SYMMETRY AND SOLUTION SET SINGULARITIES IN HAMILTONIAN FIELD THEORIES

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This paper discusses singularities in the set of solutions to the field equations in Hamiltonian field theories with first class constraints. We assume that the evolution equations are well posed, so that such singularities are caused only by the constraint equations. The latter require that the moments associated to the gauge transformations vanish. The moment map is a concept in classical mechanics which generalizes the idea of angular momentum associated to the rotation group. It is shown that level sets of certain moment maps have quadratic singularities exactly at points in phase space that are fixed under subgroups of positive dimension. A normal form for the moment map in canonical coordinates is derived. Applying these results to gravitational and Yang-Mills fields shows that the solution sets for these field equations have quadratic singularities exactly at fields with (infinitesimal) symmetries. Thus at a symmetric solution, a linearized solution must satisfy not only the linearized equations but also a quadratic condition if it is to be tangent to a curve of solutions to the full field equations.

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1. *Introduction and summary of results*

Over the past decade or so a number of researchers have studied singularities in solution sets for relativistic field equations, especially the Einstein field equations. This research culminated in results of Arms (1981), and Arms, Marsden, and Moncrief (1981, 1982) showing that for Hamiltonian field theories with first class constraints, the solution sets have quadratic singularities precisely at symmetric solutions, and are smooth manifolds elsewhere. This introduction summarizes some of the earlier research and briefly describes the quadratic singularity results and the main ideas used to derive them. Section 2 discusses some symplectic geometry (i.e. Hamiltonian mechanics); tools are developed which describe the location and structure of the singularities in level sets for (generalized) momentum. In Section 3 these symplectic geometry results are applied to classical Lorentzian field theory to give the structure of solution set singularities. The Einstein and Yang-Mills field equations serve as specific examples; extension to other fields is discussed. Most

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proofs are only outlined, and the reader is directed to the literature for details. Some generalizations of the symplectic geometry results which are new are proved in Section 4.

The study of singularities in solution sets began with the question of the validity of linear approximations. The nicest possible structure for the solution set of a nonlinear equation is a smooth manifold of solutions, approximated near each exact solution by the linear perturbations of that solution; that is, by the solutions of the linearized equation. At singularities, however, the linearized equation may have "too many" solutions. A simple example from algebra illustrates this point. Consider the nonlinear equation

$$x^2 - y^2 = 0 \tag{1.1}$$

with the corresponding linearization

$$2x(\delta x) - 2y(\delta y) = 0. \tag{1.2}$$

At most solutions (x, y) to (1.1), solutions $(\delta x, \delta y)$ to (1.2) are vectors tangent to the solution set for (1.1). For instance at $(-2,2)$, a linearized solution $(\delta x, \delta y)$ satisfies

$$-4(\delta x) - 4(\delta y) = 0,$$

so $\delta y = -\delta x$; for any such linearized solution $(\delta x, \delta y)$, there is an exact parameterized solution of the form

$$(-2, 2) + \lambda(\delta x, \delta y) + O(\lambda^2).$$

(In this simple example, the higher order terms happen to vanish.) But at the origin (0, 0), (1.2) becomes the trivial equation

$$0(\delta x) - 0(\delta y) = 0,$$

which is satisfied by arbitrary $(\delta x, \delta y)$, most of which are not tangent to the solution set for (1.1). (For example, no choice of higher order terms will make $(\lambda, 0) + O(\lambda^2)$ a solution of (1.1).)

Much as we might hope that this phenomenon is merely a mathematical curiosity, it is not: it does occur for nonlinear field equations. Brill (1972) and Brill and Deser (1973) showed that there are such spurious solutions for the linearized Einstein equations at the vacuum spacetime given by the flat three torus with zero extrinsic curvature. In fact that spacetime is an isolated solution to the nonlinear field equations even though the linearized equations have many nontrivial solutions. (Fischer and Marsden (1975)).

Fischer and Marsden (1973) coined the term *linearization instability* to describe this problem. A nonlinear equation $F(x) = 0$ is said to be *linearization stable* at a solution x_0 if every solution δx to the linearized equation $F'(x_0) \cdot \delta x = 0$ is tangent to a curve of solutions to the original nonlinear equation. If δx is considered as the first order coefficient in a power series solution, linearization stability means that there does exist a parameterized solution with that first order coefficient. For (1.1), linearization instability occurs only at the origin.

(Linearization stability is not directly related to other kinds of stability, such as dynamic stability. However, dynamic stability is often detected by examining linearized solutions. Thus linearization instability might confuse the question of dynamic stability by the presence of spurious linearized solutions.)

Fischer and Marsden (1973, 1975) established sufficient conditions for linearization stability of the Einstein field equations. For spacetimes with compact Cauchy surfaces, Moncrief (1975A) identified these sufficient conditions with the absence of Killing fields. A Killing field on a spatially compact, globally hyperbolic vacuum spacetime together with a linearized solution gives rise to an integral conserved quantity which is quadratic in the linearized field. Moncrief (1976) showed that this conserved quantity must vanish if a field that solves the linearized equations is in fact a first order approximation to a nonlinear solution. Similar results were established for the Yang-Mills and Einstein-Yang--Mills equations (including the Einstein-Maxwell system) (Moncrief (1977); Arms (1977, 1979)).

A common theme of these results is the link between symmetry and linearization instability. This link has important implications for perturbative analysis. Known explicit solutions at which perturbations can be done are almost always found by assuming some kind of symmetry. Two key characteristics of a field theory needed for this link are revealed by examining cases where symmetry exists with linearization stability. Choquet-Bruhat and Deser (1973) and O'Murchadha and York (1973) showed that Minkowski space is linearization stable, and D'Eath (1976) showed the same for Robertson-Walker universes. D'Eath's matter variables are not Hamiltonian. An essential assumption for the quadratic singularity results is formulation of the field equations as a well-posed Hamiltonian system with constraints that are first class (in the sense of Dirac): *we assume such a formulation throughout this paper.* (Recent work of Bao, Marsden, and Walton (1985) suggests that symmetric fluid universes such as the Robertson-Walker spaces will be singular points if solutions are restricted by the Lin constraints.)

Minkowski space can be described in a Hamiltonian (canonical) formalism, but causes difficulties because it has a noncompact Cauchy surface (i.e. it is an open universe). For technical reasons (although one could also argue on physical grounds) only perturbations which vanish rapidly at spatial infinity are considered. As in the compact case, infinitesimal symmetries, that is Killing fields, give rise to obstructions to linearization stability; however, only Killing fields which vanish rapidly at spatial infinity are considered. All Killing fields on Minkowski space have non-zero norm at infinity, so there is no obstruction to linearization stability. To avoid these technical details about asymptotic behavior, *we will assume that all our spacetimes have compact Cauchy surfaces.* (The reader interested in the noncompact case is referred to Choquet-Bruhat, Fischer, and Marsden (1979), and to Moncrief (1977) for some discussion of the noncompact case for Yang-Mills(-Higgs) fields. In another approach to perturbation theory for asymptotically flat spacetimes, investigated by R. Beig (1984), quadratic constraints also appear; work of L. Anderson and J. E. Marsden indicates that these quadratic constraints must be considered if the class of allowable perturbations includes those asymptotic to Poincaré transformations (private communication).)

This study of linearization stability identified the solutions with symmetry as candidates for singularities in the solution set. Furthermore the quadratic constraint was a beginning on describing the structure of the singularities. Additional work in the late 70's and early 80's was needed to show that in fact no higher order constraints are needed. Also in this time period the results were extended to other fields such as certain scalar fields coupled to gravity (Saraykar and Joshi (1981 and 1982) and classical supergravity (Bao (1984)).

Briefly stated, the complete result is as follows. Consider a Hamiltonian field theory with first class constraints, such as the Yang-Mills, vacuum Einstein, or Einstein-Yang--Mills theories, on a spacetime with a compact Cauchy surface. (Certain other technical restrictions, such as ellipticity for certain partial differential operators, and positive energy for matter fields coupled to gravity, will be detailed in Section 3.) The solution set for such

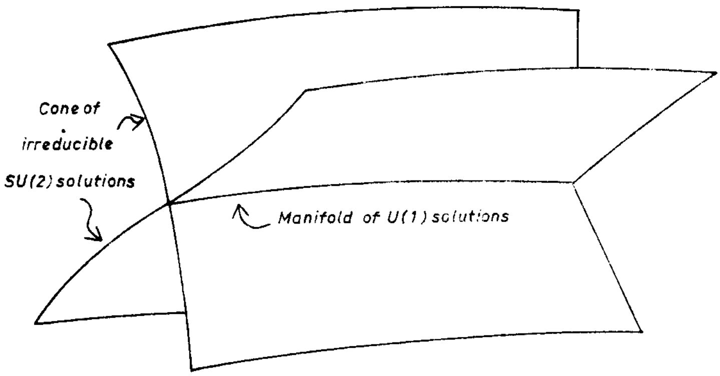

Fig. 1

a field theory has singularities only at solutions with infinitesimal gauge symmetries. An infinitesimal gauge symmetry means a one parameter family of gauge transformations that leave the fields not merely covariant but *in*variant. The singularities are quadratic; that is, they can be described by a set of homogeneous quadratic equations. These equations state that certain conserved quantities, one for each infinitesimal symmetry, vanish.

Furthermore, the structure of such a singularity can be described in some detail. It is the product of a smooth manifold of solutions with the same degree of gauge symmetry and a cone of solutions with fewer gauge symmetries. For example, consider SU(2) gauge fields. A gauge field with a gauge symmetry can be reduced to a field with a smaller gauge group, in this case U(1). Thus there are quadratic singularities at the U(1), or electromagnetic, solutions within the set of solutions to the SU(2) Yang-Mills equations. These singularities look something like Fig. 1.

The nature of the singularities in the solution set tempts one to say that there is a cone of solutions which "break symmetry". However, the relationship between this picture and the term "symmetry breaking" has not been investigated.

