FERMIONS ON A LATTICE*

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(Received December 2, 1985)

Peculiarities of fermions on a lattice are reviewed with emphasis on Wilson's method and its relation to the decoupling problem of heavy fermions.

PACS numbers: 11.15.Ha

1. Naive fermions

The fermion part of a euclidean gauge theory action reads in the continuum

$$S = -\int dx[\bar{\psi}\gamma_\mu(\partial_\mu - igA_\mu)\psi + \bar{\psi}m\psi], \tag{1}$$

where $\gamma_\mu = \gamma_\mu^\dagger$. To put this action on the lattice $x_\mu = m_\mu a$, $m_\mu = \pm 0, \pm 1, \pm 2, \ldots$, a = lattice distance, one may try a lattice derivative

$$\partial_\mu \psi(x) = \frac{1}{a}[\psi(x+a_\mu) - \psi(x)], \tag{2}$$

where $a_1 = a(1000)$, $a_2 = a(0100)$, etc. For the free fermion theory this leads to

$$S = -\sum_{x,\mu} \frac{1}{a}[\bar{\psi}(x)\gamma_\mu\psi(x+a_\mu) - \bar{\psi}(x)\gamma_\mu\psi(x)] - \sum_x \bar{\psi}(x)m\psi(x), \tag{3}$$

where $\Sigma_x = a^4\Sigma_m$. This form is not hermitian. In euclidean space we adopt a heuristic rule for hermitian conjugation: Take the part involving only spatial indices and use the Minkowski space rules to find the hermitian conjugate. The conjugate of the part with time like indices then follows by covariance. For example, $(\bar{\psi}_1\gamma_m\psi_2)^\dagger = -\bar{\psi}_2\gamma_m\psi_1$, $m = 1, 2, 3$, hence by covariance $(\bar{\psi}_1\gamma_4\psi_2)^\dagger = -\bar{\psi}_2\gamma_4\psi_1$. Similarly, $(\bar{\psi}_1\psi_2)^\dagger = \bar{\psi}_2\psi_1$,

* Presented at the XXV Cracow School of Theoretical Physics, Zakopane, Poland, June 2–14, 1985.

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$(\bar\psi_1[\gamma_\mu, \gamma_\nu]\psi_2)^\dagger = -\bar\psi_2[\gamma_\mu, \gamma_\nu]\psi_1$, $(\bar\psi_1\gamma_\mu\gamma_5\psi_2)^\dagger = -\bar\psi_2\gamma_\mu\gamma_5\psi_1$, $(\bar\psi_1\gamma_5\psi_2)^\dagger = -\bar\psi_2\gamma_5\psi_1$ $(\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4)$. Taking the hermitian part of (3) leads to

$$S = -\sum_{x,\mu} \frac{1}{2a} [\bar\psi(x)\gamma_\mu U_\mu(x)\psi(x+a_\mu) - \bar\psi(x+a_\mu)\gamma_\mu U_\mu^\dagger(x)\psi(x)] - \sum_x \bar\psi(x) m \psi(x), \tag{4}$$

where we have also written a $U(1)$ lattice gauge field

$$U_\mu(x) = \exp[-igaA_\mu(x)]. \tag{5}$$

An action that is hermitian by these rules appears to give reasonable Feynman rules such that contact terms (the regularization dependent polynominal parts) of one loop diagrams have the correct reality properties in the continuum limit. On the lattice therefore, a hermitian action is a necessary (but not sufficient) condition for a positive transfermatrix. (If it can be defined, the transfermatrix T is related to the hamiltonian H by a relation of the typical form: $T = \exp(-2aH)$).

Let us now look at some of the "Feynman rules" implied by the action (4). The fermion propagator in momentum space is given by

$$S(k) = \left(\sum_\mu i\gamma_\mu \frac{1}{a} \sin ak_\mu + m\right)^{-1}, \quad -\frac{\pi}{a} < k_\mu \leqslant \frac{\pi}{a}. \tag{6}$$

For $|k_\mu| \ll \pi/a$ this reduces to the usual covariant form $i\not{k} + m$. However, the sine functions in (6) have zeros also near $k_\mu = \pi/a$ and for ak_μ near any of the 16 four vectors π_A (mod 2π) given by

$$\begin{aligned} \pi_0 &= (0000), \\ \pi_\mu &= (\pi000), (0\pi00), (00\pi0), (000\pi), \\ \pi_{\mu\nu} &= (\pi\pi00), (\pi0\pi0), \ldots, (00\pi\pi), \\ \pi_{\lambda\mu\nu} &= (\pi\pi\pi0), (\pi\pi0\pi), (\pi0\pi\pi), (0\pi\pi\pi), \\ \pi_{1234} &= (\pi\pi\pi\pi), \end{aligned} \tag{7}$$

the propagator is relatively large. We put

$$k_\mu = \frac{1}{a}\pi_{A_\mu} + p_\mu, \quad \sin ak_\mu = \cos\pi_{A_\mu} \sin ap_\mu = \pm\sin ap_\mu. \tag{8}$$

Then, for $a \to 0$ the propagator gets a continuum form

$$S(k) \sim (i\gamma_\mu^A p_\mu + m)^{-1}, \tag{9}$$

with gamma matrices

$$\gamma_\mu^A = \cos \pi_{A_\mu} \gamma_\mu = \pm\gamma_\mu, \tag{10}$$

which are unitarily equivalent to the original γ_μ. The theory has a particle interpretation near each of the 16 points π_A of the Brillouin zone, with physical momentum p_μ related to the wave vector k_μ by (8). One naive lattice Dirac field describes 16 continuum Dirac fields — this phenomenon is sometimes called "species doubling". We shall refer to the particles or fields with $\pi_A \neq 0$ as the species doublers.

It is easy to make the unitary equivalence of the γ_μ^A explicit [1]:

$$\gamma_\mu^A = S_A^\dagger \gamma_\mu S_A, \tag{11}$$

$$S_{\pi_0} = 1, \qquad S_{\pi_\varrho} = i\gamma_\varrho\gamma_5, \qquad S_{\pi_{\varrho\sigma}} = S_{\pi_\varrho} S_{\pi_\sigma}, \text{ etc.} \tag{12}$$

The spinors $u(p)$ and $v(p)$ of the ordinary particles and antiparticles are related to those of the species doublers by

$$u_A(p) = S_A^\dagger u(p), \quad v_A(p) = S_A^\dagger v(p). \tag{13}$$

The interactions of these particles with the gauge field are described by the vertex functions (Fig. 1) which follow from the action (4)

$$\Gamma_\mu = -ig\gamma_\mu \cos \tfrac{1}{2} a(k+l)_\mu, \tag{14}$$

where a suitable choice of phases has been made. For $ak, al \to \pi_A$ this gives

$$\Gamma_\mu^A = -ig\gamma_\mu^A = -igS_A^\dagger \gamma_\mu S_A \tag{15}$$

and using the spinors (13) brings the expressions for scattering amplitudes etc. into standard form. Fig. 2 shows some examples. All objects in momentum space may be taken periodic with period $2\pi/a$. This also holds for momentum conservation at the vertices. In the tree graph continuum limit we have to take the external fermion fields wave vectors k of order π_A/a, $a \to 0$. External gauge field wave vectors have to be chosen of order 0. Then the internal gauge field momenta are of order 0 or π/a (mod $2\pi/a$). In the latter case the amplitude vanishes as $a \to 0$ because the gauge field propagator vanishes for momenta of order $1/a$. Hence processes such as Fig. 1b vanish for $A \neq B$. The pair creation process $A+\bar{A} \to B+\bar{B}$ in Fig. 2b is not suppressed as $a \to 0$ because the gauge field momentum is 0 mod $2\pi/a$. So we cannot ignore the species doublers as they can be produced in pairs in the tree graph approximation and they certainly contribute in vacuum polarization type loop diagrams.

