Vol. B17 (1986) ACTA PHYSICA POLONICA No 6

PRELIMIT EFFECTIVE POTENTIAL FROM LATTICE ϕ_4^4 THEORY

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(Received November 13, 1985)

An effective potential for lattice version of ϕ_4^4 theory is calculated in one-loop approximation. Lattices, both finite, are imposed in momentum and in the position spaces. Dependence on the lattice constants is explicitly shown.

PACS numbers: 11.15.Ha

Our aim is to evaluate an effective potential for ϕ_4^4 theory on a finite rectangular lattice, both in momentum and position spaces. A theory is specified by the Lagrangian

$$L(x) = \tfrac{1}{2}(1+A)\partial_\mu\phi(x)\partial^\mu\phi(x)-\tfrac{1}{2}(m^2+B)\phi^2(x)-\frac{1}{4!}(\lambda+C)\int dxdx'dx''$$

$$M(x, x', x'', x''')\phi(x)\phi(x')\phi(x'')\phi(x'''), \tag{1}$$

where A, B, C are renormalization constants, and $M(x, x', x'', x''')$ is a known formfactor [1], [2]. All the space-time variables x, x', x'', x''' vary in a finite region $\mathscr{B} = \bigotimes_{\mu=0}^{3} \mathscr{B}_\mu$ where $\mathscr{B}_\mu$ is a segment on μ-th axis

$$\mathscr{B}_\mu = \{x^\mu;\ -\tfrac{1}{2}L_\mu \leqslant x^\mu \leqslant \tfrac{1}{2}L_\mu\}, \tag{2}$$

$$L_\mu = \frac{2\pi}{b_\mu} = (2N_\mu+1)a_\mu, \quad 2\Lambda_\mu = \frac{2\pi}{a_\mu} = (2N_\mu+1)b_\mu, \tag{3}$$

$$\prod_{\mu=0}^{3} L_\mu = V. \tag{4}$$

Similarly, momentum variables p vary within a finite region

$$\mathfrak{a} = \bigotimes_{\mu=0}^{3} \mathfrak{a}_\mu, \quad \mathfrak{a}_\mu = \{p^\mu;\ -\Lambda_\mu \leqslant p^\mu \leqslant \Lambda_\mu\}. \tag{5}$$

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The field $\phi(x)$ interpolates between lattice values $\phi(n) \equiv \phi(n^0 a_0, n^1 a_1, n^2 a_2, n^3 a_3)$ where n^μ assume integer values $0, \pm 1, \ldots, \pm N_\mu \in \mathbf{Z}(N_\mu)$, $\bigotimes_{\mu=0}^{3} \mathbf{Z}(N_\mu) = \mathbf{Z}(N)$. One has the expansion

$$\phi(x) = \prod_{\mu=0}^{3} a_\mu \sum_{n \in Z(N)} \phi(n)\delta_{a,b}(x-na) \in QC(a, b). \tag{6}$$

Here the function $\delta_{a,b}(x-na) = \prod_{\mu=0}^{3} \delta_{a_\mu, b_\mu}(x^\mu - n^\mu a_\mu)$ plays a role of the Dirac δ-function adapted to the set $QC(a, b)$ of quasipotential interpolating functions [3–5], specified by spatial and reciprocal lattice constants a and b

$$a = (a^0, a^1, a^2, a^3), \qquad b = (b^0, b^1, b^2, b^3). \tag{7}$$

Next, we evaluate, in one-loop approximation the integral [6]

$$Z[J] = N^{-1} \int d\phi e^{\frac{i}{\hbar}(S[\phi]+J\cdot\phi)} = e^{\frac{i}{\hbar}W[J]}. \tag{8}$$

In fact we follow the method applied to the local ϕ_4^4 and carry it over to the nonlocal case. By the way, specific choice of the formfactor M ensures a complete equivalence of this theory with a lattice field theory, (with so-called SLAC derivatives [7], [8]).

We estimate the functional $Z[J]$ using classic stationary phase approximation [9], and we get for the effective action

$$\Gamma[\bar{\phi}] = S[\phi] + \frac{i\hbar}{2} \operatorname{Tr} \ln \frac{K[\bar{\phi}]}{K[0]} + \ldots, \tag{9}$$

where

$$\bar{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \phi_0(x) + \hbar \psi(x) + \ldots \tag{10}$$

and $\phi_0(x)$ solves the Landau-Ginzburg equation for a stationary point

$$S'[\phi_0](x) + J(x) = 0 \tag{11}$$

or

$$(\Box - m^2)\phi_0(x) - \frac{\lambda}{3!}\int dx'dx''dx'''M(x, x', x'', x''')\phi_0(x')\phi_0(x'')\phi_0(x''') + J(x) = 0 \tag{12}$$

and

$$K[\phi](x, y) \equiv (\Box - m^2)\delta(x-y) - \frac{\lambda}{2}\int dx'dx''M(x, y, x', x'')\phi(x')\phi(x''). \tag{13}$$