Now in nonlinear field theories such as gauge theory and general relativity, known explicit solutions, at which we try to apply linear perturbation theory, usually have some kind of symmetry. If this is gauge symmetry, then the linearized fields must satisfy not only the linearized field equations but also a quadratic, second order condition. Thus linearization stability is clearly an important question for finding classical solutions, and also for perturbative methods in quantum field theory. In fact in any quantization procedure, the classical solution set plays an important role; if the linearized solutions are not a good approximation to this set, it will cause difficulties in quantization. Some aspects of this problem have been studied by Moncrief (1978).

We close this introduction with a sketch of the proof that gauge symmetries should be linked with singularities in the solution set. Field equations such as the Einstein or Yang--Mills field equations split into evolution equations and constraint equations on the initial data. We assume that the constraint equations are well posed: given a solution to the constraint equations, the evolution equations can be solved, and the solution will preserve the constraints. Thus singularities in the space of solutions must correspond to singularities in the constraint set.

Assume that the field theory can be expressed in the canonical, (i.e. Hamiltonian) formalism. As an easy example, consider electromagnetism. The canonical coordinates (q, p) in this case are the vector potential and the electric field, (A_j, E^j). (Actually the canonically conjugate momentum for the vector potential is a *minus* the electric field *density*, but such details are unimportant for the present argument.) The one constraint, div $E = 0$, is first class.

Recall that first class constraints generate gauge transformations, that is, the Hamiltonian vector field associated to such a constraint leaves the physics invariant. If we represent the constraints by $\Phi = \{\Phi_j\} = 0$, the Hamiltonian vector field on phase space P,

$$X_{\Phi_j} = (\partial\Phi_j/\partial p)(\partial/\partial p) - (\partial\Phi_j/\partial q)(\partial/\partial p), \tag{1.3}$$

is an infinitesimal gauge transformation. (Where the phase space is more than two dimensional, a sum over all p's and q's is understood.) The flow of the vector field $X_{\Phi j}$ is a finite gauge transformation; in our example, this is an ordinary gauge transformation, $A \to A + d\lambda$.

An easy argument using the implicit function theorem from advanced calculus can be used to locate possible singularities in the constraint set $\mathscr{C} = \{\Phi = 0\}$. The implicit function theorem says that wherever $d\Phi = \Phi'$ is surjective, i.e. at any x_0 where the set $\{d\Phi_j(x_0)\}$ is linearly independent, the set $\mathscr{C}$ is a smooth manifold locally near x_0, with tangent space at x_0, $T_{x_0}\mathscr{C}$, given by ker Φ' = the set of linearized solutions. (Derivatives are to be evaluted at x_0 unless otherwise specified.) Note that (1.3) establishes a one to one correspondence between vector fields the $X_{\Phi j}$ and the forms $d\Phi_j$. The implicit function theorem fails when the $d\Phi_j$'s and thus the $X_{\Phi j}$'s are linearly dependent, that is when there are N^j's such that

$$N^j X_{\Phi_j}(x_0) = 0. \tag{1.4}$$

(Summation convention assumed.) But this last equation says precisely that there is an infinitesimal gauge symmetry that fixes x_0. When (1.4) holds, $N^j X_{\Phi j}$ is called an (*infinitesimal gauge*) *symmetry* for x_0. The dimension of the space of N^j for which (1.4) holds is called the *degree of* (*gauge*) *symmetry* for x_0.

Another easy argument suggests that at points with symmetry, linear perturbations which are tangent to curves of solutions satisfy a quadratic equation. Consider a curve

$x(\lambda)$ in phase space, $x(0) = x_0$, satisfying $\Phi(x(\lambda)) = 0$. Differentiating the linear combination $N^j\Phi_j$ with respect to λ yields

$$N^j\Phi_j'(x(\lambda)) \cdot x'(\lambda) = 0; \tag{1.5}$$

differentiating again,

$$N^j\Phi_j''(x(\lambda)) \cdot (x'(\lambda), x'(\lambda)) + N^j\Phi_j'(x(\lambda)) \cdot x''(\lambda) = 0. \tag{1.6}$$

When the latter equation is evaluated at $\lambda = 0$, the second term vanishes by the arguments of the previous paragraph. This leaves a quadratic equation,

$$N^j\Phi_j''(x_0) \cdot (\delta x, \delta x) = 0, \tag{1.7}$$

where by (1.5) evaluated at $\lambda = 0$, $\delta x = x'(0)$ is a linear perturbation of the solution x_0.

But these simple arguments raise as many questions as they answer. In the quadratic condition (1.7), together with the linearized field equations, sufficient to determine which linearized fields are actually tangent to curves of solutions? Additional differentiation, that is to say higher order perturbations, might yield additional higher order constraints. It is not even clear if the condition (1.7) is nontrivial. In the example of electromagnetism, the gauge transformations are given by $A \to A + d\lambda$, so the set of gauge transformations is parametrized by the functions λ. A constant function λ leaves the vector potential unchanged, and is therefore a symmetry for all electromagnetic fields. Although this symmetry is clearly trivial, it does lead to a homogeneous quadratic equation (1.7); the equation, $0 = 0$, is also trivial. Thus as one expects for a linear theory, there is no quadratic condition on the linear perturbations. When will (1.7) be trivial, as in this example? To answer these questions and determine the structure of the singularities in the solution set, we will need the concept of a moment map. A moment map is a function associated to a group of canonical transformations of phase space and generalizes the idea of linear momentum associated to translations and angular momentum associated to rotations. The idea that symmetries of phase space give rise to conserved quantities dates back to Noether, if not beyond, but we will use the formulation developed recently by Souriau and Kostant. It is to this topic that we turn in the next lecture.

2. *Bifurcations of moment maps: a theorem in symplectic geometry*

A moment map is an abstract form of Noether's theorem: every symmetry has associated with it a conserved quantity. A field theory with gauge transformations has associated conserved quantities. A subset of the field equations, the constraint equations, set these conserved quantities equal to zero. Thus the constraint equations can be described as the zero set of a moment map, and the problem of describing the solution set for the constraint equations can be rephrased, and somewhat generalized, as describing the level sets for a moment map. In this lecture we study this problem in finite dimensions. Lecture 3 will discuss applying the results to the infinite dimensional (function) spaces that occur in the study of field theories.

This lecture is organized as follows. First we give the abstract definition of a moment map. Applying this definition to the rotation group acting on the phase space of a single particle in three dimensional space yields the usual angular momentum as the moment map. Additional requirements, such as metrics, are discussed. Next we construct a diffeomorphism, or if you like coordinate change, that reveals the quadratic nature of singularities in level sets of moment maps. The structure of these singularities is described.

Suppose that G is a group of canonical transformations on a phase space P. Let $\mathfrak{g}$ be the Lie algebra of the group; for each $\xi \in \mathfrak{g}$, there is a vector field ξ_P on P which generates the corresponding flow on P. (Specifically, if $x \to g \cdot x$ represents the G action, then $\xi_P(x) = d/dt\{\exp(t\xi) \cdot x)\}|_{x=0}$.) A moment map for the G action is a map $\mathscr{J}\colon P \to \mathfrak{g}^*$ such that the Hamiltonian vector field associated to function $\langle\langle \mathscr{J}, \xi \rangle\rangle$ is the vector field ξ_P. (Here $\mathfrak{g}^*$ is the dual to the Lie algebra $\mathfrak{g}$ and the double brackets $\langle\langle\ ,\ \rangle\rangle$ indicate evaluation of $\mathscr{J} \in \mathfrak{g}^*$ on $\xi \in \mathfrak{g}$.) In other words, for each one parameter subgroup in G (represented by its generator ξ), there is a function $\langle\langle \mathscr{J}, \xi \rangle\rangle$ which is the Hamiltonian whose dynamics are exactly the action of that subgroup on the phase space; the moment map is simply a collective Hamiltonian for the group action.

Now in Dirac's constraint theory, the first class constraints are Hamiltonians whose dynamics are the gauge transformations. Thus the first class constraints are the moment for the gauge transformations. The study of the solution set for the field equations has been reduced in Lecture 1 to the study of the constraint set; that is now further transformed into the study of level sets for a moment map.

This definition can be given more precisely by recalling some definitions and notation from symplectic geometry, the abstraction of Hamiltonian systems. Let P be a manifold of dimension $2n$. A symplectic form ω on P is a closed, nondegenerate, covariant two tensor. Darboux's lemma says that locally there are always canonical coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ in which $\omega = dq^j \wedge dp_j$ (summation convention). The nondegeneracy of ω means that it gives a one to one correspondence between vector fields and one forms (covariant vector fields): if H is a function on P, the vector field X_H corresponding to dH is given by $dH = \omega(X_H,)$. In canonical coordinates, this means that

$$X_H = \frac{\partial H}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial H}{\partial q^j_i}\frac{\partial}{\partial p_j}.$$

Suppose that there is a group G that acts on P on the left by symplectomorphisms (i.e. canonical transformations). The moment map for this action is defined to be a map $\mathscr{J}\colon P \to \mathfrak{g}^*$ such that

$$\langle\langle \mathscr{J}'_x(V), \xi \rangle\rangle = \omega_x(\xi_P(x), V), \tag{2.1}$$

where x is a point in P and V is a vector at x.

As a simple example, consider the rotation group $G = \mathrm{SO}(3)$ acting on the phase space $P = T^*\mathbf{R}^3 \approx \mathbf{R}^6$ of a single particle. A point in P is a pair of vectors (q, p), q = position, p = momentum of the particle. The group G acts in the usual manner on each of these two vectors. Let ξ be a vector in the Lie algebra $\mathrm{SO}(3) \approx \mathbf{R}^3$; the direction and magnitude of ξ indicate the axis and magnitude, respectively, of the one parameter family of rotations it generates. From vector calculus one knows that the vector field tangent to the flow this generates on the phase space is $\xi_P = (\xi \times q, \xi \times p)$. But this is also the Hamiltonian vector field for the Hamiltonian $\xi \cdot q \times p$, so the moment for this action is the angular momentum $\mathscr{J} = q \times p$, as expected.

