & & & & & \\ \hline 0 & & & & & & & & & \\ 0 & & & & & & & & & \\ i & & & & & & & & & \end{array}\right].$$

(The empty fields are understood to be zeros.)

For each ε the system in Table II can be further reduced by eliminating auxiliary components of the wave function. If we substitute Eqs. (b), (c), (d) and (e) into Eq. (a), we are left with the single equation for the component $\varphi^{\varepsilon}(r)$

$$\left\{ -i\sigma_2 \left(\frac{d}{dr} + \frac{1}{r} \right) - \sigma_1 k \frac{1}{r} + \sigma_3 (m + \tfrac{1}{2} S) \right.$$

$$-\tfrac{1}{2}\frac{(E-V)^2+(m+\frac{1}{2}S)^2-(M+\frac{1}{2}S)^2}{(E-V)}$$

$$-\tfrac{1}{4}S'\frac{1}{M+\frac{1}{2}S}\frac{1}{E-V}\frac{1}{r}(1-k\sigma_3)+\tfrac{1}{2}(V'-\tfrac{1}{2}S'\sigma_3)\frac{1}{E-V}i\sigma_2$$

$$+\tfrac{1}{4}S'\frac{1}{M+\frac{1}{2}S}i\sigma_2+\tfrac{1}{4}S'\frac{m+\frac{1}{2}S}{M+\frac{1}{2}S}\frac{1}{E-V}\sigma_1\Big\}\varphi^\varepsilon(r)=0, \tag{13}$$

where $V'(r) = \dfrac{d}{dr}V(r)$, $S'(r) = \dfrac{d}{dr}S(r)$, $k = -\varepsilon(j+\frac{1}{2})$. Since $\varphi^\varepsilon(r)$ is a two-component spinor, Eq. (13) is actually a system of two first-order differential equations. The last four terms on the l.h.s. of Eq. (13) are specific for the interaction of a Dirac particle with the

TABLE II

The set of radial equations following from Eq. (1) in the case of Duffin-Kemmer spin equal to 0

$$-(M+\tfrac{1}{2}S)\varphi^\varepsilon+\left[(E-V)-\sigma_3(m+\tfrac{1}{2}S)+i\sigma_2\left(\frac{d}{dr}+\frac{1}{r}\right)-\sigma_1\varepsilon(j+\tfrac{1}{2})\frac{1}{r}\right]i\psi_0^\varepsilon+\left[-\tfrac{1}{2}\frac{1}{r}\right.$$
$$\left.-\tfrac{1}{2}\varepsilon\sigma_3(j+\tfrac{1}{2}+\surd)\frac{1}{r}\right]i\psi_1^\varepsilon+\left[\tfrac{1}{2}\varepsilon\sigma_3\frac{1}{r}+\tfrac{1}{2}(j+\tfrac{1}{2}-\surd)\frac{1}{r}\right]i\psi_2^\varepsilon+\left(\frac{d}{dr}+\frac{2}{r}\right)\psi_3^\varepsilon=0, \tag{a}$$

$$\left[-(E-V)+\sigma_3(m+\tfrac{1}{2}S)-i\sigma_2\left(\frac{d}{dr}+\frac{1}{r}\right)+\sigma_1\varepsilon(j+\tfrac{1}{2})\frac{1}{r}\right]i\varphi^\varepsilon-(M+\tfrac{1}{2}S)\psi_0^\varepsilon=0, \tag{b}$$

$$\left[-\tfrac{1}{2}\frac{1}{r}-\tfrac{1}{2}\varepsilon\sigma_3(j+\tfrac{1}{2}+\surd)\frac{1}{r}\right]i\varphi^\varepsilon-(M+\tfrac{1}{2}S)\psi_1^\varepsilon=0, \tag{c}$$

$$\left[\tfrac{1}{2}\varepsilon\sigma_3\frac{1}{r}+\tfrac{1}{2}(j+\tfrac{1}{2}-\surd)\frac{1}{r}\right]i\varphi^\varepsilon-(M+\tfrac{1}{2}S)\psi_2^\varepsilon=0, \tag{d}$$

$$\frac{d}{dr}\varphi^\varepsilon-(M+\tfrac{1}{2}S)\psi_3^\varepsilon=0. \tag{e}$$

Abbreviation: $\surd = \sqrt{(j+\frac{3}{2})(j-\frac{1}{2})}$.

spin-0 Duffin-Kemmer particle. In the limit $M \to \infty$ these terms can be neglected and we get the equation for a Dirac particle moving in external potentials $V(r)$ and $S(r)$.

In the nonrelativistic limit the upper component of $\varphi^\varepsilon(r)$ is the "large" one. By acting of the orbital momentum operator (10) on the wave function $\tilde{\tilde{\psi}}^\varepsilon(r)$ one can find that the upper component of $\varphi^\varepsilon(r)$ corresponds to the eigenvalue of $\tilde{\tilde{L}}^2$ equal to $(j+\frac{1}{2}\varepsilon)(j+\frac{1}{2}\varepsilon+1)$. Thus, Eq. (13) is a relativistic equation for the states denoted by nonrelativistic spectroscopic symbols $n^s l_j^P$, where $s = \frac{1}{2}$, $l = j+\frac{1}{2}\varepsilon$ and $P = \eta(-1)^l$.