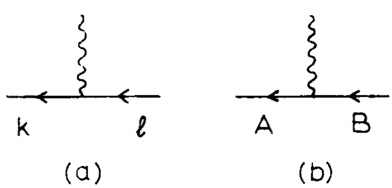

Fig. 1. Three point vertex. In (b), $k \sim \pi_A/a$, $l \sim \pi_B/a$ and the gauge field momentum is $\lll \pi/a$

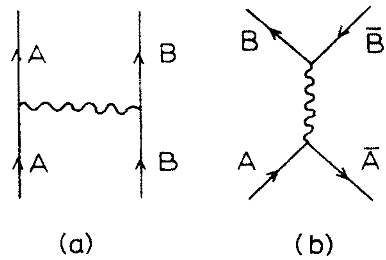

Fig. 2. Elementary tree graph scattering processes in the continuum limit

According to (15) all particles have the same vector coupling g. This is not so for axial vector gauge field couplings. Suppose we insert in (4) a $\mathrm{U}(1)_{\mathrm{A}}$ axial vector lattice gauge field

$$U_\mu(x) = \exp\left[-ig_5\gamma_5 A_\mu(x)\right]. \tag{16}$$

Then we find instead of (15) the vertex function

$$\Gamma^A_{\mu 5} = -ig_5\gamma^A_\mu\gamma_5 = -ig_5 Q^A_5 S^\dagger_A \gamma_\mu\gamma_5 S_A, \tag{17}$$

$$\begin{aligned} Q^A_5 &= +1, \qquad \pi_A = \pi_0, \pi_{\varrho\sigma}, \pi_{1234}, \\ &= -1, \qquad \pi_A = \pi_\varrho, \pi_{\varrho\sigma\tau}. \end{aligned} \tag{18}$$

Hence, there are 8 Dirac fields with axial vector charge $Q_5 = +1$ and 8 with $Q_5 = -1$:

$$\sum_A Q^A_5 = 0. \tag{19}$$

In the continuum formulation we know that a gauge field theory with fermions and axial vector couplings has to be "anomaly free" in order to maintain gauge invariance. For a $\mathrm{U}(1)_{\mathrm{A}}$ theory this means that there has to be more than one fermion and the sum of the axial vector charges has to vanish. On the lattice gauge invariance can be incorporated exactly and consistency with the continuum formulation is achieved by the presence of the species doublers. The anomalies can be calculated from the famous triangle diagram (Fig. 3): it is exactly zero in the naive lattice $\mathrm{U}(1)_{\mathrm{A}}$ theory because of (19) (all particles have the same mass).

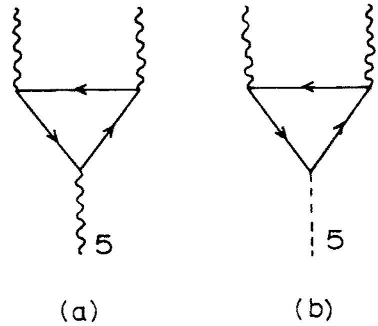

Fig. 3. Triangle diagrams: (a) Vector-Vector-Axial vector and (b) Vector-Vector-Pseudoscalar. On the lattice there are more diagrams (cf. [1]), not shown here

2. Nielson-Ninomya theorem

The fact that axial charges + and − occur in equal numbers in the naive $U(1)_A$ gauge theory suggests that there is an equal number of left and right handed particles. Suppose we use only righthanded fields $\psi_R = P_R\psi$, $\bar{\psi}_R = \bar{\psi}P_L$,

$$P_R = \tfrac{1}{2}(1+\gamma_5), \quad P_L = \tfrac{1}{2}(1-\gamma_5), \tag{20}$$

in (4), with $m = 0$. This leads to a propagator

$$S^{-1}(k) = \sum_{\mu} i\gamma_\mu \frac{1}{a} \sin ak_\mu \tfrac{1}{2}(1+\gamma_5). \tag{21}$$

Near the particle poles we find, using (8) and (12),

$$S^{-1}(k) \sim i\gamma_\mu^A p_\mu \tfrac{1}{2}(1+\gamma_5) = S_A^\dagger i\not{p}\, \tfrac{1}{2}(1\pm\gamma_5)S_A, \tag{22}$$

where the ± signs are as in (18): so this theory describes 8 right handed and 8 left handed Weyl fields. The ± signs in (22) are related to the slopes of the sine function near its zero's, which are both positive and negative. This may be interpreted as being due to the periodicity of momentum space (Fig. 4). Nielson and Ninomya have proved a beautiful theorem (NN theorem), which states that under such cherished conditions as hermiticity, locality and translation invariance the lattice theory always has an equal number of left and right handed fermion particles, in each charge sector of Hilbert space [2, 3]. They used a hamiltonian description for free fermions with continuous time and a lattice in space.

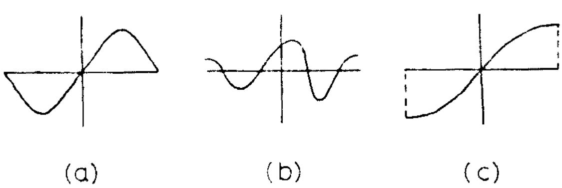

Fig. 4. Periodicity of the gradient function on the 1-torus T^1: (a) sine function, (b) a periodic function leading to more particles, (c) a jump corresponding to a non-local gradient function in position space

The NN theorem is based on topological arguments. A simple version [4] in the euclidean formulation makes use of the Poincaré-Hopf index theorem [5]. Suppose we let[1] $\sin k_\mu \to F_\mu(k)$ in (21). This corresponds to a general translation invariant effective action of the form

$$-\sum_{x,y,\mu} \bar{\psi}(x)\gamma_\mu \tfrac{1}{2}(1+\gamma_5) i\tilde{F}_\mu(x-y)\psi(y), \tag{23}$$

$$F_\mu(k) = \sum_x e^{-ikx}\tilde{F}_\mu(x). \tag{24}$$

[1] In this section we use lattice units $a = 1$.

Hermiticity requires $\tilde{F}_\mu(x) = \tilde{F}_\mu(-x)^*$, i.e. $F_\mu(k)$ is real. Locality means that $\tilde{F}_\mu(x)$ approaches zero reasonably fast as $|x| \to \infty$. This implies certain properties for $F_\mu(k)$. We shall assume that $\tilde{F}_\mu(x)$ is such that $F_\mu(k)$ is smooth (i.e. has continuous derivatives of arbitrary order). If $F_\mu(k)$ has a zero then the theory contains a Weyl field. The zero's of $F_\mu(k)$ should be isolated and of first order for a decent particle interpretation: near a zero at $k = \bar{k}$,

$$F_\mu(k) \sim F_{\mu\nu}(k-\bar{k})_\nu + O((k-\bar{k})^2), \tag{25}$$

where the coefficients $F_{\mu\nu}$ form a matrix F with $\det F \neq 0$. We may write

$$F = OP, \tag{26}$$

with O an orthogonal matrix and P a symmetric positive matrix. The matrix O can be absorbed in a similarity transformation,

$$\gamma_\mu(1+\gamma_5)O_{\mu\nu} = S^{-1}\gamma_\mu(1+\lambda\gamma_5)S, \tag{27}$$

$$\lambda = \det O = \pm 1 \tag{28}$$

(for $\lambda = 1$, S is a rotation $\exp\{\varphi_{\mu\nu}[\gamma_\mu, \gamma_\nu]\}$, for $\lambda = -1$ S can be written as the product of γ_4 and a rotation). Near $k = \bar{k}$ the propagator is equivalent to

$$S(k) \sim -i\,\frac{\not{p}}{p^2}\,\tfrac{1}{2}\,(1+\lambda\gamma_5), \tag{29}$$

$$p_\mu \equiv P_{\mu\nu}(k-\bar{k})_\nu. \tag{30}$$

A non-trivial $P_{\mu\nu}$ cannot be absorbed in a redefinition of the gamma matrices. All we can do is absorb $\det P$ in a wave function renormalization.