Not all group actions admit moment maps, at least not globally. There are several conditions on the phase space, the group, or the action that will insure the global existence of the moment. (Cf. for instance section 26 of Guillemin and Sternberg (1984 A).) One which applies in all of our examples is the following. If the group action on the phase space P is the lift of an action on the configuration space M, then there is a natural moment map:

$$\langle\!\langle \mathscr{J}(q, p), \xi \rangle\!\rangle = \langle\!\langle p, \xi_M(q) \rangle\!\rangle,$$

where ξ_M indicates the generator of the group action on M and the brackets on the right hand side indicate evaluation (or inner product, if you regard the canonical moment p as a contravariant vector).

Thus the moment map is a modern formulation, as presented here due to Souriau, of Noether's theorem. Recently there has been much interest in the momentum map, not only in mathematical physics but also in several areas of pure mathematics. For instance Guillemin and Sternberg (1982), and independently Atiyah (1982), have shown that the image of certain moment maps for torus actions is a convex polytope. This generalizes old results of Schur and Kostant on eigenvalues for Hermitian matrices, and has applications to solving polynomial equations and geometric invariant theory. (See e.g. the survey article of Atiyah (1983); Guillemin and Sternberg (1984 B) and Kirwan (1984 A, B). Other applications include work on action angle variables by Duistermaat (1980), and reduction of Hamiltonian systems with symmetry, for instance by Meyer (1973) and Marsden and Weinstein (1974) and Bos and Gotay (1984), which generalizes Jacobi's "elimination of the node" in celestial mechanics. Guillemin and Sternberg (1984 A) discuss many physics applications of symplectic geometry in general and the moment map in particular.

An additional requirement on a moment map which is often present in applications is *equivariance*. The group has a canonical action on its dual Lie algebra, the coadjoint action defined by $\langle\!\langle Ad_g^* \mu, \xi \rangle\!\rangle = \langle\!\langle \mu, Ad_{g^{-1}} \xi \rangle\!\rangle$. (For matrix groups, the adjoint action is given by $Ad_{g^{-1}}\, \xi = g^{-1} \xi g$, and the coadjoint action by $Ad_g^* \mu = g \mu g^{-1}$.) The moment map $\mathscr{J}$ is said to be (Ad^*)-equivariant if $\mathscr{J}(g \cdot x) = Ad_g^*(\mathscr{J}(x))$; in other words the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\mathscr{J}} & g^* \\ g \cdot \downarrow & & \downarrow Ad_g^*. \\ P & \xrightarrow[\mathscr{J}]{} & g^* \end{array}$$

Most moment maps that arise naturally in physics are equivariant. For instance, when the group action on phase space is the natural lift of the action on configuration space, the canonical moment map described above is equivariant.

In addition to the symplectic (canonical) structure, the phase space P often carries a natural Riemannian metric, which we will denote by $\langle,\rangle$. We will require that the group action also preserves the metric, and that the symplectic and metric structures together give an *(almost) complex structure* or *complex rotation* $\mathbf{J}$. (The qualifier almost indicates that the structure exists on each fiber of the tangent bundle, but may not be integrable.) This structure is defined by the requirements ahat $\mathbf{J}^2 = -$ identity and $\langle X, Y\rangle = \omega(X, \mathbf{J}Y)$. In local coordinates, $\mathbf{J}(\partial/\partial q^j) = \partial/\partial p_j$, and $\mathbf{J}(\partial/\partial p_j) = -\partial/\partial q^j$. The complex structure will be preserved by the group action because the metric and symplectic structures are. In terms of these metric and complex structures, we can rephrase the definition of a Hamiltonian vector field as $X_H = -\mathbf{J}(\nabla H)$. On a finite dimensional manifold, one can always choose such compatible metric and complex structures (Weinstein (1977), p. 8), and if the group is compact by averaging one may assume that all these structures are G-invariant.

We assume the group action on phase space has closed orbits, which will follow for instance if the group is compact. This assumption plus the assumption that the action preserves the metric will insure the existence of a slice for the action. We recall the basic properties of a slice S_x at a point x:

(*i*) $g \cdot x = x \Rightarrow g \cdot S_x = S_x$;

(*ii*) $g \cdot S_x \cap S_x \neq \varnothing \Rightarrow g \cdot x = x$;

(*iii*) if G_x is the stabilizer subgroup of x (i.e., $G_x = \{g \in G: g \cdot x = x\}$), then there are neighborhoods N and N_x of the identity in G and G_x, respectively, such that if $N \approx N_x \times H$, then $S_x \times H \approx$ a neighborhood of x. Roughly speaking, this means that the slice is fixed by and only by the stabilizer subgroup of x, and locally near x, P is the product of the slice and a complement for the stabilizer subgroup in a natural way. As an example, consider the action of SO(3) on $\mathbf{R}^3$. At any point other than the origin, a slice is a radial line segment. At the origin, a slice is an SO(3)-invariant neighborhood. Note that at points where the dimension of the orbits is locally constant, the slice parametrizes the orbits, as one would expect; this is not true at a point where the dimension of the orbits jumps, like the origin.

For the rest of this lecture, derivatives of the moment map are to be evaluated at a fixed point x unless otherwise specified. It is useful to introduce the adjoint map to $\mathscr{J}'$, $\mathscr{J}'^*: \mathfrak{g} \to T_xP^*$, defined by

$$\langle\langle \mathscr{J}'(V), \xi \rangle\rangle = \langle V, \mathscr{J}'^*(\xi)\rangle;$$

in other words, $d\langle\langle \mathscr{J}, \xi\rangle\rangle = \mathscr{J}'^*(\xi)$. We will assume that there is an Ad^*-invariant metric on the dual Lie algebra. (All that is actually needed is invariance under the stabilizer group of the point in phase space at which we are working, but often there is invariance under the full group.) This metric gives a canonical identification of the tangent and cotangent spaces; with some abuse of notation, we shall also use $\mathscr{J}'^*$ to represent its composition with this identification. Then the dual Lie algebra splits as

$$\mathfrak{g}^* = \ker \mathscr{J}'^* \oplus \operatorname{Im} \mathscr{J}'. \tag{2.2}$$

(Such splittings are simply the statement in linear algebra that the equation $AX = Y$ is solvable iff Y is perpendicular to ker A^t.) Using the adjoint map, (2.1) becomes

$$\xi_P = -\mathbf{J} \circ \mathscr{J}(\xi). \tag{2.3}$$

Thus the tangent space to the orbit $G \cdot x_0$ is $\operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*$. An obvious candidate for the tangent space to the slice is the orthogonal complement, $\ker \mathscr{J}' \circ \mathbf{J}$. To obtain the slice, we can exponentiate this tangent space and take a G-invariant neighborhood of x. The properties of the slice then follow because the action is isometric. In many applications, however, the phase space is naturally an affine space on which the G action is affine. Then the tangent space at x can be identified with P; using this identification we may use an "affine" slice, a G-invariant neighborhood of the origin in $\ker \mathscr{J}' \circ \mathbf{J}$.

There are two obvious splittings of the tangent space at x:

$$TP_x = \ker \mathscr{J}' \oplus \operatorname{Im} \mathscr{J}'^* = \ker \mathscr{J}' \circ \mathbf{J} \oplus \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*. \tag{2.4}$$

Suppose that $\mu = \mathscr{J}(x)$ is a fixed point of the coadjoint action on $\mathfrak{g}^*$, i.e. that the orbit $G \cdot \mu = \{\mu\}$. (In the field theory applications, such as general relativity or Yang-Mills, $\mu = 0$, so this condition is clearly satisfied.) Equivariance implies that $T_x G \cdot x = \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^* \subset \ker \mathscr{J}'$. Thus the splittings in (2.4) can be combined to get

$$T_x P = \ker \mathscr{J}' \cap \ker \mathscr{J}' \circ \mathbf{J} \oplus \operatorname{Im} \mathscr{J}'^* \oplus \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*. \tag{2.5}$$

This splitting is orthogonal. The summands have the following interpretation at regular points of $\mathscr{J}$. Since $\ker \mathscr{J}'$ is the tangent space to the set of solutions to the constra intequation $\mathscr{J} = \mu$, the first summand is tangent to intersection of the constraint set with the slice, and as such can be thought of as the "true degrees of freedom", the tangent space to the "Meyer-Marsden-Weinstein" reduced space of G orbits in the level set of $\mathscr{J}$. (Cf. references to reduction mentioned above.) We shall call the first summand the *linear reduced space.* The second summand is orthogonal to the constraint set, and the third summand is tangent to the orbit $G \cdot x$. This splitting of the tangent space is called Moncrief's decomposition (Moncrief (1975B)) and generalizes a decomposition of gravitational waves on Minkowski space in general relativity due to Deser (1967). (In the case that $G \cdot \mu \neq \{\mu\}$, then the tangent space splits into four components; see Theorem 4.2. For the rest of this lecture and in the applications in lecture 3, however, we shall assume that $G \cdot \mu = \{\mu\}$.)

Let us summarize the structure we have to work with. The group action is canonical and isometric, has an equivariant moment map and admits a slice at x. The tangent space at x splits as in (2.5), and the dual Lie algebra splits as in (2.2).

Our goal is to describe the constraint set $\mathscr{C} = \mathscr{J}^{-1}(\mu)$. The first step is to characterize the smooth points of $\mathscr{C}$. Actually this has already been done in Section 1, but we repeat the argument in the current language. Let $\mathfrak{g}_x$ be the Lie algebra of G_x.

Proposition 2.1. If $\mathfrak{g}_x = \{0\}$, then $\mathscr{C}$ is a smooth manifold near x with tangent space at x given by $\ker \mathscr{J}'$.