For the Duffin-Kemmer spin equal to 0 the total spin is equal to 1/2 and the constraint (6) does not appear.

3. *The case of the Duffin-Kemmer spin equal to 1*

In the case of Duffin-Kemmer spin equal to 1 we will use the 10-dimensional representation of Duffin-Kemmer matrices written down in Table I. The 40-component wave function $\tilde{\tilde{\psi}}(r)$ expressed by 4-component spinors has the form $\tilde{\tilde{\psi}}(r) = (\psi_0, \psi_1, \psi_2, \psi_3, \varphi_1, \varphi_2, \varphi_3, \chi_1, \chi_2, \chi_3)$. For the spinors $\psi_\mu(r)$, $\varphi_k(r)$ and $\chi_k(r)$ we assume the following 2-component notation: $\psi_0 = (\sigma_3\psi_0^-, \psi_0^+)$, $\psi_1 = (\psi_1^+, \sigma_3\psi_1^-)$, $\psi_2 = (\sigma_3\psi_2^-, \psi_2^+)$, $\psi_3 = (\sigma_3\psi_3^-, \psi_3^+)$, $\varphi_1 = (\sigma_3\varphi_1^-, \varphi_1^+)$, $\varphi_2 = (\varphi_2^+, \sigma_3\varphi_2^-)$, $\varphi_3 = (\varphi_3^+, \sigma_3\varphi_3^-)$, $\chi_1 = (\chi_1^+, \sigma_3\chi_1^-)$, $\chi_2 = (\sigma_3\chi_2^-, \chi_2^+)$, $\chi_3 = (\sigma_3\chi_3^-, \chi_3^+)$, Substituting the wave function $\tilde{\tilde{\psi}}(r)$ expressed by $\psi_\mu^\varepsilon(r)$, $\varphi_k^\varepsilon(r)$ and $\chi_k^\varepsilon(r)$ ($\varepsilon = \pm 1$) into Eq. (7) one can get two separated systems of equations corresponding to two values of ε. As previously, this splitting is connected with the relation between ε and the total parity (Eq. (12)).

For given j, m_j, and ε the resulting system contains 20 equations. The simplest form of this system can be obtained when introducing a Jordan basis for the matrix $\beta^0\alpha_3 - \beta^3$ standing in Eq. (7) at the differential operator $\frac{d}{dr}$. According to this, let us define new components $f_\mu^\varepsilon(r)$, $g_k^\varepsilon(r)$ and $h_k^\varepsilon(r)$ by the relations

$$\begin{aligned}
\psi_0^\varepsilon &= \frac{1}{\sqrt{2}} f_1^\varepsilon + \frac{1}{\sqrt{2}} h_1^\varepsilon,\\
\psi_1^\varepsilon &= -\frac{1}{\sqrt{2}} g_2^\varepsilon - \frac{\varepsilon}{\sqrt{2}} g_3^\varepsilon,\\
\psi_2^\varepsilon &= -\frac{\varepsilon}{\sqrt{2}} \sigma_3 g_2^\varepsilon + \frac{1}{\sqrt{2}} \sigma_3 g_3^\varepsilon,\\
\psi_3^\varepsilon &= \frac{1}{\sqrt{2}} \sigma_2 f_1^\varepsilon - \frac{1}{\sqrt{2}} \sigma_2 h_1^\varepsilon,\\
\varphi_1^\varepsilon &= -\frac{\varepsilon}{2} \sigma_3 f_2^\varepsilon - \frac{\varepsilon}{2} \sigma_3 h_2^\varepsilon + \tfrac{1}{2} \sigma_3 f_3^\varepsilon + \tfrac{1}{2} \sigma_3 h_3^\varepsilon,\\
\varphi_2^\varepsilon &= \tfrac{1}{2} f_2^\varepsilon + \tfrac{1}{2} h_2^\varepsilon + \frac{\varepsilon}{2} f_3^\varepsilon + \frac{\varepsilon}{2} h_3^\varepsilon,\\
\varphi_3^\varepsilon &= i\varepsilon f_0^\varepsilon,\\
\chi_1^\varepsilon &= -\tfrac{1}{2} \sigma_2 f_2^\varepsilon + \tfrac{1}{2} \sigma_2 h_2^\varepsilon - \frac{\varepsilon}{2} \sigma_2 f_3^\varepsilon + \frac{\varepsilon}{2} \sigma_2 h_3^\varepsilon,\\
\chi_2^\varepsilon &= i\frac{\varepsilon}{2} \sigma_1 f_2^\varepsilon - i\frac{\varepsilon}{2} \sigma_1 h_2^\varepsilon - i\tfrac{1}{2} \sigma_1 f_3^\varepsilon + i\tfrac{1}{2} \sigma_1 h_3^\varepsilon,\\
\chi_3^\varepsilon &= g_1^\varepsilon,
\end{aligned} \tag{14}$$

(all these functions are 2-component spinors). Then we obtain the system of equations listed in Table III.