One finds the effective potential $V_{\text{eff}}(\phi)$ in usual way

$$\Gamma[\phi] = -V_{\text{eff}}(\phi) \int dx \qquad \phi - \text{constant}. \tag{14}$$

The renormalization constants B, C are determined by requirements that m and λ should be physical parameters

$$V''_{\text{eff}}(\phi)|_{\phi=0} = m^2, \tag{15}$$

$$V^{(\text{IV})}_{\text{eff}}(\phi)|_{\phi=0} = \lambda. \tag{16}$$

One gets from it the results

$$B = -\frac{i\hbar}{2}\prod_{\mu=0}^{3}\frac{b_\mu}{2\pi}\sum_p \frac{\lambda}{p^2-m^2}, \tag{17}$$

$$C = -\frac{i\hbar}{2}\prod_{\mu=0}^{3}\frac{b_\mu}{2\pi}\sum_p \frac{3\lambda^2}{(p^2-m^2)^2}. \tag{18}$$

The constant A does not contribute to the considered first order approximation in $\hbar$. In this way we find the desired expression for the prelimit effective potential

$$V_{\text{eff}}(\phi) = \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{i\hbar}{2}\prod_{\mu=0}^{3}\frac{b_\mu}{2\pi}\sum_p\left\{\ln\left(1-\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right) + \frac{\frac{\lambda\phi^2}{2}}{p^2-m^2} + \tfrac{1}{2}\left(\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right)^2\right\} + \ldots, \tag{19}$$

where

$$p = mb = (m^0 b_0, m^1 b_1, m^2 b_2, m^3 b_3);\ m^\mu \in \mathbf{Z}(N_\mu). \tag{20}$$

It is surprising perhaps that the result does not depend on the formfactor M. It comes about due to its special property of all the formfactors which appear in the quasicontinual formulation of a field theory (conf. [1], [2])

$$\begin{gathered}\int dx_n M(x_1, \ldots, x_n) = M(x_1, \ldots, x_{n-1}),\ n > 3,\\ \int dx'' M(x, x', x'') = \delta_{a,b}(x-x'),\\ \int dx' \delta_{a,b}(x-x') = 1.\end{gathered} \tag{21}$$

The effective potential does however depend on the constants a and b. Performing the thermodynamic limit $b \to 0$ we get

$$\lim_{b\to 0} V_{\text{eff}}(\phi) = \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \frac{i\hbar}{2}\int\frac{dp}{(2\pi)^4}\left\{\ln\left(1-\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right) + \frac{\frac{\lambda\phi^2}{2}}{p^2-m^2} + \tfrac{1}{2}\left(\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right)^2\right\} + \ldots \tag{22}$$

Momenta of integrations vary here within the sets

$$-\Lambda_\mu \leqslant p^\mu \leqslant \Lambda_\mu = \frac{\pi}{a_\mu}, \qquad \mu = 0, 1, 2, 3. \tag{23}$$

Now, we remove the spatial lattice by performing the limit $a \to 0$ and we obtain in this way

$$\lim_{a\to 0}\lim_{b\to 0} V_{\text{eff}}(\phi) = \tfrac{1}{2} m^2\phi^2 + \frac{\lambda}{4!}\phi^4,$$

$$-\frac{i\hbar}{2}\int\frac{dp}{(2\pi)^4}\left\{\ln\left(1-\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right)+\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}+\tfrac{1}{2}\left(\frac{\frac{\lambda\phi^2}{2}}{p^2-m^2}\right)^2\right\}+\dots. \tag{24}$$

Performing the Wick rotation and taking the integral yields the result

$$\lim_{a\to 0}\lim_{b\to 0} V_{\text{eff}}(\phi) = \tfrac{1}{2} m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\hbar m^2}{(8\pi)^2}\left\{\left(1+\frac{\lambda\phi^2}{2m^2}\right)^2 \ln\left(1+\frac{\lambda\phi^2}{2m^2}\right)\right.$$
$$\left. -\frac{\lambda\phi^2}{2m^2}\left(1+\frac{3\lambda\phi^2}{4m^2}\right)\right\}+\dots. \tag{25}$$

Apparent singularity at $m^2 = 0$ can be removed by reparametrization of the theory. Namely, defining the new coupling constant λ_M by imposing normalization condition on nonvanishing mass $M \neq 0$

$$V_{\text{eff}}^{(\text{IV})}(M) = \lambda_M \tag{26}$$

and reexpressing λ by λ_M, one gets in the massless limit the Coleman-Weinberg result [10]

$$\lim_{m^2\to 0}\lim_{a\to 0}\lim_{b\to 0} V_{\text{eff}}(\phi) = \frac{\lambda_M}{4!}\phi^4 + \frac{\hbar\lambda_M^2}{256\pi^2}\phi^4\left(\ln\frac{\phi^2}{M^2} - \tfrac{25}{6}\right) + \dots \tag{27}$$

Our main result consists of the formula (19) for prelimit value of the potential. It contains a more detailed information about the system than the Coleman-Weinberg limiting formula (27), [11, 12].

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