The propagator (29) corresponds to a left ($\lambda = -1$) or right ($\lambda = +1$) handed Weyl field, provided we interpret $p_\mu = P_{\mu\nu}(k-\bar{k})_\nu$ as the physical momentum. Now λ is the index of the vector field $F_\mu(k)$ at the zero $k = \bar{k}$ (the degree of the mapping $F_\mu/|F| = O_{\mu\nu}p_\nu/|p|$ onto S^4). The Poincaré-Hopf index theorem states that the global sum of the indices equals the Euler characteristic χ_E of the manifold. This is zero for the momentum space torus T^4: $\Sigma\lambda = \chi_E(T^4) = 0$. Hence there are necessarily as many left as right handed Weyl fields in the continuum limit.

Suppose we introduce interactions with a U(1) gauge field. Then the particles may scatter and translation invariance guarantees wave vector conservation (mod 2π): $\Sigma_{\text{final}}k_\mu = \Sigma_{\text{initial}}k_\mu$, in particular $\Sigma_f\bar{k}_\mu = \Sigma_i\bar{k}_\mu$. For the particle momenta this implies (cf. (30)) $\Sigma_f P^{-1}_{\mu\nu}p_\nu = \Sigma_i P^{-1}_{\mu\nu}p_\nu$, which is equivalent to ordinary momentum conservation $\Sigma_f p_\mu = \Sigma_i p_\mu$ only if all the $P_{\mu\nu}$ are identical (including possible $P_{\mu\nu}$'s for the gauge particle). We have to choose the action such that this is the case. Symmetries of the action should guarantee that all $P_{\mu\nu}$ are identical to all orders in the loop expansion. (It is easy to violate these requirements; consider e.g. a gradient operator with a nearest and next-nearest neighbor interaction; Ref. [6] gives another example).

The action (23) has a global U(1) invariance which serves to distinguish particles from antiparticles. Extension to more general invariance groups is more or less [3] straightforward. We take $F_\mu(k)$ to commute with the invariance transformations. Then there has to be an equal number of left and right handed particles (transforming in an irreducible representation (irrep) say, r), and similarly for the antiparticles (transforming in an irrep r^*). These conclusions do not hold if here is no symmetry at the level of the action ("on the lattice"), but only at the level of particles (continuum limit). For example, suppose we take two actions of the form (23) with two different functions $F_\mu(k)$ having one coinciding zero with $\lambda = -1$ and each having one more non-coinciding zero with $\lambda = +1$. Then the continuum limit describes one left handed SU(2) doublet field and two right handed singlet fields. However, with no symmetry at the level of the action it seems impossible to formulate gauge invariance.

The NN theorem has been interpreted as a no-go theorem for putting the weak interactions on the lattice. One argument was, that the lattice always implied a right handed neutrino (assuming neutrinos to be massless). However, the right handed neutrino may decouple in the continuum limit. A better argument was that any gauge field would always couple equally to left and right handed Weyl fields. However, this argument assumed that we use all the particles suggested by the zeros of the gradient function $F_\mu(k)$, use all the species doublers. This is not necessary as they can be given an arbitrary large mass. This is what Wilson's fermion method does.

3. *Wilson's fermion method*

Three fermion methods have been used to reduce the number of fermions: Wilson's method [7], the staggered fermion method [8], and the non-local method [9]. Here we shall only describe Wilson's method.

The heuristic idea is to add $ar\partial_\mu\bar{\psi}\partial_\mu\psi$ to the naive fermion action (2), (3). Since this term is proportional to the lattice distance a it vanishes in the continuum limit for the wanted particles which have a finite ∂_μ. However, for the species doublers $\partial_\mu = O(1/a)$ and the term is of order r/a. It leads to a mass of order r/a for the species doublers such that they do not really exist as particles. The mass terms in the lattice action can be written in the form $(aM = 1/K_{\text{Wilson}})$

$$S_{\text{mass}} = -\sum_x \bar{\psi}(x)M\psi(x)$$
$$+\frac{r}{2a}\sum_{x,\mu}[\bar{\psi}(x)U_\mu(x)\psi(x+a_\mu)+\bar{\psi}(x+a_\mu)U_\mu^\dagger(x)\psi(x)], \tag{31}$$

where the gauge field U_μ has been inserted in the second term (called the Wilson mass term) to make it gauge invariant. The free fermion propagator is now given by

$$S^{-1}(k) = \sum_\mu i\gamma_\mu\frac{1}{a}\sin ak_\mu + M - \frac{r}{a}\sum_\mu \cos ak_\mu. \tag{32}$$

Putting $k_\mu = p_\mu$ and letting $a \to 0$ leads to the continuum form $i\not{p}+m$, provided we choose M as

$$M = 4\frac{r}{a}+m. \tag{33}$$

With this choice the mass parameters of the species doublers are of order $1/a$:

$$M-\frac{r}{a}\sum_\mu \cos \pi_{A_\mu} = m+\frac{r}{a}\sum_\mu (1-\cos \pi_{A_\mu}) \tag{34}$$

We see from (33) that the ordinary mass parameter M is of order $1/a$ in the continuum limit. The value $4r/a$ can be understood from (31) as the value, where S_{mass} vanishes, for a zero momentum ψ field and a free gauge field $U_\mu \to 1$. In the coupled theory the fluctuations of the U_μ's reduces the effective r value, suggesting that in the continuum limit we must let aM approach a critical value aM_c,

$$M \to M_c(r, g), \tag{35}$$

with $aM_c < 4r$ for $g^2 \neq 0$. A one loop calculation of the fermion self energy verifies this. In the Feynman gauge [1]

$$\Sigma(p) = g^2\Big\{\sigma_1(r)a^{-1}+\sigma_2(r)m+\sigma_3(r)i\not{p}$$

$$-\frac{1}{4\pi}\int_0^1 dx[m+\tfrac{1}{2}(1-x)i\not{p}]\ln[a^2(xm^2+x(1-x)p^2)]\Big\}+O(a), \tag{36}$$

$$S^{-1}(p) = M-4ra^{-1}+i\not{p}+\Sigma(p). \tag{37}$$

Here the σ's are numbers depending on r but not on m which contain all the lattice artifacts: For example, $\sigma_1(0) = 0$, $\sigma_1(1) = 0.326$ [10, 28]. The expression (36) for Σ shows that there is an $O(a^{-1})$ mass divergence in addition to the usual logarithmic divergence. This $O(a^{-1})$ divergence determines M_c,

$$aM_c = 4r-g^2\sigma_1(r)+O(g^4). \tag{38}$$

The bare fermion mass which can be compared with the usual one in the continuum theory is identified as

$$m = M-M_c(r, g). \tag{39}$$

We see furthermore from (36) that all r-dependence can be absorbed in renormalization. The physics depends only on g and m (or rather, on the renormalized g_R and m_R).