TABLE III

The set of radial equations following from Eq. (1) in the case of Duffin-Kemmer spin equal to 1

$$-\varkappa_2 f_1^\varepsilon+\left[\frac{3}{\sqrt{2}}\frac{1}{r}+\sqrt{2}\frac{d}{dr}+\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2+\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_3\right]g_1^\varepsilon$$
$$+\left[\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_1+\frac{1}{\sqrt{2}}\frac{1}{r}i\sigma_2\right]f_2^\varepsilon-\sqrt{2}\frac{1}{r}i\sigma_2h_2^\varepsilon-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_1f_3^\varepsilon=0, \qquad \text{(a)}$$

$$\left[-\frac{1}{\sqrt{2}}\frac{1}{r}+\frac{1}{\sqrt{2}}\varkappa_1\sigma_1-\frac{1}{\sqrt{2}}vi\sigma_2+\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_3\right]f_1^\varepsilon+\varkappa_2g_1^\varepsilon$$
$$+\left[\frac{1}{\sqrt{2}}\frac{1}{r}+\sqrt{2}\frac{d}{dr}-\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2-\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_3\right]h_1^\varepsilon-\sqrt{2}\frac{1}{r}i\sigma_2g_2^\varepsilon=0, \qquad \text{(b)}$$

$$\left[\frac{1}{\sqrt{2}}\frac{1}{r}-\frac{1}{\sqrt{2}}\varkappa_1\sigma_1-\frac{1}{\sqrt{2}}vi\sigma_2-\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_3\right]g_1^\varepsilon-\varkappa_2h_1^\varepsilon$$
$$+\left[-\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_1+\frac{1}{\sqrt{2}}\frac{1}{r}i\sigma_2\right]h_2^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_1h_3^\varepsilon=0, \qquad \text{(c)}$$

$$\left[-\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_1-\frac{1}{\sqrt{2}}\frac{1}{r}i\sigma_2\right]f_1^\varepsilon+\sqrt{2}\frac{1}{r}i\sigma_2h_1^\varepsilon-\varkappa_2f_2^\varepsilon$$
$$+\left[\sqrt{2}\frac{1}{r}+\sqrt{2}\frac{d}{dr}-\frac{1}{\sqrt{2}}\varkappa_1\sigma_1-\frac{1}{\sqrt{2}}vi\sigma_2\right]g_2^\varepsilon-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3g_3^\varepsilon=0, \qquad \text{(d)}$$

$$\sqrt{2}\frac{1}{r}i\sigma_2g_1^\varepsilon+\left[-\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2\right]f_2^\varepsilon-\varkappa_2g_2^\varepsilon+\left[\sqrt{2}\frac{1}{r}+\sqrt{2}\frac{d}{dr}+\frac{1}{\sqrt{2}}\varkappa_1\sigma_1-\frac{1}{\sqrt{2}}vi\sigma_2\right]h_2^\varepsilon$$
$$-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3f_3^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3h_3^\varepsilon+\left[-\frac{1}{\sqrt{2}}k\frac{1}{r}+\frac{1}{\sqrt{2}}\frac{1}{r}\sigma_3\right]f_0^\varepsilon=0, \qquad \text{(e)}$$

$$\left[\frac{1}{\sqrt{2}}k\frac{1}{r}\sigma_1-\frac{1}{\sqrt{2}}\frac{1}{r}i\sigma_2\right]h_1^\varepsilon+\left[\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2\right]g_2^\varepsilon-\varkappa_2h_2^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3g_3^\varepsilon=0, \qquad \text{(f)}$$

$$\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_1f_1^\varepsilon-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3g_2^\varepsilon-\varkappa_2f_3^\varepsilon+\left[\sqrt{2}\frac{1}{r}+\sqrt{2}\frac{d}{dr}-\frac{1}{\sqrt{2}}\varkappa_1-\frac{1}{\sqrt{2}}vi\sigma_2\right]g_3^\varepsilon=0, \qquad \text{(g)}$$

$$-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3f_2^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3h_2^\varepsilon+\left[-\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2\right]f_3^\varepsilon$$
$$-\varkappa_2g_3^\varepsilon+\left[\sqrt{2}\frac{1}{r}+\sqrt{2}\frac{d}{dr}+\frac{1}{\sqrt{2}}\varkappa_1\sigma_1-\frac{1}{\sqrt{2}}vi\sigma_2\right]h_3^\varepsilon-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}f_0^\varepsilon=0, \qquad \text{(h)}$$

$$-\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_1h_1^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}\sigma_3g_2^\varepsilon+\left[\frac{1}{\sqrt{2}}\varkappa_1\sigma_1+\frac{1}{\sqrt{2}}vi\sigma_2\right]g_3^\varepsilon-\varkappa_2h_3^\varepsilon=0, \qquad \text{(i)}$$

$$\left[\frac{1}{\sqrt{2}}k\frac{1}{r}-\frac{1}{\sqrt{2}}\sigma_3\frac{1}{r}\right]g_2^\varepsilon+\frac{1}{\sqrt{2}}\sqrt{}\frac{1}{r}g_3^\varepsilon-\varkappa_2f_0^\varepsilon=0. \qquad \text{(j)}$$

Notation: $\varkappa_1=m+\frac{1}{2}S(r)$, $\quad\varkappa_2=M+\frac{1}{2}S(r)$, $\quad v=E-V(r)$, $\quad k=-\varepsilon(j+\frac{1}{2})$,

$\sqrt{}=\sqrt{(j+\frac{3}{2})(j-\frac{1}{2})}=\sqrt{k^2-1}$.