Proof. From (2.3), $\mathfrak{g}_x = \ker \mathscr{J}'^*$. Then the result follows from the implicit function theorem and (2.2). ■

Thus the implicit function theorem fails at a solution with symmetry. At these points, we apply the "Liaponov-Schmidt" procedure: that is, we split the moment map into two pieces, to one of which we can apply the implicit function theorem and a second for which we must find another approach. Let π_1 be the orthogonal projection of $\mathfrak{g}^*$ onto $\ker \mathscr{J}'^*$ and π_2 the complementary projection onto $\operatorname{Im} \mathscr{J}'$. Let $\mathscr{J}_1 = \pi_1 \circ \mathscr{J}$; $\mathscr{J}_1$ is just the moment map for the G_x action, and $\mathscr{J}_1'(x) = 0$. Let $\mathscr{J}_2 = \pi_2 \circ \mathscr{J}$; $\mathscr{J}_2$ is the nonsingular part of $\mathscr{J}$ at x in the sense that $\mathscr{J}_2'(x_0)$ is surjective. The implicit function theorem tells us that the set $\mathscr{C}_2 = \{\mathscr{J}_2 = 0\}$ is a manifold near x with tangent space given by $\ker \mathscr{J}_2' = \ker \mathscr{J}'$. To show that there are quadratic singularities in $\mathscr{C} = \mathscr{C}_1 \cap \mathscr{C}_2$, where $\mathscr{C}_1 = \{\mathscr{J}_1 = 0\}$, we need only find coordinates on $\mathscr{C}_2$ such that the defining equation for $\mathscr{C}_1$ is a homogeneous quadratic equation in these coordinates.

In finite dimensions, there is an equivariant form of the Darboux lemma which gives this result. Briefly, restrict attention to the set $P' = S \cap \mathscr{C}_2$, where S is a slice for the G action at x. The tangent space T_xP' is the first component of the splitting in (2.5), which is symplectic. Since being symplectic is an open condition, P' is a symplectic submanifold near x. Now the equivariant Darboux lemma, by Weinstein (1977) (see also Guillemin and Sternberg (1984 A) Theorem 22.2), may be applied to the G_x action on P', giving canonical coordinates in which the G_x action is linear. (The Weinstein-Moser proof of the Darboux lemma uses the constant symplectic form on the tangent space to obtain the canonical coordinates. Since the group action is linear on this tangent space, a simple check that the construction may be done in a way that preserves the group action gives the results.) Since the G_x action is linear, the corresponding moment map $\mathscr{J}_1$ is quadratic, and further more since its derivative vanishes at x, it is homogeneous quadratic. Thus $S \cap \mathscr{C}$ is given by a homogeneous quadratic equation; the properties of the slice and the equivariance of $\mathscr{J}$ imply that $\mathscr{C} \approx S \cap \mathscr{C} \times G \cdot x$.

But this elegant Darboux approach is unsatisfactory in applications to field theory. In infinite dimensions it is difficult to check several technical details needed to make this approach work; in fact, there is no guarantee that it will work on arbitrary Banach manifolds, although I suspect that the field theory cases probably contain enough structure to make it work. In any case, no one has checked these details, because a simpler approach is available, which we are about to discuss. Also, although the Darboux argument is in a sense constructive (the required coordinate change is obtained by flowing along a vector field), the coordinate change we shall use below is more explicit.

This alternate construction applies in the case that the phase space P is affine, e.g. an open subset of a linear space, and the moment map $\mathscr{J}$ is already quadratic (but not necessarily homogeneous quadratic) in the affine coordinates. In fact, it is only necessary that $\mathscr{J}_1$ be quadratic. This is the case for Yang-Mills and for general relativity in the case of spacelike Killing fields. (As is often true in relativity, the timelike case must be handled separately and patched together with the results of applying some general theory for Hamiltonian systems to the spacelike case; see Lecture 3.)

Now it may seem at first glance that if $\mathscr{J}$ is already quadratic then there is nothing to prove. A simple example shows that a *system* of quadratic equations may have a higher order singularity. Let $F(x, y, z) = (x^2+y^2-z, x^2-yz)$. Although F is quadratic, $F^{-1}(0, 0)$ near $(0, 0, 0)$ does not have a quadratic, conical singularity, but rather a cubic one: $\{x = \pm y^{3/2}, z = y^2+y^3+ \ldots\}$.

We will find a diffeomorphism, that is a coordinate change, that flattens $\mathscr{C}_2$ onto its tangent space ker $\mathscr{J}'$. Then we will show that this diffeomorphism maps the set $\mathscr{C}_1$ onto a cone of solutions to a homogeneous quadratic equation. (In fact the discussion below can be refined to show that there are canonical coordinates on the slice in which the moment splits into the direct sum of a linear map and a homogeneous quadratic map; see theorem 4.3.)

For simplicity we assume that $\mathscr{J}(x) = 0$. If $\mathscr{J}(x) = \mu \neq 0$, but $G \cdot \mu = \{\mu\}$, then the arguments below apply with the constant μ added or subtracted in various equations as appropriate. (Equivalently, modify the moment map by subtracting the constant μ.) For the case $G \cdot \mu \neq \{\mu\}$, see Theorem 4.1.

The affine structure of P can be used to identify P and the tangent space at x locally near x. In these coordinates,

$$\mathscr{J}(x+h) = \mathscr{J}'(h)+B(h, h),$$

where $B(h, h)$ is the bilinear function $\mathscr{J}''(h, h)/2$. Recall that derivatives are to be evaluated at x; h is a vector on which the linear map $\mathscr{J}'$ acts. Consider the "Laplacian" $\Delta = \mathscr{J}' \circ \mathscr{J}'^*$, which is an isomorphism of Im $\mathscr{J}'$ onto itself. (In the field theory applications, this map is a second order differential operator whose highest order term is the ordinary Laplacian.) Let $\Psi = \Delta^{-1} \circ \pi_2 : \mathfrak{g}^* \to \text{Im}\,\mathscr{J}'$, the "Green's function" for Δ. The required diffeomorphism $F: P \to P$ is

$$F(x+h) = x+h+\mathscr{J}'^* \circ \Psi \circ B(h, h). \tag{2.6}$$

We call this map the "Kuranishi map"; a construction by Kuranishi (1965) in his study of the deformation of complex structures inspired the use of a similar map by Atiyah, Hitchin, and Singer (1978) in their study of Euclidean Yang-Mills fields. (Note, however, that the type of singularities we obtain here do not occur in the Euclidean theory.)

Proposition 2.2: *Properties of the Kuranishi map*: *A. It is a local diffeomorphism near x*, because near x it differs from the identity by a quadratic perturbation. *B. It flattens* $\mathscr{C}_2$ *onto* ker $\mathscr{J}$, as desired, for

$$\begin{aligned}\mathscr{J}'(F(x+h)-x) &= \mathscr{J}'(h)+\Delta \circ \Psi \circ B(h, h)\\ &= \pi_2(\mathscr{J}'(h)+B(h, h)) = \mathscr{J}_2(x+h).\end{aligned} \tag{2.7}$$

C. It preserves the slice. Equivariance gives Im $\mathbf{J} \circ \mathscr{J}'^* \subset \ker \mathscr{J}'$, and applying the rotation $\mathbf{J}$ to this inclusion gives Im $\mathscr{J}'^* \subset \ker \mathscr{J}' \circ \mathbf{J}$. The former contains the difference (F — identity) and the latter is the slice, so F moves points within the slice. *D. It preserves the symplectic form restricted to the slice*, for Im $\mathscr{J}'^*$ is the kernel of ω on S. (Simply consider the splitting (2.5).) Thus F is in some sense a canonical map of the coisotropic subspace S. (Remark: The proof of D. in Prop. 1.4 of Arms, Marsden, and Moncrief (1982) is in error: it assumes a constant metric, which is not true in the case considered there, general relativity. The present proof applies to that case.) The intersection $\mathscr{C}_2 \cap S$ is symplectic, and combining the properties above we obtain the following. *E. The Kuranishi map is a symplectomorphism* (*canonical transformation*) *of a neighborhood of x in* $\mathscr{C}_2 \cap S$ *to a neighborhood* *of x in* $\ker \mathscr{J} \cap S$. Thus we have a canonical coordinate system in which a subset of the constraints are satisfied ($\mathscr{J}_2 = 0$). The key result which takes care of the rest of the constraints is the following.

Theorem 2.3. The Kuranishi map F maps the zero set for the moment in the slice and near x to the cone

$$Q = \{x+h : h \in \ker \mathscr{J}' \cap \ker \mathscr{J}' \circ \mathbf{J},$$

$$\text{and for all symmetries } \xi \in \ker \mathscr{J}' \text{ of } x, \quad \{B(h, h), \xi\} = 0\}. \tag{2.8}$$

(The quadratic equation (2.8) may be rewritten

$$\pi_1 \circ B(h, h) = 0.) \tag{2.8'}$$

Using the group action to move off the slice, we obtain

$$\mathscr{J}^{-1}(0) = \mathscr{C} \approx (\text{locally near } x)\, Q \times G \cdot x = Q \times G/G_x.$$

Proof. Given Proposition 2.2, it suffices to show that for $x+h \in S$,

$$\pi_1 \circ B[F(x+h)-x, F(x+h)-x] = \mathscr{J}_1(x+h). \tag{2.9}$$

Now $F(x+h)-x = h+k$ where $h \in \ker \mathscr{J}'$ and $k \in \operatorname{Im} \mathscr{J}'^*(x)$.
Lemma: *For* $k \in \operatorname{Im} \mathscr{J}'^*$, $\pi_1 \circ B(h+k, h+k) = \pi_1 \circ B(h, h)$.
Proof. Let $B_1(h, h) = \pi_1 \circ B(h, h)$. The proofs of lemmas 3 and 4 of Arms, Marsden and Moncrief (1981) apply to show that B_1 is gauge invariant, i.e.

$$B_1(h, f) = 0 \text{ if } f \in \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*,$$

and B_1 is $\mathbf{J}$-invariant, i.e.

$$B_1(\mathbf{J}(h), \mathbf{J}(f)) = B(h, f).$$

Now $B(h+k, h+k) = B(h, h)+B(k, 2h+k)$. But $\mathbf{J}(k) \in \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*$, so the second term vanishes. ●

Thus the left hand side of (2.9) reduces to $\pi_1 \circ B(h, h)$. The right hand side is

$$\pi_1[\mathscr{J}(x+h)] = \pi_1[\mathscr{J}'(h)+B(h, h)] = \pi_1[B(h, h)].$$

so (2.9) is established. ■

Remarks: (*i*) As discussed above, this theorem generalizes to nonzero values of the moment; see Theorem 4.1. (*ii*) When applied to a single Hamiltonian $H : \mathbf{R}^2 \to \mathbf{R}^1$ with closed orbits of common period, the theorem says that H is homogeneous quadratic in coordinates centered at a critical point, and so is the Hamiltonian of a harmonic oscillator. (*iii*) The quadratic equations in (2.8) may be degenerate so that Q in fact turns out to be a manifold. An important example is the electromagnetic field: see the discussions at the end of Lecture 1 and below. Another example is the harmonic oscillator, with Hamiltonian = moment for the one dimensional group of dynamics = q^2+p^2; here the zero level set is also a manifold, but of much smaller dimension than expected.