4. QCD

The asymptotic freedom of QCD implies that the bare gauge coupling $g \to 0$ in the continuum limit. In other words: the critical value of g is zero, $g_c = 0$. Similarly $aM_c = 4r + O(g^2) = 4r$, the free field value. It is possible to define an $M_c(r, g)$ as the value where the pseudoscalar mesons become massless, for any g (even strong coupling), but we shall not elaborate on this here.

In QCD with N_f flavors M can be a matrix of the form

$$M = M_L P_L + M_L^\dagger P_R, \qquad M_L = N_f \times N_f \text{ matrix.} \tag{40}$$

There is no a priori reason not to generalize r to a matrix R of the same form as (40). By a chiral transformation we can bring R into positive diagonal form, $R = \text{diag}\,(r_1, \ldots, r_{N_f})$. Under this transformation M keeps its general form (40). Note that on the lattice the path integral measure is *invariant* under chiral $U(N_f) \times U(N_f)$ transformations. This is different from the continuum lore where such transformations may induce a θ-parameter through a jacobian factor [11]. How does the θ-parameter arise in this lattice theory?

As mentioned above, in taking the continuum limit aM has to approach a critical form aM_c, which will be positive and diagonal ($aM_c = 4R + O(g^2)$). The difference $m = M - M_c$ plays the role of the quark mass matrix which may still be of the form (40). To find the θ parameter one considers the fermion determinant

$$\exp \text{Tr} \ln (D + M - W) = \exp \text{Tr} \ln (D + m + M_c - W), \tag{41}$$

for a smooth external gauge field configuration. Here we have written the fermion action S_ψ as $-\bar{\psi}(D + M - W)\psi$, $\bar{\psi}W\psi$ being a shorthand form for the Wilson mass term in (31). Let V be a chiral transformation which diagonalizes m,

$$\bar{V} m V = m_d = \text{positive diagonal,} \tag{42}$$

$$V = V_L P_L + V_R P_R, \qquad \bar{V} = V_R^\dagger P_L + V_L^\dagger P_R, \tag{43}$$

with unitary V_L and V_R. Then

$$\text{Tr} \ln (D + m + M_c - W) = \text{Tr} \ln [D + m_d + \bar{V}(M_c - W)V]. \tag{44}$$

The question is really, how does the fermion determinant depend on V? It is easiest to calculate the change of (44) under a change δV of V:

$$\text{Tr}\, [\delta\bar{V}(M_c - W)V + \bar{V}(M_c - W)\delta V]\, [D + m_d + \bar{V}(M_c - W)V]^{-1}. \tag{45}$$

This expression leads to the infinite sum of diagrams in Fig. 5. The crucial point is now that the factor $M_c - W$ in the numerator of (45) suppresses the region where m_d has any influence. In the continuum limit we may as well set $m_d = 0$, in (45). ($M_c - W$ leads only vertices which vanish in the classical continuum limit (e.g. $ra^{-1}\Sigma_\mu(1 - \cos ak_\mu)$ and which therefore need a fermion loop momentum of order $a^{-1} \gg m_d$ to give non-vanishing contribu-

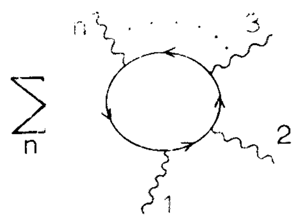

Fig. 5. Diagrams for the fermion determinant in an external gauge field

tions to (45)). With $m_d = 0$, (45) can be rewritten as

$$\mathrm{Tr}\,[\bar{V}^{-1}\delta\bar{V}(M_c-W)(D+M_c-W)^{-1}+\delta VV^{-1}(D+M_c-W)^{-1}(M_c-W)]$$
$$= \mathrm{Tr}\,(V_L\delta V_L^{-1}-V_R\delta V_R^{-1})\gamma_5(M_c-W)(D+M_c-W)^{-1}, \tag{46}$$

where the last form uses the fact that D, M_c and W are all flavor diagonal, and the cyclic property of the trace in space-time (including Dirac indices). Denoting the latter trace by Tr^{st}, we have the result [1, 12, 13]

$$\mathrm{Tr}^{st}\gamma_5(M_c-W)(D+M_c-W)^{-1} \sim Q, \quad a\to 0, \tag{47}$$

where Q is the topological charge

$$Q = \int dx q(x), \tag{48}$$

$$q(x) = \frac{1}{32\pi^2}\int dx \varepsilon_{\mu\nu\varrho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}F_{\varrho\sigma}. \tag{49}$$

Note that (47) is independent of value of the r parameters [1] (as long as they are non-zero), i.e. (47) is a multiple of the identity in flavor space. It follows that the coefficient of Q is given by

$$\mathrm{Tr}^{f}\,(V_L\delta V_L^{-1}-V_R\delta V_R^{-1}) = \delta\ln\det V_RV_L^{-1} = i\delta\arg\det m_L, \tag{50}$$

where Tr^{f} is the trace in flavor space. Integrating δV finally gives the result

$$\exp\mathrm{Tr}\ln(D+m+M_c-W) \sim \exp i\theta Q\exp\mathrm{Tr}\ln(D+m_d+M_c-W), \tag{51}$$

$$\theta = \arg\det m_L, \tag{52}$$

as in the continuum theory. On the lattice the θ-parameter arises from the mis-match between M and M_c.

The expression (47) is related to the divergence of the axial vector currents. The axial vector Ward-Takahashi (WT) identities follow from the response to position dependent chiral transformations,

$$\langle\partial_\mu A^\mu_\alpha - D^A_\alpha + i\bar{\eta}\tfrac{1}{2}\lambda_\alpha\gamma_5\psi + i\bar{\psi}\tfrac{1}{2}\lambda_\alpha\gamma_5\eta\rangle = 0, \tag{53}$$

where the η's are sources which allow for the construction of any WT identity and the $\frac{1}{2}\lambda_\alpha$ are the flavor generators. On the lattice [1, 14]

$$A^\mu_\alpha(x) = \frac{i}{4}\bar{\psi}(x)\gamma_\mu\gamma_5\,\tfrac{1}{2}\lambda_\alpha U_\mu(x)\psi(x+a_\mu)+\text{h.c.}, \tag{54}$$

$$D^A_\alpha(x) = \bar{\psi}(x)i\gamma_5\{\tfrac{1}{2}\lambda_\alpha, M\}\psi(x)$$
$$-\frac{r}{2a}\sum_\mu\{[\bar{\psi}(x)i\gamma_5\,\tfrac{1}{2}\lambda_\alpha U_\mu(x)\psi(x+a_\mu)+\text{h.c.}]+[x\to x-a_\mu]\}. \tag{55}$$

The chiral symmetry breaking consists of two terms, corresponding to the two mass terms in the action (31). The two terms are dimension 4 operators (recall that $M = O(a^{-1})$) so the chiral symmetry breaking appears to be strong. However, in the continuum limit the two terms cancel, except when there are anomalies in the continuum. This is verified in one loop diagrams [1]. It is natural to write

$$D^A_\alpha(x) = \bar{\psi}(x)i\gamma_5\{\tfrac{1}{2}\lambda_\alpha, m\}\psi(x)+D^A_\alpha(x)_c, \tag{56}$$

$$D^A_\alpha(x)_c = D^A_\alpha(x)|_{M=M_c}\ \text{(explicit } M \text{ dependence in (55))}. \tag{57}$$