We can see that in Table III the equations (c), (f) and (i) are purely algebraic, while the components $f^{\varepsilon}_{\mu}(r)$ are not differentiated anywhere. Due to this we can eliminate all the functions f^{ε}_{μ} and also h^{ε}_{k}. This elimination consists of two steps. In the first step we remove the components f^{ε}_{2} $h^{\varepsilon}_{2}, f^{\varepsilon}_{3}$, h^{ε}_{3} and f^{ε}_{0} using the equations (d), (f), (g), (i) and (j), in the second we take away f^{ε}_{1} and h^{ε}_{1} substituting (a) into (b) and then (c) into (b). The practical calculations were performed with the help of the computer program for algebraic manipulations "SCHOONSCHIP" [3]. The results are given in Table IV. We are left with 3 equations for the 2-component spinors g^{ε}_{1}, g^{ε}_{2} and g^{ε}_{3}, so we have altogether 6 differential equations of the first order.

TABLE IV

The equations following from the set in Table III by eliminating auxiliary components of the wave function (see text)

$$
\begin{aligned}
&\Bigg\{-2v\frac{d}{dr}-2v\frac{1}{r}(2-\varrho\varkappa_2^2)-S'\frac{1}{\varkappa_2}v(\tfrac{1}{2}-\varrho\varkappa_2^2)+V'+\Bigg[-kS'\frac{1}{\varkappa_2}\frac{1}{r}(\tfrac{1}{2}-\varrho\varkappa_2^2)-k\varrho\frac{1}{r^2}(v^2+\varkappa_1^2-\varkappa_2^2)\\
&-2\varkappa_1 v\left(1+\varrho\frac{1}{r^2}\right)\Bigg]\sigma_1+\Bigg[-(v^2+\varkappa_1^2-\varkappa_2^2)\left(1+\varrho\frac{1}{r^2}\right)-2k\varrho\varkappa_1 v\frac{1}{r^2}-S'\frac{1}{\varkappa_2}\frac{1}{r}(\tfrac{1}{2}-\varrho\varkappa_2^2)\Bigg]i\sigma_2\\
&+\Bigg[-2kv\frac{1}{r}+S'\frac{1}{\varkappa_2}\varkappa_1(\tfrac{1}{2}-\varrho\varkappa_2^2)+\tfrac{1}{2}S'\Bigg]\sigma_3\Bigg\}g^{\varepsilon}_1(r)+\frac{1}{\sqrt{2}}\Bigg\{-\Bigg[V'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho v\frac{1}{r}\Bigg]\sigma_1(k-\sigma_3)\\
&-\Bigg[-\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho\varkappa_1\frac{1}{r}\Bigg]i\sigma_2(k-\sigma_3)+S'(1-\varrho\varkappa_2^2)-\frac{1}{\varkappa_2}\frac{1}{r}(1+\varrho\varkappa_2^2)(v^2-\varkappa_1^2+\varkappa_2^2)\Bigg\}g^{\varepsilon}_2(r)\\
&\qquad+\frac{1}{\sqrt{2}}\sqrt{\ }\Bigg\{\Bigg[V'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho v\frac{1}{r}\Bigg]\sigma_1+\Bigg[-\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho\varkappa_1\frac{1}{r}\Bigg]i\sigma_2\\
&\qquad\qquad-\Bigg[S'\varrho\frac{1}{r^2}+\varrho\frac{1}{\varkappa_2}\frac{1}{r^3}(v^2-\varkappa_1^2+\varkappa_2^2)\Bigg](k+\sigma_3)\Bigg\}g^{\varepsilon}_3(r)=0, \qquad \text{(a)}
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{\sqrt{2}}\Bigg\{\Bigg[V'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho v\frac{1}{r}\Bigg]\sigma_1(k-\sigma_3)+\Bigg[-\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho\varkappa_1\frac{1}{r}\Bigg]i\sigma_2(k-\sigma_3)+S'(1-\varrho\varkappa_2^2)\\
&-2\varrho\varkappa_2\frac{1}{r}(v^2-\varkappa_1^2+\varkappa_2^2)\Bigg\}g^{\varepsilon}_1(r)+\Bigg\{-2v\frac{d}{dr}-2v\frac{1}{r}(2-\varrho\varkappa_2^2)+\tfrac{1}{2}S'\varrho\varkappa_2 v+V'+\Bigg[\tfrac{1}{2}kS'\frac{1}{\varkappa_2}\frac{1}{r}(1-\varrho\varkappa_2^2)\\
&+k\varrho\frac{1}{r^2}(v^2+\varkappa_1^2-\varkappa_2^2)+2\varkappa_1 v\left(1+\varrho\frac{1}{r^2}\right)\Bigg]\sigma_1+\Bigg[\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}(1-\varrho\varkappa_2^2)+(v^2+\varkappa_1^2+\varkappa_2^2)\left(1+\varrho\frac{1}{r^2}\right)\\
&+2k\varrho\frac{1}{r^2}\varkappa_1 v\Bigg]i\sigma_2+[-\tfrac{1}{2}S'\varrho\varkappa_1\varkappa_2+\tfrac{1}{2}S']\sigma_3\Bigg\}g^{\varepsilon}_2(r)+\sqrt{\ }\Bigg\{\Bigg[\tfrac{1}{2}S'\varrho\varkappa_2\frac{1}{r}-\varrho\frac{1}{r^2}(v^2+\varkappa_1^2-\varkappa_2^2)\Bigg]\sigma_1\\
&-2\varrho\varkappa_1 v\frac{1}{r^2}i\sigma_2+\Bigg[-\tfrac{1}{2}kS'\varrho\frac{1}{\varkappa_2}\varkappa_1\frac{1}{r^2}+\tfrac{1}{2}S'\varrho\frac{1}{\varkappa_2}v\frac{1}{r^2}+2v\frac{1}{r}\left(1+\varrho\frac{1}{r^2}\right)\Bigg]\sigma_3-\tfrac{1}{2}S'\varrho\frac{1}{\varkappa_2}\varkappa_1\frac{1}{r^2}\\
&\qquad\qquad+\tfrac{1}{2}kS'\varrho\frac{1}{\varkappa_2}v\frac{1}{r^2}+2k\varrho v\frac{1}{r^3}\Bigg\}g^{\varepsilon}_3(r)=0, \qquad \text{(b)}
\end{aligned}
$$