By study of the quadratic map B, Arms, Marsden and Moncrief (1981) are able to describe the structure of the constraint set $\mathscr{C}$ in some detail. It is a product of a smooth manifold of solutions with the same degree of symmetry and a cone of solutions with a lower degree of symmetry. As an illustration, suppose that the quadratic condition (2.8) is $x^2 - y^2 = 0$ on a four dimensional space with coordinates (x, y, z, w). The set $\{(x, y, z, w) : x = y = 0\}$ is the degeneracy space of the quadratic function B, and will be solutions to the constraint equations with *exactly* the same (infinitesimal) symmetries. (By property (*ii*) of the slice, the stabilizer subgroup of any point in the slice is a subgroup of G_x. When $F^{-1}(Q)$ is swept out by the group action to fill out all of $\mathscr{C}$, the degeneracy space spreads out to include all nearby points with a group of symmetries conjugate to G_x.) Solutions out on the cone, where $(x, y) \neq (0, 0)$, have smaller stabilizer subgroups. Another example is given in the discussion of Fig. 1. More details are given in Arms, Marsden, and Moncrief (1981).

In some cases, all solutions have some symmetry. For instance, in electromagnetism, a constant function gives rise to a gauge transformation that fixes all potentials A, and no potential has any other symmetries besides the constant functions. The corresponding quadratic equations (2.8) turns out to be $0 = 0$ and the constraint set is a manifold defined by the linearized equations alone, as we knew it must be in this case since we know that electromagnetism is a linear theory. Looking at this from the perspective of the description of the constraint set, we see that there are no solutions of lower symmetry to branch to, so there can be no cone, and the whole linear reduced space is the degeneracy space for (2.8).

A similar situation arises for angular momentum. Away from the origin $(q, p) = (0, 0)$, a point of phase space has zero angular momentum if and only if the position and linear momentum vectors are proportional, and rotations around their common axis give such a point one degree of symmetry. The corresponding quadratic condition (2.8) is again trivial; this reflects the fact that it suffices to require that the two components of the angular momentum vanish — the two perpendicular to either the position or the velocity. The origin, however, has a higher degree of symmetry (three) than the surrounding solutions, and (2.8) is nontrivial. In fact the set of points with zero angular momentum is a cone over the smooth manifold made by identifying antipodal points in the product of spheres $S^1 \times S^2$. (Arms, Marsden, and Moncrief (1981) and Bos and Gotay (1984)).

If we consider the total angular momentum for two or more particles (which for simplicity we will assume can be in the same state at the same time), the picture is more interesting and harder to describe. The origin is the point of maximal symmetry. From there we can "break symmetry" to a set of particles with one degree of symmetry (all positions and linear momenta proportional); or we can move to a point with no symmetry at all (e.g., two particles with nonzero but opposite angular momentum). Thus we get "cones over cones".

It is tempting to suggest that this picture has something to do with symmetry breaking as the term is used in the physics literature. Unfortunately the relationship, if any, has not been established.

The set of solutions with the same degree of symmetry as x is a symplectic manifold. Thus the phase space is a stratified set of symplectic manifolds. For purposes of studying the dynamics of one particular solution to the constraint equations, the singularities in the solution set have no significance: the dynamics will preserve the degree of symmetry, so we may restrict our attention to the manifold of solutions with the same degree of symmetry. Moncrief (1980) discusses this situation for Yang-Mills fields. However, in so doing we ignore solutions with fewer symmetries. If we wish to study the set of all classical solutions, we are forced to deal with quadratic singularities.

For more examples of nontrival quadratic singularities, we turn to field theory, the subject of the final lecture.

3. The structure of the solution set for the classical Einstein and Yang-Mills equations

The theory in Lecture 2 is developed on a finite dimensional phase space. In this lecture, we apply these results to field theory, in particular to gravitational and gauge fields. The phase spaces for field theories are infinite dimensional, so some complications arise. Thus there are several goals for this lecture: (*i*) to show how the abstract results of Lecture 2 translate into statements about the Einstein and Yang-Mills field equations; (*ii*) to list the technical details that must be checked in applying the theory in infinite dimensions; (*iii*) and to discuss briefly how the theory can be extended to other field theories.

First consider the example of a Yang-Mills field. Let G be the gauge group, say SU(2), with Lie algebra $\mathfrak{g}$. The basic field variable is the $\mathfrak{g}$-valued vector potential on spacetime. Recall that the field theory must be expressed in Hamiltonian form. This means we split the four dimensional vector potential into two pieces, using a spacelike Cauchy surface Σ. A point in configuration space is the restriction of the four dimensional vector potential to Σ to obtain the three dimensional vector potential A_i^a. The superscript a (b, c, etc.) indexes the Lie algebra values of the potential and the subscript i(j, k, etc.) indicates that the potential is a one form on Σ. (Of course one may ignore the Lie algebra indices and pretend the field is electromagnetic.) The canonically conjugate variable is (minus) the electric field (density), which we shall represent by η_a^i. (From a mathematical viewpoint, it is more natural to use the two form dual to this vector density; cf. Arms (1981).) The action of the group of gauge transformations $\mathcal{G}$ on the configuration space lifts to an action on the phase space $P = \{(A, \eta)\}$; A is pseudotensorial and η is tensorial. If we represent the group G as a matrix group, a gauge transformation is given (at least locally) by $g: \Sigma \to G$, which acts on the fields by

$$A \to g^{-1}Ag + g^{-1}dg, \qquad \eta \to g^{-1}\eta g.$$

The corresponding moment is the gauge covariant divergence of η:

$$K_a(A, \eta) = (\nabla\eta)_a = \eta^i_{a\|i} + C^c_{ab}A^b_i\eta^i_c, \tag{3.1}$$

where the double bar in the subscript indicates the metric covariant derivative. For a gauge field with no sources, Gauss's Law says this moment must vanish.

For gravity, we use the formulation of Arnowitt, Deser, and Misner (1962). The configuration space is the set of (positive definite) metrics $\{g_{ij}\}$ on the spacelike Cauchy surface Σ. The conjugate variable is the symmetric covariant two tensor π^{ij}, which is essentially the extrinsic curvature of Σ. The Einstein field equations and a point in the phase space $P = \{(g, \pi)\}$ determine the spacetime metric. The diffeomorphisms of spacetime are the gauge transformations, acting on configuration space and hence on P by pulling back the tensors g and π. The equivariant moment which generates these transformations is the pair $(\boldsymbol{H}, \boldsymbol{J})$, where $\boldsymbol{H}$ is the super Hamiltonian and $\boldsymbol{J}$ is the supermomentum. (Explicit formulas are given for instance in the ADM reference, and in Fischer, Marsden and Moncrief (1980).) For vacuum relativity, the constraint equations on the initial data require that both $\boldsymbol{H}$ and $\boldsymbol{J}$ vanish.

Recall the restrictions on the field theory mentioned in Lecture 1. Essential assumptions include that the theory must be Hamiltonian with constraints all first class; this condition is satisfied in our two examples. The phase space must carry, in addition to its canonical (symplectic) structure compatible positive definite (Riemannian) metric and almost complex structures. We require that the gauge transformations preserve the metric as well as the canonical structure. In our examples, this metric is obtained from metrics on the spacetime and $\mathfrak{g}$ by integrating over Σ. It will be compatible and gauge invariant, and positive definite if the metric on $\mathfrak{g}$ is. This puts a mild restriction on the gauge group G, but the gauge groups of interest all satisfy this condition. For instance, it suffices for G to be compact semisimple, abelian, or the direct or semidirect product of such groups. In the case of general relativity, the metric is pulled back by the diffeomorphism, but so are the fields (g, π); the end result is that the scalar product is invariant. For convenience, we assume that the spacetime has a compact Cauchy surface; see Lecture 1 for a discussion of the consequences of removing this assumption.

Using this metric we can define the adjoint map $\mathscr{J}'^*$ from the Lie algebra of the gauge transformation group back to the tangent space of phase space. The splitting (2.5) then follows. Note that in both our examples, the group action is affine, so that we can use the first two components of the splitting, that is the orthogonal complement of the orbit, as the affine slice. We remark also that the evolution equations for relativity (or gauge fields coupled to gravity) can be expressed in terms of the adjoint map; see for instance Fischer, Marsden, and Moncrief (1980).