In the continuum limit ($\lambda_0 \propto \mathbf{1}$)

$$D^A_\alpha(x)_c \sim \delta_{\alpha 0}(\text{tr}\,\tfrac{1}{2}\lambda_0)2q(x), \qquad a\to 0, \tag{58}$$

with q the topological charge density given in (49). For example, for the triangle diagrams in Fig. 3 one finds that $A^\mu_\alpha(x)$ and $\bar{\psi}(x)i\gamma_5\frac{1}{2}\lambda_\alpha\psi(x)$ lead to finite well defined vertex functions as $a\to 0$ which are identical to the continuum forms with vector gauge invariance imposed. These expressions are therefore independent of r. It follows that $D^A_\alpha(x)_c$ also has to have an r-independent limit which is precisely the anomaly in the WT identity of the continuum. This is verified by explicit calculation. For $\alpha \neq 0$, $D^A_\alpha(x)_c$ vanishes on-shell. Off-shell it only generates contact terms, such as, for example, the non-abelian flavor anomaly in the VVA vertex function [1].

5. *Dynamical symmetry breaking*

When all mass terms are sent to zero (i.e. $M\to 0$ and $r\to 0$) there is ample evidence that the theory undergoes dynamical symmetry breaking, from strong coupling calculations as well as computer simulations of the $\langle\bar{\psi}\psi\rangle$ indicator. The symmetry that is actually present for $M = r = 0$ is $U(4N_f)\times U(4N_f)$ [15, 16] and it is dynamically broken to the diagonal subgroup $U(4N_f)$, with $16N_f^2$ corresponding to Nambu-Goldstone (NG) bosons. Increasing r from 0 to 1 and keeping at the same time $M = M_c(r, g)$ one finds [7, 14, 15] at strong coupling that N_f^2-1 NG bosons remain massless (the flavor non-singlet pseudo scalar mesons), 1 NG boson becomes massive (the flavor singlet pseudo scalar meson), $3N_f^2$ become massive vector mesons (the "dormant" Goldstone bosons [17]), and $12N_f^2$ get an infinite mass (as $r \to 1$). Furthermore

$$m_\pi^2 \sim Cm = C(M - M_c), \tag{59}$$

for small m.

Although this is all very nice, there still remains a crucial question to be answered: is chiral $SU(N_f) \times SU(N_f)$ symmetry realized non-linearly in the effective action of QCD? The answer is negative [15] at strong coupling $g \to \infty$, but should hopefully be positive in the true continuum limit which is at $g \to 0$. At the moment all we have to support this conjecture is the evidence mentioned above from weak coupling perturbation theory, that chiral symmetry is restored in the continuum limit.

6. *Decoupling*

The species doublers of the naive lattice fermion theory never really appear in Wilson's formulation as their mass parameters $m + 2nra^{-1}$, $n = 1, 2, 3, 4$ (cf. (34)) are of order of the cut off. Indeed, the r-dependence can be absorbed completely in the renormalization procedure. It is instructive, however, to understand some of the peculiarities of Wilson's fermion method by first keeping the species doublers and sending their masses to infinity after taking the continuum limit. This can be accomplished by putting [1, 18]

$$r = \tilde{r}a, \tag{60}$$

with $\tilde{r}$ some mass, which leads to finite masses for the species doublers:

$$m_A = m + 2n\tilde{r}, \qquad n = 0, 1, \dots, 4. \tag{61}$$

Letting $\tilde{r} \to \infty$ after $a \to 0$ leads to a decoupling problem: can fermions be decoupled by making them infinitely heavy, such that they have no effect on the remaining particles? In a vector gauge theory the answer appears to be yes provided we make some adaptive renormalizations [19]. Some effects of the decoupling particles cannot and need not be absorbed in renormalization: these are related to the anomalies in the WT identities. For example, the axial vector divergence D_0^A leads to a vertex function in Fig. 3 given by (in case of flavor symmetry)

$$2m\Gamma_{\mu\nu}(p, q) = \sum_A Q_5^A 2m_A \Gamma_{\mu\nu}^{\text{cont}}(p, q, m_A) + O(a), \tag{62}$$

with $\Gamma_{\mu\nu}^{\text{cont}}$ the standard continuum form given by

$$2m_A\Gamma_{\mu\nu}^{\text{cont}}(p, q, m_A) = \frac{1}{2\pi^2}\,\varepsilon_{\mu\nu\varrho\sigma}p_\varrho q_\sigma \int_0^1 dx \int_0^{1-x} dy\, \frac{2m_A^2}{m_A^2 + y(1-y)p^2 + 2xypq + x(1-x)q^2}, \tag{63}$$

and the Q_5^A given by (18). There is no anomaly in (62) which is consistent with the fact (19) that the Q_5^A sum up to zero. For $\tilde{r} \to \infty$, $m_A(\pi_A \neq 0) \to \infty$ and the contributions of the species doublers go over into a finite polynomial which is precisely the anomaly, because $\Sigma_{\pi_A \neq 0} Q_5^A = -1$,

$$2m\Gamma_{\mu\nu}(p, q)_{\tilde{r}=\infty} = 2m\Gamma_{\mu\nu}^{\text{cont}}(p, q, m) - \frac{1}{2\pi^2} \varepsilon_{\mu\nu\varrho\sigma} p_\varrho q_\sigma. \tag{64}$$

Decoupling of heavy fermions is much less easy to discuss in theories where the fermion masses m are generated by Yukawa couplings G to Higgs fields ϕ, $m = G\langle\phi\rangle$, as in the theory of electroweak interactions [20, 21]. The basic problem can be seen from Fig. 6: For $m \to \infty$ we need $G \to \infty$ since $\langle\phi\rangle$ (which determines the weak vector masses) is to remain finite in perturbation theory. Since the fermion-Higgs couplings are also proportional to G this implies strong self couplings among the Higgs fields. The problem can no longer be analyzed within weak coupling perturbation theory.

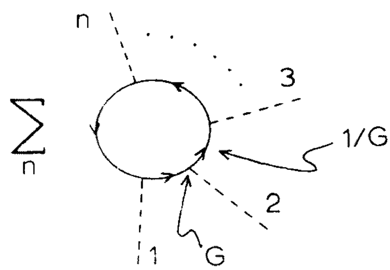

Fig. 6. Diagrams for the induced Higgs couplings due to a heavy fermion, $G \to \infty$. There is also a contribution with external gauge field lines

7. *Glashow-Weinberg-Salam (GWS) theory*

At the time that the connection between species doubling and the triangle anomaly was cleared up, a lattice regularization of the GWS theory, which is equivalent to the continuum formulation in perturbation theory, did not seem possible, since decoupling was not supposed to hold. In the GWS theory, fermion masses are generated by Yukawa couplings to the Higgs field. The same can be done for the Wilson mass term. We can give the species doublers an infinite mass by putting the Yukawa couplings to infinity. This implies a strongly interacting Higgs sector, out of reach of perturbation theory.

A related problem concerns the cancellation of anomalies. One can put only quarks or only leptons on the lattice in a manifestly gauge invariant way. The standard continuum formulation requires complete families of both quarks and leptons, in order to cancel the anomalies. How could a lattice theory of the electroweak interactions with leptons only, say, ever have a satisfactory continuum limit if that limit is not anomaly free?