TABLE IV (Continued)

$$\frac{1}{\sqrt{2}}\sqrt{\ }\left\{-\left[V'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho v\frac{1}{r}\right]\sigma_1-\left[-\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}+S'\varrho\varkappa_1\frac{1}{r}\right]i\sigma_2-S'\varrho\frac{1}{r^2}(k-\sigma_3)\right\}g_1^\varepsilon(r)$$

$$+\sqrt{\ }\left\{\tfrac{1}{2}S'\varrho\varkappa_2\frac{1}{r}\sigma_1+\left[-\tfrac{1}{2}kS'\varrho\frac{1}{\varkappa_2}\varkappa_1\frac{1}{r^2}-\tfrac{1}{2}S'\varrho\frac{1}{\varkappa_2}v\frac{1}{r^2}+2v\frac{1}{r}\right]\sigma_3+\tfrac{1}{2}S'\varrho\frac{1}{\varkappa_2}\varkappa_1\frac{1}{r^2}\right.$$

$$\left.+\tfrac{1}{2}kS'\varrho\frac{1}{\varkappa_2}v\frac{1}{r^2}\right\}g_2^\varepsilon(r)+\left\{-2v\frac{d}{dr}-2v\frac{1}{r}+\tfrac{1}{2}S'\varrho\varkappa_2 v+V'+\left[\tfrac{1}{2}kS'\frac{1}{\varkappa_2}\frac{1}{r}(1-\varrho\varkappa_2^2)+2\varkappa_1 v\right]\sigma_1\right.$$

$$\left.+\left[-\tfrac{1}{2}S'\frac{1}{\varkappa_2}\frac{1}{r}(1-\varrho\varkappa_2^2)+(v^2+\varkappa_1^2-\varkappa_2^2)\right]i\sigma_2+[-\tfrac{1}{2}S'\varrho\varkappa_1\varkappa_2+\tfrac{1}{2}S']\sigma_3\right\}g_3^\varepsilon(r)=0. \qquad \text{(c)}$$

Notation: $\varkappa_1 = m+\frac{1}{2}S(r)$, $\quad \varkappa_2 = M+\frac{1}{2}S(r)$, $\quad v = E-V(r)$, $\quad k = -\varepsilon(j+\frac{1}{2})$, $\quad V' = \frac{d}{dr}V(r)$,

$$S' = \frac{d}{dr}S(r), \qquad \varrho = \left[\varkappa_2^2+\frac{1}{r^2}(k^2-1)\right]^{-1}, \qquad \sqrt{\ } = \sqrt{(j+\tfrac{3}{2})(j-\tfrac{1}{2})} = \sqrt{k^2-1}\,.$$

In the nonrelativistic limit ($|\vec{p}|$, $V(r)$, $S(r) \ll M$, m) the system listed in Table IV splits into 3 separated subsystems, each one corresponding to the Schrödinger-like equation for a combination of the upper ("large") components of spinors g_k^ε. Acting by the operators $\tilde{\tilde{S}}^2$ and $\tilde{\tilde{L}}^2$ (Eqs. (9) and (10)) on these combinations one can recognize that they correspond to $s = \frac{1}{2}$ and $l = j+\frac{1}{2}\varepsilon$ *or* $s = \frac{3}{2}$ and $l = j+\frac{1}{2}\varepsilon$ *or* $s = \frac{3}{2}$ and $l = j-\frac{3}{2}\varepsilon$. In the relativistic case these states are mixed because of the spin-orbit interactions (there is also a "small" admixture of other states). Nevertheless, the total number of degrees of freedom should be preserved, i.e. for fixed j, m_j and ε the system in Table IV should describe *three* sequences of states.