There also must be a positive definite metric on the dual to the Lie algebra of the group og diffeomorphisms or gauge transformations. This metric must have certain invariance properties with respect to the natural coadjoint action of the group on the dual Lie algebra. Choose a particular (gravitational or gauge) field; call this the *background field*, and call the subgroup of gauge transformations that leave this field invariant the *symmetry subgroup*. The metric must be invariant under the action of the symmetry subgroup. For the gauge fields, the Lie algebra of $\mathfrak{g}$ is the space $\mathfrak{v}$ of gauge covariant Lie algebra valued functions; these can be exponentiated to obtain a gauge transformation. As above, metrics on the spacetime fields give rise to a positive definite metric on the dual space $\mathfrak{v}^*$ which is invariant under $\mathfrak{g}$. For general relativity, a fixed vector field on spacetime is an element of the Lie algebra for the group of diffeomorphisms; it is represented on Σ by its normal and tangential components, say (N, X^i). (Moncrief (1975A) first showed that (N, X^i) in the kernel of the adjoint map can be evolved to a unique Killing field.) The scalar product of two such vector fields is given by $\int_\Sigma (NM + X^i Y_i) dV$, where dV is the volume form on Σ. This scalar product is *not* invariant under all diffeomorphisms, but is for those which fix the spacetime metric; in other words, it is invariant under the action of the symmetry subgroup. Using this scalar product, we get the splitting (2.2) of the dual Lie algebra.

A few technical points. The exact function spaces used must be carefully chosen. In order to make the arguments from finite dimensions carry over to infinite dimensions, Sobolev spaces of some minimum degree of differentiability must be used. Then by taking the "inverse limit" as the degree of differentiability goes to infinity, the results are proved for smooth fields. These differentiability arguments become particularly ticklish in the case of gravity. If a diffeomorphism has k continuous derivatives, its derivative, used in the pullback action, has only $k-1$ derivatives. Nevertheless, all the constructions in Lecture 2 can be made to work, modulo some modifications mentioned below. The metric on the phase space need only be a *weak* metric; that is, the associated map from the tangent to the cotangent spaces of the phase space need only be injective, not necessarily surjective. The existence of adjoint operators and the various splittings are not automatic; in the cases of general relativity and gauge theory, they are obtained using the Fredholm alternative from the fact that the constraint equations are elliptic in some generalized sense. See for instance Fischer, Marsden and Moncrief (1980) for details.

For gauge theory, the moment map (3.1) is quadratic in (A, η). Thus the quadratic version of Theorem 2.3 can be applied with little difficulty. An infinitesimal symmetry for a gauge field is a covariant constant function. Now some such symmetries are trivial in the following sense. If the center of the Lie algebra of the gauge group is nonzero, any constant in this center is covariant constant for any connection, and is therefore a symmetry for all fields and in particular all solutions. The corresponding quadratic equation (2.8) is the trivial equation $0 = 0$. There is no singularity because there is no way to break symmetry to branch off the symmetric solution set to the cone. In fact it is possible to modify the implicit function theorem argument in this case: since the covariant constant functions are symmetries for all fields, they are in the kernel of the adjoint map for all fields. Therefore the image of the moment map has no projection on the subspace of constant functions in the center of the Lie algebra. We can use the complementary subspace as the codomain of the moment map, compute that the derivative is surjective, and conclude that at any field with no extra symmetry the solution set is a manifold, as in Proposition 2.1. This is another way to treat electromagnetic fields; cf. Arms (1977) and (1979).

Excluding such trivial symmetries, we get the following result. Let the term Yang-Mills field indicate a gauge field which satisfies the Yang-Mills equations. At any Yang-Mills field with a nontrivial symmetry, the constraint set has a quadratic singularity. The singularity consists of a product of a manifold of solutions with the same degree of symmetry and a cone of solutions with less symmetry. The cone is diffeomorphic to the solution set of the following homogeneous quadratic equations: for each covariant constant φ^c,

$$\int_{\Sigma} \varphi^c K_c''(\delta A, \delta\eta)dV = \int_{\Sigma} C^a_{bc}(\delta\eta)_a(\delta A)^b \varphi^c dV = 0, \tag{3.2}$$

where the C^a_{bc} are the structure constants of the Lie algebra and $(\delta A, \delta\eta)$ is a solution to the linearized field equations.

Additional information is gained by noticing that when there are covariant constant functions, the gauge group may be reduced to the subgroup which commutes with the covariant constant functions. For example, suppose the group is SU(2). Local gauge transformations can be used so that a covariant constant function takes its values in a one dimensional subspace of the Lie algebra (the tangent space to a U(1) subgroup); in fact, the local gauges can be chosen so that the gauge group is reduced to U(1). We get a manifold of U(1), i.e. electromagnetic solutions, with a cone of true SU(2) solutions branching over each U(1) solution. (See Fig. 1.) Thus to obtain approximations of SU(2) gauge fields by perturbing a U(1) field, one must consider (3.2) as well as the linearized field equations.

As usual, applying general results about field theories to gravitational fields involves some additional complications. We sketch the arguments and refer the reader to Fischer, Marsden, and Moncrief (1980) and Arms, Marsden, and Moncrief (1982) for details. A symmetry in this case is a vector field whose flow leaves the gravitational field invariant, that is a Killing field. We assume that we may choose the Cauchy surface Σ to have trace constant extrinsic curvature, for instance a maximal hypersurface. (This is analogous to working in a slice for the time translations.) It is well known that the space of Killing fields for a given metric is finite dimensional; furthermore if we choose Σ as above, the space of Killing fields will have a basis that has at most one timelike field, orthogonal to Σ, and the rest of the fields spacelike and tangent to Σ. The spacelike component of the moment map, the supermomentum, is in fact quadratic in the canonical variables, so that the techniques of Lecture 2 (with attention the technical details mentioned above) suffice in the case of spacelike Killing fields. The super Hamiltonian, however, is not quadratic in the canonical variables because of the scalar curvature term. If there is a single timelike Killing field, then an infinite dimensional version of the Morse lemma gives the result. To combine the two cases, one must find coordinates on the phase space that decouple the components of the moment. This turns out to be possible on a maximal hypersurface; the desired coordinates are a splitting of g and π into transverse traceless tensors, hessians, etc. We recover the same picture as before: there is a manifold of solutions with the same number of independent Killing fields, and over each such symmetric solution is a cone of solutions with less symmetry. The cone is diffeomorphic to the solutions of a set of quadratic equations, one for each independent Killing field X^μ:

$$\int_\Sigma (X^0 H'' + X^i J_i'')\,dV = 0, \tag{3.3}$$

where the double primes indicate a second order functional derivative evaluated on (and quadratic in) $(\delta g, \delta\pi)$ satisfying the linearized field equations and a gauge fixing condition.

All the calculations for the theorem are done in a particularly nice gauge — that is to say working in the slice for the diffeomorphism group action. This restriction even applies to the perturbation $(\delta g, \delta\pi)$. Thus in order to check (3.3), it seems that one would have to work on a maximal hypersurface, in a particular gauge for the spacelike diffeomorphisms, and consider only perturbations that preserve these conditions as well. It turns out, however, that the integral in (3.3) is a conserved quantity, that is independent of hypersurface Σ, and furthermore is gauge invariant. Thus the condition (3.3) can be checked on any Cauchy surface and in any gauge.

For the coupled case, the gauge transformation group includes simultaneous gauge transformations of the Yang-Mills field and diffeomorphisms of spacetime. In terms of the bundle formulation of gauge fields, these are bundle automorphisms that do not necessarily preserve the base space. A symmetry is a Killing field X^μ and a function φ^a whose covariant derivative cancels the Lie derivative of the connection with respect to X: in other words, there is a gauge transformation that compensates for changes in the gauge field caused by the change of (external) coordinates. The moment map is $(\boldsymbol{H}_{\text{tot}}, \boldsymbol{J}_{\text{tot}}, \boldsymbol{K})$, where the subscript indicates the total superHamiltonian and supermomentum for the coupled fields. As for gravity alone, the cases where X is spacelike and timelike have to be handled separately. When both kinds of symmetry are present, coordinates must be found which decouple the spacelike and timelike pieces of the moment map. This is possible exactly because the Yang-Mills field adds positive energy term to the total superHamiltonian. The quadratic condition on linear perturbations is given by

$$\int_{\Sigma} (X^0 \boldsymbol{H}'' + X^i \boldsymbol{J}_i'' + \varphi^a \boldsymbol{K}_a'') dV = 0. \tag{3.4}$$

As with gravity, this condition may be checked in any gauge and on any Cauchy surface; the necessary invariance conditions are established in Anderson and Arms (1986).

Let us summarize briefly the characteristics needed in a field theory to obtain these quadratic singularity results. We need a Hamiltonian field theory with first class constraints. The gauge transformations generated by these constraints must preserve a positive definite metric on the phase space. Certain technical details must be verified, most easily if the constraint equations are elliptic in some suitable sense. For coupling to gravity, we need, at least for the proof that has been used so far, a trace constant extrinsic curvature Cauchy surface and some kind of positive energy condition. For example, in the case of the scalar field coupled to gravity, the full program has only been carried out for the massless case, because the massive scalar field adds a negative term to the energy. (Cf. Saraykar and Joshi (1981, 1982).) Other candidates for this program include other formulations of gravity, classical supergravity, and spinor fields. Anderson and Arms (1986) list the conserved quantities for various possible fields and list the relevant literature.

4. *Some additional geometry*

This section contains proofs of several previously unpublished results. Theorems 4.1 and 4.2 generalize the quadratic singularity, Theorem 2.3 and the splitting (2.5), respectively, to the case of nonzero moment values. Theorem 4.3 indicates how level sets for different moment values fit together by giving a normal form for the moment in canonical coordinates on the slice.

A. Nonzero values of the moment map

The results on the structure of the level sets for the moment map can be generalized to nonzero values of momentum. For vacuum or coupled dynamic fields, the field equations always specify that the (total) moment vanishes, so the nonzero values are not of interest in those cases. However, if a static source field is given, then the moment value is nonzero.

The generalization has not, to my knowledge, appeared in the literature, but is "known", e.g. to Marsden and Weinstein. Similar techniques are used by Guillemin and Sternberg (1984 A, § 26) to construct the reduced phase space. For convenience the result is stated here for compact groups and finite dimensional spaces; the compactness assumption may be replaced by the various assumptions discussed in Section 2.