Recently, this latter obstruction has been removed in the continuum theory: decoupling of a fermion from an anomaly free set in the GWS theory induces so-called Wess-Zumino terms and Goldstone-Wilczek currents in the effective Higgs couplings, which cancel all anomalies of the remaining fermions [21]. Hence, a GWS theory with only leptons does not seem to be impossible. A problem could be that the theory after decoupling is not renormalizable by the usual power counting arguments. However, experience with the lattice regularization has taught us not to shy away from so-called nonrenormalizable models. For example non-linear σ-models may in the continuum limit simply lead to free field theories [22]. Let us now first review the continuum formulation of the quark or lepton part of the GWS action.

The theory for quarks or leptons can be formulated in terms of n left handed doublet fields $\begin{pmatrix} \psi_L^1 \\ \psi_L^2 \end{pmatrix}_k$ and n right handed singlets ψ_{Rk}^1, ψ_{Rk}^2, $k = 1, 2, \ldots, n$; n is the number of families. The fermion part of the action can be written as

$$-S_\psi = \bar{\psi}_{Lk} D_L \psi_{Lk} + \bar{\psi}_{Rk}^1 D_R^1 \psi_{Rk}^1 + \bar{\psi}_{Rk}^2 D_R^2 \psi_{Rk}^2$$
$$+ G_{kl}^1 \bar{\psi}_{Lk} \tilde{\varphi} \psi_{Rl}^1 + G_{kl}^2 \bar{\psi}_{Lk} \varphi \psi_{Rl}^2 + G_{kl}^{1*} \bar{\psi}_{Rl}^1 \tilde{\varphi}^\dagger \psi_{Lk} + G_{kl}^{2*} \bar{\psi}_{Rl}^2 \varphi^\dagger \psi_{Lk}, \tag{65}$$

where

$$D_L = \gamma_\mu(\partial_\mu - igA_\mu - ig' \tfrac{1}{2} Y_L B_\mu), \quad A_\mu = \sum_{m=1}^{3} A_\mu^m \tfrac{1}{2} \tau_m, \tag{66}$$

$$D_R^{1,2} = \gamma_\mu(\partial_\mu - ig' \tfrac{1}{2} Y_R^{1,2} B_\mu), \quad Y_R^1 = Y_L + 1, \quad Y_R^2 = Y_L - 1, \tag{67}$$

and φ is the Higgs doublet with hypercharge $Y_\varphi = 1$,

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} \varphi_2^* \\ -\varphi_1^* \end{pmatrix}. \tag{68}$$

In (65) a standard form is chosen where there is no family mixing in the gradient part of the action. We can put the Yukawa couplings in a suitable standard form by a change of variables which preserves the form of the gradient part,

$$\psi_{Lk} \to W_{Lkl} \psi_{Ll}, \quad \bar{\psi}_{Lk} \to \bar{\psi}_{Ll} W_{Llk}^\dagger,$$
$$\psi_{Rk}^{1,2} \to W_{Rkl}^{1,2} \psi_{Rl}^{1,2}, \quad \bar{\psi}_{Rk}^{1,2} \to \bar{\psi}_{Rl}^{1,2} W_{Rlk}^{\dagger 1,2}, \tag{69}$$

such that, in matrix notation,

$$G^1 \to W_L^\dagger G^1 W_R^1, \quad G^2 \to W_L^\dagger G^2 W_R^2. \tag{70}$$

Convenient standard forms are such that

$$G^1 = D = \text{diagonal}, \quad G^2 = P = \text{hermitian}, \tag{71}$$

or vice-versa,

$$G^1 = P, \quad G^2 = D, \tag{72}$$

with only $n^2 - n + 1$ free parameters in P and with all eigenvalues of P and $D \geqslant 0$.

In the unitary gauge

$$\varphi = \begin{pmatrix} 0 \\ \varrho \end{pmatrix}, \quad \tilde{\varphi} = \begin{pmatrix} \varrho \\ 0 \end{pmatrix}, \tag{73}$$

the Yukawa couplings can be written in the form

$$\varrho(\bar{\psi}^1 G^1 \psi^1 + \bar{\psi}^2 G^2 \psi^2), \qquad \psi^{1,2}_{\substack{\mathrm{L}\\ \mathrm{R}}} = P_{\substack{\mathrm{L}\\ \mathrm{R}}} \psi^{1,2}, \qquad \bar{\psi}^{1,2}_{\substack{\mathrm{L}\\ \mathrm{R}}} = \bar{\psi}^{1,2} P_{\substack{\mathrm{R}\\ \mathrm{L}}}, \tag{74}$$

where the Dirac fields are constructed out of the L and R fields in the usual way. Dynamical symmetry breakdown leads in this gauge in the tree graph approximation to the fermion mass matrix

$$m = \varrho_0 G, \qquad \varrho_0 = \langle \varrho(x) \rangle, \qquad G = \begin{pmatrix} G^1 & 0 \\ 0 & G^2 \end{pmatrix}. \tag{75}$$

For the leptons one may choose the standard form (72) ($\psi^1_k = \nu_e, \nu_\mu, \ldots;\ \psi^2_k = e, \mu, \ldots$) and for quarks the form (71) ($\psi^1_k = u, c, \ldots;\ \psi^2_k = d, s, \ldots$). For example, for $n = 3$ a possible form for P is

$$P = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e^* & f \end{pmatrix}, \qquad a, b, c, d, f \text{ real}. \tag{76}$$

The diagonalization of P introduces the generalized Cabbibo matrix in the gauge field couplings. If the neutrinos are massless, $G^1 = P = 0$ for the leptons. Then the right handed neutrinos decouple.

We can attempt to transcribe this theory to the lattice using Wilson's fermion method to get rid of the species doublers. The D parts are straightforward. Using matrix notation for all indices except space-time:

$$\bar{\psi} D \psi \rightarrow \sum_{x,\mu} \frac{1}{2a} [\bar{\psi}(x) \gamma_\mu U_\mu(x) \psi(x + a_\mu) - \bar{\psi}(x + a_\mu) \gamma_\mu U^\dagger_\mu(x) \psi(x)], \tag{77}$$

$$U_\mu = U_{\mu\mathrm{L}} P_\mathrm{L} + U_{\mu\mathrm{R}} P_\mathrm{R}, \tag{78}$$

$$U_{\mu\mathrm{L}} = \exp\left[-ia(g A_\mu + g' \tfrac{1}{2} y_\mathrm{L} B_\mu)\right], \tag{79}$$

$$U_{\mu\mathrm{R}} = \exp(-iag' \tfrac{1}{2} Y_\mathrm{R} B_\mu), \qquad Y_\mathrm{R} = Y^1_\mathrm{R} \tfrac{1}{2}(1 + \tau_3) + Y^2_\mathrm{R} \tfrac{1}{2}(1 - \tau_3). \tag{80}$$

For the quarks we have to multiply U_μ by the SU(3) color matrix field. The Yukawa couplings in (65) can be transcribed as

$$\sum_x \bar{\psi}(x) [\phi(x) G P_\mathrm{R} + G^\dagger \phi^\dagger(x) P_\mathrm{L}] \psi(x), \tag{81}$$

with

$$\phi = \begin{pmatrix} \varphi^*_2 & \varphi_1 \\ -\varphi^*_1 & \varphi_2 \end{pmatrix}, \qquad G = \begin{pmatrix} G^1 & 0 \\ 0 & G^2 \end{pmatrix}. \tag{82}$$