The case $j = 1/2$ needs a special attention. The system in Table IV divides into two parts: the first containing Eqs. (a) and (b) for spinors g_1^ε and g_2^ε and the second one given by Eq. (c) for the spinor g_3^ε. This separation enables us to impose in a consistent way the constraint (6) which now assumes the simple form: $g_3^\varepsilon \equiv 0$. Physically, this condition means that the states with $l = -1$ *or* $l = 0$, $s = \frac{3}{2}$, $j = \frac{1}{2}$ do not exist. In fact, one can easily find, that the spinor g_3^ε corresponds to these unwanted states. Concluding, in the case of $j = 1/2$ the system contains Eqs. (a) and (b) and describes *two* sequences of states (corresponding in the nonrelativistic limit to $l = 1$, $s = \frac{1}{2}$ *or* $l = 1$, $s = \frac{3}{2}$ for $\varepsilon = +1$ and $l = 0$, $s = \frac{1}{2}$ *or* $l = 2$, $s = \frac{3}{2}$ for $\varepsilon = -1$).

4. *Behaviour for $r \to 0$*

Generally, the number of solutions regular at $r = 0$ should be equal to the number of degrees of freedom. It means, that Eq. (13) should have one regular solution, but the system in Table IV should get three regular solutions. Obviously, such a situation takes place if the potentials $V(r)$ and $S(r)$ are regular at $r = 0$, since then the equations become asymptotically free. Let us now investigate the case, when the scalar potential $S(r)$ is regular (the value $S(0)$ can be put zero by the appropriate redefinition of masses), but the vector potential $V(r)$ has a weak power-like singularity:

$$V(r) \underset{r\to 0}{\sim} b_0 r^{-a_0}, \quad 0 < a_0 < 1,$$

$$S(r) \underset{r\to 0}{\to} 0. \tag{15}$$

Substituting the behaviour (15) into Eq. (13) and keeping only the leading terms one can obtain the asymptotic form of the wave equation for the case of Duffin-Kemmer spin equal to 0:

$$\left\{-i\sigma_2\left(\frac{d}{dr}+\frac{1}{r}\right)-\sigma_1 k\frac{1}{r}-\tfrac{1}{2}\frac{V'(r)}{V(r)}i\sigma_2+o(V(r))\right\}\varphi^{\varepsilon}(r)=0. \tag{16}$$

Now, it is easy to show that there is one regular solution (with behaviour $r^{j-\frac{1}{2}+\frac{1}{2}a_0}$) and one irregular (with singularity $r^{-j-\frac{3}{2}+\frac{1}{2}a_0}$). Thus, the number of regular solutions is consistent with our expectations.

The situation is different in the case of Duffin-Kemmer spin equal to one. The asymptotic form of the equations listed in Table IV is the following:

$$2V(r)\frac{d}{dr}g_1^{\varepsilon}(r)-\frac{1}{\sqrt{2}}V'(r)\frac{1}{M}\frac{1}{r}\sigma_1(k-\sigma_3)g_2^{\varepsilon}(r)$$

$$+\frac{1}{\sqrt{2}}\sqrt{k^2-1}\,V'(r)\frac{1}{M}\frac{1}{r}\sigma_1 g_3^{\varepsilon}(r)=0, \tag{17a}$$

$$\frac{1}{\sqrt{2}}V'(r)\frac{1}{M}\frac{1}{r}\sigma_1(k-\sigma_3)g_1^{\varepsilon}(r)+2V(r)\frac{d}{dr}g_2^{\varepsilon}(r)=0, \tag{17b}$$

$$-\frac{1}{\sqrt{2}}\sqrt{k^2-1}\,V'(r)\frac{1}{M}\frac{1}{r}\sigma_1 g_1^{\varepsilon}(r)+2V(r)\frac{d}{dr}g_3^{\varepsilon}(r)=0. \tag{17c}$$

Assuming the solution in the form

$$(g_1^{\varepsilon}, g_2^{\varepsilon}, g_3^{\varepsilon}) \sim (A_1, A_2, A_3)\exp\left(\frac{\kappa}{r}\right) \tag{18}$$

one can obtain that six independent solutions correspond to the following values of κ:

$$\kappa_1=\kappa_2=0,$$

$$\kappa_3=\kappa_4=+i\tfrac{1}{2}\sqrt{k^2-1}\frac{1}{M}, \qquad \kappa_5=\kappa_6=-i\tfrac{1}{2}\sqrt{k^2-1}\frac{1}{M}. \tag{19}$$

Solutions "1" and "2" have the asymptotic form different from (18). By a careful investigation of higher order terms it is possible to show that the leading behaviour of these solutions is power-like with the exponent $j-\frac{1}{2}+\frac{1}{2}a_0$ (regular behaviour) or $-j-\frac{3}{2}+\frac{1}{2}a_0$ (irregular behaviour). The remaining solutions "3" to "6" are oscillating with the phase rising to infinity as r approaches zero. So, we have only one regular solution instead of three.