Theorem 4.1. Theorem 2.3 generalizes to nonzero moment maps. That is, let $\mathscr{J}$ *be the equivariant moment map for a compact group action on a finite dimensional space. The level sets for* $\mathscr{J}$ *have quadratic singularities at symmetric points. The singularities are diffeomorphic to the product of the solution set for a set of homogeneous quadratic equations, one for each independent symmetry, and a manifold of symmetric points.*

Proof. The notation is as in Section 2. If $\mu = \mathscr{J}(x)$, let $G \cdot \mu$ be the orbit of μ under the action of G. If $G \cdot \mu$ is a single point, i.e. if the subgroup G_μ which leaves μ fixed is equal to the entire group, then the proofs for $\mu = 0$ carry through unchanged, as noted above. If $G \cdot \mu$ is nontrivial, then it carries a symplectic structure, canonical up to sign (cf. Abraham and Marsden (1978)), as follows. The tangent space to $G \cdot \mu$ at μ is spanned by the generators ξ_* of the G action on $\mathfrak{g}^*$, where $\xi \in \mathfrak{g}$. The symplectic form at μ is given by

$$\omega_\mu(\xi_*, \zeta_*) = \mu([\xi, \zeta]).$$

Let $\tilde{P} = P \times G \cdot \mu^-$; that is, $\tilde{P}$ as a manifold is the product of P and $G \cdot \mu$, and the symplectic form on $\tilde{P}$ is the pullback of the symplectic form on P minus the pullback of the symplectic form on $G \cdot \mu$. The group action on P is the product of the actions on the factors. The moment for this action is given by

$$\mathscr{J}(x, v) = \mathscr{J}(x) - v.$$

Thus $\tilde{\mathscr{J}}^{-1}(0) \cap (P \times \{v\}) = \mathscr{J}^{-1}(v) \times \{v\}$, and by the equivariance of $\mathscr{J}$,

$$\tilde{\mathscr{J}}^{-1}(0) = \mathscr{J}^{-1}(\mu) \times G \cdot \mu.$$

The singularities in $\mathscr{J}^{-1}(\mu)$ will be essentially those in $\tilde{\mathscr{J}}^{-1}(0)$.

Now we apply Theorem 2.3 to $\tilde{\mathscr{J}}$. Note that $\tilde{\mathscr{J}}'(x, v) = \mathscr{J}'(x) - \mathrm{Id}$, where Id is the identity on the second factor, and that $\tilde{\mathscr{J}}''(x, v) = \tilde{\mathscr{J}}''(x)$. Equivariance implies that

$$\ker \tilde{\mathscr{J}}'(x, v) = \ker \mathscr{J}'(x) \times \mathscr{G} \cdot \mu.$$

All these facts together imply that

$$\mathscr{J}^{-1}(0) \approx (\ker \mathscr{J}'(x, v) \cap \{\mathscr{J}''(x) = 0\}) \times G \cdot \mu. \ \blacksquare$$

Unlike the singularity results, the splitting (2.5) does change when $G \cdot \mu$ is nontrivial: there are four components instead of three. Let H be the subgroup of G that fixes $\mu = \mathscr{J}(x)$ and $\mathfrak{h}$ its Lie algebra. We have assumed an adjoint action invariant metric on the dual Lie algebra $\mathfrak{g}^*$; this metric can be used to identify a copy of $\mathfrak{h}^*$ inside $\mathfrak{g}^*$. (The cross-section construction of Guillemin and Sternberg (1984 A, § 41) gives a nice formulation of this identification without using the metric, although the metric is used in the proof.) Let $\mathscr{K}$ be the moment map for the action of H on P; note that $\mathscr{K}$ is just the composition of $\mathscr{J}$ with projection onto $\mathfrak{h}^*$.

Theorem 4.2. In the notation just described, the tangent space splits as follows:

$$T_xP = \ker \mathscr{J}' \cap \ker \mathscr{J}' \circ \mathbf{J} \oplus \operatorname{Im} \mathbf{J} \circ \mathscr{K}'^* \oplus \operatorname{Im} \mathscr{J}'^* \cap \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^* \oplus \operatorname{Im} \mathscr{K}'^*. \tag{4.1}$$

Discussion. The first summand, obviously symplectic, is the space of "true degrees of freedom" or *linear reduced space*. In the nonsingular case, it is tangent to the reduced space, either the level set $\mathscr{J}^{-1}(\mu)$ reduced by the action of H or the inverse image of the orbit, $\mathscr{J}^{-1}(G \cdot \mu)$, reduced by G. The second summand gives the generalized local angle coordinates; it is tangent to $H \cdot x$, the orbit of x under H; this orbit is the intersection of the level set $\mathscr{J}^{-1}(\mu)$ with the full orbit $G \cdot x$. The complementary orbit directions are given by the third summand; angle and action in these directions cannot be separated because this space is symplectic. It can be thought of as the tangent space to $G \cdot x/H \cdot x$. But $G \cdot x/H \cdot x \approx G/H \approx G \cdot \mu$, so the third summand $\approx T_\mu G \cdot \mu (= \mathfrak{h}^{\perp *})$. In fact $\mathscr{J}'$ is a symplectomorphism between these two spaces; this follows from the definition of equivariant moment, the choice of symplectic structure on $G \cdot \mu$, and the fact that $\mathscr{J}$ is one to one on this space. (Thus $\mathscr{J}$ is a symplectic covering of $G \cdot \mu$, as Kostant showed; cf. Abraham and Marsden (1978).) The third summand can be rewritten as $\mathscr{J}'^*(\mathfrak{h}^\perp)$, because $\mathscr{J}'^*$ preserves orthogonality between $\mathfrak{h}$ and $\mathfrak{h}^\perp$, as follows. Using the metric to identify $\mathfrak{g} \approx \mathfrak{g}^*$, the tangent space to the orbit of any $v \in \mathfrak{g}^*$ is $\mathfrak{g}_v^\perp = [v, \mathfrak{g}] = [v, \mathfrak{g}_v^\perp]$. Then

$$\langle \mathscr{J}'^*(\mathfrak{g}_v), \mathscr{J}'^*(\mathfrak{g}_v^\perp) \rangle = \langle \mathfrak{g}_{vp}, [v_p, \mathfrak{g}_{vp}^\perp] \rangle = \langle [v, \mathfrak{g}_v]_p, \mathfrak{g}_{vp}^\perp \rangle = 0, \tag{4.2}$$

using the facts that $\mathbf{J}$ and v_p are infinitesimal isometries and that $\xi \to \xi_p = \mathbf{J} \circ \mathscr{J}'^*(\xi)$ is a Lie algebra (anti-) homomorphism. The fourth summand is perpendicular to $\mathscr{J}^{-1}(G \cdot \mu)$ and gives the local action variables corresponding to the angle variables in the second summand. This action angle description was given by Marsden [1981], pp. 30–33, but the present derivation is more direct.

Another result in the literature which follows immediately from (4.1) is the following theorem of Kostant and Sternberg. (See e.g. Guillemin and Sternberg [1984 A], Theorem 26.8). The orbit $G \cdot x$ is symplectic iff the connected components of G_x and G_μ coincide, i.e. iff $\mathfrak{g}_x = \mathfrak{g}_\mu$. But the tangent space to the orbit is the sum of the second and third summads in (4.1), which is symplectic iff the second summand is trivial. The latter means that $\mathscr{K}'^*$ is the zero map, so

$$\mathfrak{g}_x = \ker \mathscr{K}'^* = \mathfrak{h} = \mathfrak{g}_\mu.$$

Proof. The original splitting result in Arms, Fischer, and Marsden (1975) gives

$$T_xP = \ker \mathscr{J}' \cap \ker \mathscr{K}' \circ \mathbf{J} \oplus \operatorname{Im} \mathbf{J} \circ \mathscr{K}'^* \oplus \operatorname{Im} \mathscr{J}'^*. \tag{4.3}$$

Applying the rotation $\mathbf{J}$ to (4.3) gives

$$T_xP = \ker \mathscr{J}' \circ \mathbf{J} \cap \ker \mathscr{K}' \oplus \operatorname{Im} \mathscr{K}'^* \oplus \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*. \tag{4.4}$$

To combine these two splittings, we will need a

Lemma: $\ker \mathscr{K}' = \ker \mathscr{J}' + \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*$ (not direct sum).

Proof: First note that $Y \in \ker \mathscr{K}'$ iff $\mathscr{J}'(Y) \in \mathfrak{h}^\perp$ iff $\mathscr{J}'(Y) = \xi_*$ for some $\xi \in \mathfrak{g}$ because $\mathfrak{h}^\perp$ is tangent to $G \cdot \mu$. Now by equivariance $\mathscr{J}'(\xi_P) = \xi_*$, so that $Y - \xi_P \in \ker \mathscr{J}'$. Since $\{\xi_P : \xi \in \mathfrak{g}\} = \operatorname{Im} \mathbf{J} \circ \mathscr{J}'^*$, the lemma is proved. ●

By the lemma the first summands in (4.3) and (4.4) both become ker $\mathcal{J}' \cap$ ker $\mathcal{J}' \circ \mathbf{J}$. Now from the third and fourth summands we get

$$\operatorname{Im} \mathbf{J} \circ \mathcal{K}'^* \oplus \operatorname{Im} \mathcal{J}'^* = \operatorname{Im} \mathcal{K}'^* \oplus \operatorname{Im} \mathbf{J} \circ \mathcal{J}'^*. \tag{4.5}$$

But $\operatorname{Im} K'^* \subset \operatorname{Im} \mathcal{J}'^*$, so (A.5) is equal to

$$\operatorname{Im} \mathbf{J} \circ \mathcal{K}'^* \oplus \operatorname{Im} \mathcal{J}'^* \cap \operatorname{Im} \mathbf{J} \circ \mathcal{J}'^* \oplus \operatorname{Im} \mathcal{K}'^*. \blacksquare$$

B. A normal form for the moment map

From Theorems 2.3 and 4.1, we know that under certain assumptions the level sets of the moment map have only quadratic singularities. This suggests that the moment map ought to be a quadratic function in canonical coordinates on P and linear coordinates on $\mathfrak{g}^*$. In fact one probably cannot obtain exactly this result: the singularity results from quadratic dependence on the linear reduced space and equivariance forces the moment to be quadratic in the coordinates along the orbit. Thus in any canonical coordinate system, the moment is probably fourth order near any singularity that has a nontrivial orbit. (It is possible that in some cases nonlinear coordinates on the dual Lie algebra, say that linearized the orbit, would eliminate this problem). On a slice for G action, however, canonical coordinates can be found that make $\mathcal{J}$ quadratic. Furthermore, the moment (more precisely, the moment minus its value around the point of singularity) splits into a direct sum of a homogeneous quadratic function on a symplectic subspace and a linear function on an isotropic subspace. This result applies to both the zero and nonzero moment values; the compact group case is implicit in the moment map reconstruction argument of Guillemin and Sterberg (1984 A, § 41).