Since $\bar{\psi}_L(x)\phi(x)$ and $\phi^\dagger(x)\psi_L(x)$ are SU(2) singlets, we can invent a gauge invariant hermitian Wilson mass term as follows,

$$-\tfrac{1}{2}\sum_{x,\mu}[\bar{\psi}(x)\phi(x)G_W U_{\mu R}(x)P_R\psi(x+a_\mu)+\bar{\psi}(x+a_\mu)U^\dagger_{\mu R}(x)G^\dagger_W\phi^\dagger(x)P_L\psi(x)$$
$$+\bar{\psi}(x+a_\mu)\phi(x+a_\mu)G_W U^\dagger_{\mu R}(x)P_R\psi(x)+\bar{\psi}(x)U_{\mu R}(x)G^\dagger_W\phi^\dagger(x+a_\mu)P_L\psi(x+a_\mu)]. \qquad (83)$$

Without loss of generality G_W can be chosen in the standard form (71, 72)

$$G_W=\begin{pmatrix}D & 0\\ 0 & P\end{pmatrix}_W \quad \text{or} \quad G_W=\begin{pmatrix}P & 0\\ 0 & D\end{pmatrix}_W. \qquad (84)$$

A suitable Higgs part of the action is easily constructed:

$$S_\phi=\frac{1}{2a^2}\sum_{x,\mu}\mathrm{tr}\,[\phi^\dagger(x)U_{\mu L}(x)\phi(x+a_\mu)U^\dagger_{\mu R}(x)+\mathrm{h.c.}]$$
$$-\tfrac{1}{2}\sum_x\left[\left(\frac{8}{a^2}+\sigma\right)\mathrm{tr}\,\phi^\dagger(x)\phi(x)+\lambda\,\mathrm{tr}\,(\phi^\dagger(x)\phi(x))^2\right], \qquad (85)$$

where now the parameter Y_L in $U_{\mu L,R}$ is set equal to 1. In the classical continuum limit (85) reduces to the usual form

$$S_\phi \sim -\int dx[(D_\mu\varphi)^\dagger D_\mu\varphi+\sigma\varphi^\dagger\varphi+\lambda(\varphi^\dagger\varphi)^2], \qquad (86)$$

$$D_\mu\varphi=(\partial_\mu-igA_\mu-ig'\tfrac{1}{2}B_\mu)\varphi. \qquad (87)$$

The action for the gauge fields can be taken in the usual standard form. An interesting possibility is that the SU(2) and U(1) gauge fields can be combined into a U(2) field, such that det $U_\mu(x)$ plays the role of the U(1) gauge field. On the lattice the U(2) theory may be different from the SU(2) × U(1) theory. The model (77)–(84) was constructed some time ago and is referred to in [14]. It is constructed independently in [23]. The generation of Wilson mass terms by Yukawa couplings with a Higgs field is discussed before in [18].

In the tree graph approximation the continuum limit of this lattice model is straightforward. The fermion masses follow from the Yukawa couplings (81), (83), by substituting $\phi=\varrho_0\mathbf{1}$, $\varrho_0\approx 250$ GeV, which leads to mass terms of the form (31), (60), with

$$\tilde{r}=G_W\varrho_0. \qquad (88)$$

Hence the continuum mass matrix is given by $m=(G-4G_W)\varrho_0$, while the masses of the species doublers are given by $m+2G_W\varrho_0, \ldots, m+8G_W\varrho_0$, analogous to (61). Removing the species doublers can be done by letting $G_W\to\infty$ (i.e. in the standard representation (71, 72) all eigenvalues $\to\infty$), keeping m and ϱ_0 fixed (since m_W and m_Z are of order ϱ_0 in the tree graph approximation). This implies that also $G=4G_W+m/\varrho_0\to\infty$, while $G^{\mathrm{cont}}\equiv G-4G_W=m/\varrho_0$ stays fixed. Once the continuum limit is taken, the dependence on G_W drops out and G^{cont} may be put into the standard form (71, 72). Further diagonalization leads to the generalized Cabbibo matrix; G^{cont} may be identified with the Yukawa coupling matrix in the continuum.

In the tree graph approximation the lattice theory is equivalent to the continuum theory with fermion decoupling. In higher orders the theory is nonrenormalizable by continuum standards, as mentioned earlier. Hopefully, a nonperturbative treatment can make sense out of the lattice model.

Above, the continuum limit was taken while keeping the species doublers. Subsequently the latter were removed by sending their Yukawa couplings to infinity. We can then appeal to results in the continuum formulation such as the appearance of Wess-Zumino and Goldstone-Wilczek terms in the effective Higgs action. However, it is more in the spirit of Wilson's fermion method never to have to deal with the species doublers, by giving them masses of order $1/a$ and removing them in the process of taking the continuum limit. This can be done, for instance, by choosing G_W such, that for every a the lowest energy E of the species doublers (as calculated from the large time behavior of the fermion propagators) is fixed as $E = \text{const}/a$. Or we could require $\langle\bar{\psi}(x)\phi(x)G_W U_{\mu R}(x)\psi(x+a_\mu)\rangle + \text{c.c.} = \text{const}/a^4$, since the expectation value of the Wilson mass term density should be of order a^{-4}.

Another approach could be to use the decomposition

$$\phi = \varrho V, \quad V^\dagger V = \mathbf{1}, \quad \det V = 1, \tag{89}$$

define

$$\varrho_0 = \langle\varrho\rangle \tag{90}$$

and require

$$G_W\varrho_0 = \frac{R'}{a}, \ R' \text{ in standard form with eigenvalues of order } a^0. \tag{91}$$

It seems a plausible assumption that (91) leads to an E of order $1/a$. Experience with QCD suggests that

$$M' = G\varrho_0 \tag{92}$$

has to approach a critical value $M'_c \propto R'/a$ and that the difference $m' = M' - M'_c$ may be interpreted as the bare fermion mass matrix. The physical fermion masses vanish as $m' \to 0$ (by definition of M'_c).

8. *Radially frozen Higgs field*

It may be awkward to have to compute first ϱ_0 and then tune G_W accordingly. A drastic way of avoiding this problem is to "freeze" $\varrho(x)$, which can be done by tuning the bare Higgs selfcoupling parameters in (85) ($\lambda \to \infty$, $-\sigma/2\lambda$ fixed), such that

$$\varrho(x) \equiv \varrho_0, \tag{93}$$

where we are still free to choose ϱ_0 as we wish.