In the case of $j = 1/2$ the situation is similar. The asymptotic behaviour of the wave function is of the type $\exp(\kappa r^{-a_0/2})$. From the four possible values of κ one is negative (providing regular solution), one is positive (leading to exponentially divergent solution) and two are imaginary (giving irregular oscillating solutions). Thus, in this case we get only one regular solution instead of two.

The above considerations formally do not apply to the potentials with the Coulomb singularity. Nonetheless, carefully investigating the case of Coulomb potential one can find out that in this case the asymptotic solutions at $r = 0$, though differing in details, are qualitatively very similar. In particular, there appear oscillating behaviours instead of regular ones when the Duffin-Kemmer spin is equal to 1.

If the number of regular solutions is smaller than the number of degrees of freedom, we fall into a contradiction and the equation (1) cannot be used in a consistent way. The lack of a sufficient number of regular solutions in the above examples is connected with the appearance of oscillating solutions. Oscillating solutions of other Dirac-type wave equations are known to be characteristic for the Klein paradox. The possible interpretation of the contradiction found in this paper as a consequence of the Klein paradox will be discussed in the next section.

5. *A new Klein paradox?*

In Ref. [4] it was shown, how to obtain from Eq. (1) the equation consistent with the hole theory. The modification has the form

$$\beta^0 V(\vec{r}) \to \beta^0 P(\vec{p}) V(\vec{r}), \tag{20}$$

where[1]

$$\beta^0 P(\vec{p}) = \tfrac{1}{2}\,\beta^0(\vec{\alpha}\vec{p}+\beta m)\frac{1}{\sqrt{p^2+m^2}} + \tfrac{1}{2}\,(-\vec{\beta}\vec{p}+M)\frac{1}{\sqrt{p^2+M^2}}\,. \tag{21}$$

The nonlocal projecting operator a la Salpeter, $P(\vec{p})$, eliminates the sea of particle-antiparticle states responsible for the Klein paradox. In order to show that the oscillating solu-

[1] When in particular there is no scalar potential $S(r)$, the operator $P(\vec{p})$ can be written in the form

$$P(\vec{p}) = \Lambda^{\rm D}_{+}(\vec{p})\Lambda^{\rm DK}_{+}(-\vec{p})-\Lambda^{\rm D}_{-}(\vec{p})\Lambda^{\rm DK}_{-}(-\vec{p}),$$

where

$$\Lambda^{\rm DK}_{\pm}(\vec{p}) = \left(1-\frac{\vec{\beta}\vec{p}}{M}\right)\left(\tfrac{1}{2}(\beta^0)^2 \pm \tfrac{1}{2}\beta^0\frac{\vec{\beta}\vec{p}+M}{\sqrt{p^2+M^2}}\right).$$

Here, $\Lambda^{\rm D}_{\pm}$ and $\Lambda^{\rm DK}_{\pm}$ are projectors onto the subspaces spanned by the positive (+) or negative (−) energy solutions of the Dirac (D) and Duffin-Kemmer (DK) free equations. In this argument we were able to assume consistently (when $S(r) = 0$), that the wave function satisfies the constraint $\vec{\beta}\vec{p}(\beta^0)^2\psi = M(1-(\beta^0)^2)\psi$.

tions found in the previous section are connected with the appearance of the Klein paradox it is sufficient to prove that $\beta^0 P(\vec{p})$ vanishes asymptotically in acting on $\psi(\vec{r})$:

$$|\beta^0 P(\vec{p})\psi(\vec{r})/\psi(\vec{r})| \underset{r\to 0}{\to} 0. \tag{22}$$

Let us assume that the function $\psi(\vec{r})$ satisfies the conditions:

1. $\psi(\vec{r})$ is the solution of Eq. (1);
2. $\psi(\vec{r})$ has the asymptotic behaviour $\exp[i\int f(r)dr]$, where the function $f(r)$ is power--like:

$$\frac{d}{dr} f(r) \underset{r\to 0}{\sim} \frac{1}{r} f(r) \tag{23}$$

and singular:

$$|rf(r)| \underset{r\to 0}{\to} \infty. \tag{24}$$

All oscillating solutions found in the previous section satisfy the above conditions. We will prove that Eq. (22) follows from these conditions.