Theorem 4.3. Let $\mathcal{J}$ be an equivariant moment map. Suppose either that the moment is quadratic in some canonical coordinates on an affine space (the field theory case) and the various assumptions discussed in Sections 2 and 3 hold, or that all spaces are finite dimensional and G is compact. Then there exist local coordinates (q, p, r) on a slice for the action at x, where the pair (q, p) is symplectic and r is symplectically orthogonal to (x, y), in which

$$\mathcal{J}(q, p, r) = \mu + \begin{bmatrix} Q(q, p) \\ L(r) \end{bmatrix},$$

where $x = (0, 0, 0)$, $\mu = g(x)$, Q is homogeneous quadratic and L is one to one and linear.

Remarks. If $G = G_\mu$, then L is an isomorphism. In terms of the discussion of Theorem 4.2, (q, p) are coordinates on the linear reduced space and r is the set of action variables with corresponding angle variables along $G_\mu \cdot x$.

Proof. First assume that $\mathcal{J}(x)$ is invariant under the group action. We wish to use the affine slice, ker $\mathcal{J}'(x) \circ \mathbf{J}$, where local coordinates are used to identify the tangent space T_xP with P. In the field theory case this works because the momentum is quadratic so the group action is linear. In the finite dimensional case we must first apply the equivariant Darboux lemma. Let H be the stabilizer subgroup of the point x. The equivariant Darboux lemma implies that there are local canonical coordinates at x in which the H action in linear. Thus the H action preserves $\ker \mathscr{J}'(x_0) \circ \mathbf{J}$. From this and the compactness of the group it follows that $\ker \mathscr{J}'(x_0) \circ \mathbf{J}$ satisfies the properties of a slice.

In a neighborhood of x in P, define the Kuranishi map as before, replacing the second derivative by the remainder

$$R(h) = \mathscr{J}(x+h) - \mathscr{J}(x) - \mathscr{J}'(h)$$

(all derivatives are evaluated at x unless otherwise states). Thus

$$F(x+h) = x + \mathscr{J}'^* \circ \Psi \circ R(h).$$

As before Ψ is the "Green's" operator $(\mathscr{J}' \circ \mathscr{J}'^*)^{-1}$ on $\operatorname{Im} \mathscr{J}'$, and π_1 and π_2 are the orthogonal projections on $\ker \mathscr{J}'^*$ and $\operatorname{Im} \mathscr{J}'$ in $\mathfrak{g}^*$, respectively. All the properties of the Kuranishi map listed in Proposition 2.2 still hold; in particular F is a symplectomorphism on the slice and linearizes the π_2 components of $\mathscr{J}$; that is,

$$\pi_2(\mathscr{J}(x+h) - \mathscr{J}(x)) = \mathscr{J}'(F(x+h) - x).$$

Note that $\pi_1 \circ \mathscr{J} = \pi_1(\mathscr{J}(x)) + \pi_1 \circ R$ is the moment for the H action, and is therefore homogeneous quadratic in the coordinates we are using on the slice. Thus we can apply the lemma in Theorem 2.3 to $\pi_1 \circ R$: For $k \in \operatorname{Im} \mathscr{J}'^*$, $\pi_1 \circ R(h+k) = \pi_1 \circ R(h)$. Therefore $\pi_1 \circ R$ is invariant under F. Let Π_1 and Π_2 be the orthogonal projections on $\ker \mathscr{J}'$ and $\operatorname{Im} \mathscr{J}'^*$ in P, respectively. The lemma also implies that $\pi_1 \circ R(h)$ is Π_1 invariant: $\pi_1 \circ R(h) = \pi_1 \circ R \circ \Pi_1(h)$, since $\Pi_1(h) = h - \Pi_2(h)$ and $\Pi_2(h) \in \operatorname{Im} \mathscr{J}'$. Thus F reduces the π_1 components of $\mathscr{J}$ to a homogeneous quadratic map on the linear reduced space:

$$\pi_1(\mathscr{J}(x+h) - \mathscr{J}(x)) = \pi_1 \circ R \circ \Pi_1(F(x+h) - x) \quad \text{for } x \in \text{slice}.$$

All together this gives us

$$\mathscr{J} \circ F^{-1}(x+h) = \begin{bmatrix} \pi_1 \circ R \\ \mathscr{J}' \end{bmatrix}(h) = \begin{bmatrix} \pi_1 \circ R \circ \Pi_1 \\ \mathscr{J}' \circ \Pi_2 \end{bmatrix}(h).$$

To finish the proof in this case, choose (q, p) to be any symplectic coordinates on the symplectic subspace $\Pi_1(\ker \mathscr{J}' \circ \mathbf{J}) = \ker \mathscr{J}' \cap \ker \mathscr{J}' \circ \mathbf{J}$. Note that $\mathscr{J}'$ is a linear isomorphism on the isotropic subspace $\Pi_2(\ker \mathscr{J}' \circ \mathbf{J}) = \operatorname{Im} \mathscr{J}'^*$.

If $\mu = \mathscr{J}(x)$ is not invariant, let G_μ be its stabilizer subgroup. Let π_3 be the orthogonal projection onto $\mathfrak{g}_\mu^{\perp *}$, and let $\mathscr{J}_3 = \pi_3 \circ \mathscr{J}$. Consider the set $M = \{\mathscr{J}_3 = 0\}$. By the implicit function theorem, M is a manifold, and it is G_μ-invariant by equivariance and the Ad^* invariance of the metric on $\mathfrak{g}^*$. If $\ker \mathscr{J}_3'(y) = T_y M$ is symplectic and $\mathbf{J}$-invariant for each $y \in M$, then we can apply the previous case to the G_μ action on M. (In the field theory case, we assume that all the subspaces in the argument have closed orthogonal complements and that there are canonical coordinates on M in which the moment $\mathscr{J}_3$ is quadratic.) Since a slice for the G_μ action on M is a slice for the G action on P, this gives the result in the general case.

It remains to show that $\ker \mathscr{J}_3'(y)$ is $\mathbf{J}$-invariant and therefore symplectic at each $y \in M$. Now let all derivatives be evaluated at y. It is obvious that $\ker \mathscr{J}' \subset \ker \mathscr{J}_3'$.

Also $\mathscr{J}'_3(\mathscr{J}'^*(\mathfrak{g}_\mu)) = 0$, because

$$\langle \mathscr{J}'(\mathscr{J}'^*(\mathfrak{g}_\mu)), \mathfrak{g}_\mu^\perp \rangle = \langle \mathscr{J}'^*(\mathfrak{g}_\mu), \mathscr{J}'^*(\mathfrak{g}_\mu^\perp)\rangle = 0$$

by (4.2). Conversely, suppose $\mathscr{J}'_3(v) = 0$. Then $\mathscr{J}'(v) = \xi_1 + \xi_2 \in \mathfrak{g}_\mu$, where $\xi_1 \in \mathfrak{g}_y^\perp \cap \mathfrak{g}_\nu$, $\nu = \mathscr{J}(y)$, and $\xi_2 \in \mathfrak{g}_\nu^\perp \cap \mathfrak{g}_\mu$. By the splitting (2.4) applied to the G_ν action, there is a $\zeta_1 \in \mathfrak{g}_\nu$ so that $\mathscr{J}'(\mathscr{J}'^*(\zeta_1)) = \xi_1$. By the discussion of (4.1) applied to the G_μ action, there is a $\zeta_3 \in \mathfrak{g}_\nu^\perp \cap \mathfrak{g}_\mu$ so that $\mathscr{J}'^*(\zeta_2)$ is in the third summand in (4.1) and $\mathscr{J}'(\mathscr{J}'^*)\zeta_2)) = \xi_2$. Thus $v - \mathscr{J}'^*(\zeta_1) - \mathscr{J}'^*(\zeta_2) \in \ker \mathscr{J}'$, and

$$\ker \mathscr{J}_3 = \ker \mathscr{J}' \oplus \mathscr{J}'^*(\mathfrak{g}_\mu) = \ker \mathscr{J}' \cap \ker \mathscr{J}' \circ \mathbf{J} \oplus \mathbf{J} \circ \mathscr{J}'^*(\mathfrak{g}_\nu) \oplus \mathscr{J}'^*(\mathfrak{g}_\mu) \tag{4.6}$$

using (4.1) applied to the G_μ action. By the equivariance of the G_μ action, $\mathbf{J} \circ \mathscr{J}'^*(\mathfrak{g}_\mu) \subset \ker \mathscr{J}'$; this fact combined with (4.6) shows that $\ker \mathscr{J}'_3$ is **J**-invariant. ■

Remark. Actually in the field theory case it is not necessary for the entire momentum map to be quadratic; it suffices for the components corresponding to the symmetry subgroup to be quadratic. For example, this proof can be applied to the vacuum Einstein equations at a solution with only spacelike symmetries. For the nonquadratic, infinite dimensional case, this proof may also be applicable if the action admits some (not necessarily affine) slice and the equivariant Darboux lemma can be made to work.

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