Let us rewrite the Yukawa couplings in the fermion action, using the notations (89)–(92):

$$S_{\psi}^{\mathrm{Yu}} = -\sum_{x} \bar{\psi}(x)\,[V(x)M'P_{\mathrm{R}}+M'^{\dagger}V^{\dagger}(x)P_{\mathrm{L}}]\psi(x)$$

$$+\frac{1}{2a}\sum_{x,\mu}\{\bar{\psi}(x)\,[V(x)R'U_{\mu\mathrm{R}}(x)P_{\mathrm{R}}+U_{\mu\mathrm{R}}(x)R'V^{\dagger}(x+a_{\mu})P_{\mathrm{L}}]\psi(x+a_{\mu})$$

$$+\bar{\psi}(x+a_{\mu})\,[V(x+a_{\mu})R'U^{\dagger}_{\mu\mathrm{R}}(x)P_{\mathrm{R}}+U^{\dagger}_{\mu\mathrm{R}}(x)R'V^{\dagger}(x)P_{\mathrm{L}}]\psi(x)\}. \qquad (94)$$

In the unitary gauge, $V(x)\to\mathbf{1}$, this action has the same form as the mass terms of the action (31) (except for the generalization $r\to$ matrix R'). The residual gauge invariance corresponds to QED ($\times$ QCD in case of quarks). The physically relevant parameters of the theory are g, g', M' and $\kappa\equiv a^2\varrho_0^2$ (R' is supposed to be irrelevant). The parameter κ determines the strength of the nearest neighbor coupling in the radially frozen Higgs field action

$$S_V = \frac{\kappa}{a^4}\sum_{x,\mu}\tfrac{1}{2}\,\mathrm{tr}\,[V^{\dagger}(x)U_{\mu\mathrm{L}}(x)V(x+a_{\mu})U^{\dagger}_{\mu\mathrm{R}}(x)+\mathrm{h.c.}], \qquad (95)$$

which follows from (85), (89), (93). In taking the continuum limit, if it exists, we may think of g, g' and κ as determining the vector boson masses m_{W} and m_{Z}, and M' as determining the fermion masses.

It is not clear yet how to do this. Simulations of the SU(2) Higgs model without fermions [24] suggest that in the Higgs phase, for a given g, there will be a $\kappa_m(g)$ where am_{W} is minimal, but non-zero. Presumably, to reach $am_{\mathrm{W}}\to 0$, g has to approach a critical value g_{c} and $\kappa\to\kappa_{\mathrm{c}}\equiv\kappa_m(g_{\mathrm{c}})$. Perhaps $g_{\mathrm{c}}=0$ and $\kappa_{\mathrm{c}}\approx 0.37$ is the critical value of the nonlinear SU(2) $\times$ SU(2) sigma model. In the full GWS theory the situation may be similar, although the value of κ_{c} can be strongly influenced by the presence of fermions. Then g'/g can be chosen to fix the ratio $m_{\mathrm{Z}}/m_{\mathrm{W}}$ and $m'=M'-M'_{\mathrm{c}}$ can be chosen to fix the fermion/m_{W} mass ratios. The mass of the Higgs particle would be a prediction of the theory. Simulations with non-radially frozen Higgs models suggest that the physics may be independent of λ, in which case we might as well use the more economical radially frozen model where $\lambda\to\infty$.

Assuming that a continuum limit exists, can we say something about the effect of the electroweak interactions on the quarks, in particular, about the QCD θ-parameter? Let us use the framework of a simple mean field approximation, where the SU(2) $\times$ U(1) gauge fields in the action are replaced by a selfconsistent mean field value. Since the action is gauge invariant, we can do this in any gauge we like, for example the unitary gauge, supplemented by a suitable gauge condition on the electromagnetic field. We assume, of course, that the system is in the electromagnetic Coulomb phase. Hence, in the action we replace $V(x)\to\mathbf{1}$ and approximate

$$U_{\mu\mathrm{L},\mathrm{R}}(\mathrm{SU}(2)\times\mathrm{U}(1))\to u_{\mathrm{L},\mathrm{R}}, \qquad (96)$$

where u_L and u_R are matrix functions in flavor space depending on the parameters g, g' and κ, as well as on the matrices M', R' and τ_3. The QCD action may be transformed into a form where we can recognize the M and R matrices and in general there will be a mismatch between M and M_c ($\propto R$) with a corresponding induced θ-parameter.

However, suppose we choose M' in the standard form (71) as well, such that it commutes with R' (recall that the latter has already taken in the standard form (71), without loss of generality) and with τ_3. With such a choice one expects u_L and u_R to have the commuting standard form too. A simple wave function transformation $\psi_{L,R} \to u_{L,R}^{-1/2}\psi_{L,R}$, $\bar{\psi}_{L,R} \to \bar{\psi}_{L,R}u_{R,L}^{-1/2}$ now brings the D part of the QCD in action in normalized form and we recognize

$$M = M'(u_R u_L)^{-1/2}, \qquad R = R'(u_R u_L^{-1})^{1/2}. \tag{97}$$

It follows that M and M_c have the standard form and M' may be chosen such that $m = M - M_c$ has the standard form as well (m positive). Consequently det m is positive and the θ-parameter is zero. The essential point is that once M' and R' are chosen in commuting standard forms, θ will come out zero. Apart from the requirement that u_L and u_R are positive, no special tuning of g and g' is required. This seems to be different in the continuum theory where it appears that θ has to be adjusted order by order in perturbation theory [25].

9. *Electroweak currents*

Another implication of the lattice GWS model is a definite prescription for the electroweak current operators in lattice QCD. Within lattice QCD there are several ways to define vector and axial vector currents [1, 14], which appear to be equivalent at weak but not at strong coupling [26]. The currents are defined by the coefficients linear in gA_μ and $g'B_\mu$ in the fermionic part of the GWS action, with $\phi \to \varrho_0 \mathbf{1}$. Taking $R' = r\mathbf{1}$ for simplicity (at lowest order in g' and g) one finds that the currents for $W_\mu^\pm$ have the V$-$A form, with A_α^μ given by (54) and V_α^μ by the analogous form

$$V_\alpha^\mu(x) = \frac{i}{4}\,[\bar{\psi}(x)\gamma_\mu\lambda_\alpha U_\mu(x)\psi(x+a_\mu) + \text{h.c.}], \tag{98}$$

where $U_\mu(x)$ is the QCD gauge field. These vector currents are not conserved on the lattice although they become conserved in the continuum limit [1, 26]. For the electromagnetic current one finds the form

$$j_{em}^\mu(x) = \frac{i}{2}\,[-\bar{\psi}(x)\,(r-\gamma_\mu)QU_\mu(x)\psi(x+a_\mu)$$

$$+\bar{\psi}(x+a_\mu)\,(r+\gamma_\mu)QU_\mu^\dagger(x)\psi(x)], \tag{99}$$

with Q the quark charge matrix. This current is exactly conserved on the lattice, which is related to the presence of the r-dependent terms in (99). Finally the current for Z_μ is given by

$$j_Z^\mu(x) = V_3^\mu(x) - A_3^\mu(x) - \sin^2\theta_W j_{em}^\mu(x), \tag{100}$$

as might be expected. Hence, as mentioned in [14], both V_α^μ and $\hat{V}_\alpha^\mu \equiv j_{em}^\mu(Q \to \frac{1}{2}\lambda_\alpha)$ are used by the electroweak interactions.

10. Conclusion

The lattice regularization has been stimulating in investigating the foundations of quantum field theory, especially in the case of fermions. It appears that the Nielson--Ninomya theorem is applicable to any regularization using a compactifiable momentum space. The anomalies appear as remnants of the unavoidable explicit symmetry breaking of the regularization. The properties of Wilson's fermion method may be investigated from the point of view of heavy fermion decoupling, which is an important physical problem in the continuum approach as well.

For the GWS theory the lattice formulation suggests that anomaly cancellation between quarks and leptons is not necessary and that no awkward tuning is required for keeping the induced θ-parameter in QCD equal to zero. If correct, these properties may have implications for model building.

Part of this review is based on work some time ago in which conversations with L. H. Karsten and M. Veltman were very helpful. J. C. Vink contributed with discussions on the θ-parameter of QCD. I would like to thank the organizers of the Cracow School of Physics for their invitation and warm hospitality. This work is financially supported by the "Stichting voor Fundamenteel Onderzoek der Materie (FOM)".

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