Assumption 2 implies

$$p^2\psi(\vec{r}) \underset{r\to 0}{\approx} -\frac{d^2}{dr^2}\psi(r, \theta, \varphi) \underset{r\to 0}{\approx} [f(r)]^2\psi(\vec{r}) \tag{25}$$

and similarly

$$\frac{1}{p^2}\psi(\vec{r}) \underset{r\to 0}{\approx} \frac{1}{[f(r)]^2}\psi(r), \tag{26}$$

$$\frac{1}{\sqrt{p^2}}\psi(\vec{r}) \underset{r\to 0}{\approx} \frac{1}{f(r)}\psi(r). \tag{27}$$

By expanding the operator $1/\sqrt{p^2+m^2}$ in a formal series in m^2/p^2 one can write

$$\frac{1}{\sqrt{p^2+m^2}}\psi(\vec{r}) = \frac{1}{\sqrt{p^2}}\left(1-\tfrac{1}{2}\frac{m^2}{p^2}+\dots\right)\psi(\vec{r}) \underset{r\to 0}{\approx} \frac{1}{f(r)}\psi(\vec{r}), \tag{28}$$

independently of the mass value. Substituting (28) in (21) we obtain

$$\beta^0 P(\vec{p})\psi(\vec{r}) \underset{r\to 0}{\approx} \frac{1}{f(r)}\tfrac{1}{2}(\beta^0\vec{\alpha}\vec{p}-\vec{\beta}\vec{p})\psi(\vec{r}), \tag{29}$$

where only the terms of order $\psi(\vec{r})$ are left. On the other hand, if the potentials $V(r)$ and $S(r)$ have the behaviours not more singular than $1/r$, then it follows from the assumption 1 that $\psi(\vec{r})$ satisfies asymptotically the equation

$$(\beta^0\vec{\alpha}\vec{p}-\vec{\beta}\vec{p})\psi(\vec{r}) \underset{r\to 0}{\approx} 0+o\left(\frac{1}{r}\psi(r)\right). \tag{30}$$

Substituting (30) into (29) and using the assumption (24) we see that the condition (22) is fulfilled.

In Ref. [4] it was generally proved that Eq. (1) modified by the substitution (20) is free from the Klein paradox at $r = 0$. In particular, using the present method, it is easy to show that such an equation does not admit solutions satisfying the condition 2. To this end let us suppose that such a solution exists. From the asymptotic condition

$$\frac{1}{\sqrt{p^2+m^2}} V(\vec{r})\psi(\vec{r}) \underset{r\to 0}{\approx} \frac{1}{f(r)} V(r)\psi(\vec{r}) \tag{31}$$

we see that the term $\beta^0 P(\vec{p})V(\vec{r})\psi(\vec{r})$ can be neglected in comparison with $\{-\beta^0(\vec{\alpha}\vec{p}+\beta m) +\vec{\beta}\vec{p}-M\}\psi(\vec{r})$. The resulting equation is asymptotically a free equation (if $S(r)\underset{r\to 0}{\longrightarrow} 0$), so it has no solutions satisfying condition 2. In this way we have proved by the reduction ad absurdum that also for the initial equation such solutions cannot exist.

It is surprising that the Klein paradox in Eq. (1) appears even for potentials with a very weak power singularity. Another known two-body relativistic equation with static vector interactions, the Breit equation (without Breit terms), also suffers from the Klein paradox, but only for the Coulomb potential with a strong coupling constant. Some analogy of the paradox found in this paper is provided by the drastic Klein paradox appearing in the three-body Dirac equation [5]. Note that both in the case of three-body Dirac equation and in the case of Eq. (1) with Duffin-Kemmer spin equal to 1 we have formally three spins 1/2.

6. *Final remarks*

A natural field of application for Eq. (1) and the radial equations derived in the present paper (Eq. (7) and Table IV) is the quark-diquark model of baryons. In the first approximation one may try to treat the diquark as a point-like particle and to use the colour triplet-antitriplet potentials emerging from the quark-antiquark model of mesons [6]. In such an approximation the vector potential at small distances is dominated by the one-gluon exchange and has a Coulomb-like singularity (with an additional log-dependence if the asymptotic freedom is taken into account). As it was shown in Sect. 4, in the case of the diquark spin equal to 1 there appear troubles related to Klein paradox, as the potential here is singular. To remove this paradox we should modify Eq. (1) by the substitution (20) [4]. Nevertheless, the modified equation contains the nonlocal operator and is difficult to solve practically.

Another approach may be based on the finite size of diquarks. When the diquark structure is taken into account, then the interaction is smeared out by the diquark extension and the potential is regular at $r = 0$. In such a case the Klein paradox does not appear and our radial equation can be used directly. There are also physical motivations for this approach: computations based on the Breit equation have showed [8] that the treatment of diquarks as point-like objects is physically unacceptable.

A priori it is an open question, whether it is necessary to use the modification (20) if the finite spatial extension of diquarks is taken into account. Acting on the wave function by the projector (21) one removes the non-desired particle-antiparticle states. If the Klein paradox does not appear in Eq. (1), then the wave function has only a small admixture of such states and so the action of the projector (21) introduces few changes. Thus, the solutions of Eq. *with* and *without* the modification (20) should be in the first approximation similar. Nonetheless, higher order differences may be significant. An analogous problem for the Dirac equation was discussed in Ref. [7].

Concluding, the derived radial equation listed in Eq. (7) and Table IV can be helpful in relativistic calculations for energy levels of quark-diquark or other fermion-boson systems. If the complete wave function is needed (e.g. for a perturbative treatment), then the auxiliary components of the wave function can be found from the sets of equations given in Tables II and III.

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