ALGEBRAIC CONSTRUCTION OF THE EFFECTIVE MASS MATRIX FOR LEPTONS AND QUARKS

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An algebra of some "pseudo-annihilation" and "pseudo-creation" operators is found which generates an exponentially rising spectrum bounded from below. In terms of these operators an effective mass matrix is constructed for leptons and quarks in consistency with their observed rising mass spectra and decreasing generation mixing in the case of quarks.

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The phenomenon of lepton and quark generations appears nowaday as a genuine puzzle of the fundamental physical research. In particular, the rapidly rising lepton and quark mass spectra accompanied by the decreasing Cabibbo-Kobayashi-Maskawa mixing of quarks [1] is a theoretical key problem to be solved, though its solution is generally believed to lie beyond the standard model.

Sometimes, the proper solution of a physical problem is suggested by finding an adequate formal description of the physical phenomenon involved. Having in mind such a possibility we propose in this note an algebraic formalism to describe the effective mass matrix for leptons and quarks.

Our formalism is based on the "pseudo-annihilation" and "pseudo-creation" operators, a and a^+, defined by the commutation relation [2]

$$aa^+ - \lambda^2 a^+ a = 1, \tag{1}$$

where $\lambda^2 > 1$ is a dimensionless constant. This relation can be also rewritten as $[a, a^+] = 1+(\lambda^2-1)a^+a$. Note that a and a^+ become the usual Bose annihilation and creation operators if $\lambda^2 \to 1$.

Making use of operators a and a^+ defined in Eq. (1) we can exactly solve the eigenvalue equation

$$N|n\rangle = N_n|n\rangle, \quad \langle n|n\rangle = 1 \quad (n = 0, 1, 2, \ldots) \tag{2}$$

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for the Hermitian operator $N = a^{+}a$ (we anticipate that the spectrum of N is discrete). In fact, Eqs. (1) and (2) imply

$$Na^{+}|n\rangle = (\lambda^2 N_n + 1)a^{+}|n\rangle,$$

$$Na|n\rangle = \frac{1}{\lambda^2}(N_n - 1)a|n\rangle. \tag{3}$$

Hence, we can conclude that [3]

$$a^{+}|n\rangle = \sqrt{\lambda^2 N_n + 1}\,|n+1\rangle,$$

$$a|n\rangle = \sqrt{N_n}\,|n-1\rangle \tag{4}$$

and

$$N_{n+1} = \lambda^2 N_n + 1. \tag{5}$$

Defining $|0\rangle$ through the condition $a|0\rangle = 0$ which gives $N_0 = 0$ and then solving the spectral recurrence equation (5) we obtain the following spectrum for N:

$$N_n = \frac{\lambda^{2n} - 1}{\lambda^2 - 1} = \begin{cases} 0 & \text{for} \quad n = 0, \\ 1 + \lambda^2 + \ldots + \lambda^{2n-2} & \text{for} \quad n \geqslant 1. \end{cases} \tag{6}$$

Since $\lambda^2 > 1$, it is an exponentially raising spectrum $N_n \sim \exp(2n \ln \lambda) + \text{const}$, where $\lambda = \sqrt{\lambda^2} > 1$. To give a numerical example: with $\lambda = 4$ we have $N_n = 0, 1, 17, 273, \ldots$ for $n = 0, 1, 2, 3, \ldots$. In the limit of $\lambda^2 \to 1$ we get $N_n = n$. Note that the spectrum (6) also satisfies the three-term recurrence formula

$$N_{n+2} - N_{n+1} = \lambda^2(N_{n+1} - N_n) \tag{7}$$

which is less informative than the two-term recurrence equation (5) (equivalent to the spectrum (6) if $N_0 = 0$).

Now, define the Hermitian operator

$$M = m_0 + \omega N, \tag{8}$$

where $m_0 > 0$ and $\omega > 0$ are mass-dimensional constants. For the eigenvalues m_n ($n = 0, 1, 2, \ldots$) of M we get from Eqs. (5), (6) and (7) the formulae

$$m_{n+1} - m_0 = \lambda^2(m_n - m_0) + \omega, \tag{9}$$

$$m_n = m_0 + \omega \frac{\lambda^{2n} - 1}{\lambda^2 - 1} \tag{10}$$

and

$$m_{n+2} - m_{n+1} = \lambda^2(m_{n+1} - m_n), \tag{11}$$

respectively. Let us try to apply these formulae describing an exponentially rising spectrum $m_n \sim \exp(2n \ln \lambda) + \text{const}$ to the observed lepton and quark mass spectra.

In the case of charged leptons, identifying m_0, m_1, m_2 with m_e, m_μ, m_τ, respectively, we reproduce these masses by means of Eq. (10) with

$$\omega^{(e)} = m_\mu - m_e \simeq 105 \text{ MeV}, \quad \lambda^{(e)} = \left(\frac{m_\tau - m_\mu}{m_\mu - m_e}\right)^{1/2} \simeq 3.99. \tag{12}$$

Then, the hypothetical next charged lepton (call it "ω") is predicted at the mass $m_\omega = m_3 \simeq 28.5$ GeV.

In the case of neutrinos, the degeneracy $m_{\nu_e} = m_{\nu_\mu}$, if it appears, implies $\omega^{(\nu)} = 0$ and hence the mass degeneracy of all neutrinos (and so no neutrino oscillations).

In the case of up quarks, putting m_0, m_1, m_2 equal to $m_u \simeq 0$, $m_c \simeq 1.5$ GeV, m_t, respectively, we get

$$\omega^{(u)} = m_c - m_u \simeq 1.5 \text{ GeV}, \quad \lambda^{(u)} = \left(\frac{m_t - m_c}{m_c - m_u}\right)^{1/2} \simeq \left(\frac{m_t}{1.5 \text{ GeV}} - 1\right)^{1/2}. \tag{13}$$

Hence, taking as the top quark mass

$$m_t \simeq 25 \text{ GeV}, \ 30 \text{ GeV}, \ 35 \text{ GeV}, \ 40 \text{ GeV}, \ 45 \text{ GeV} \tag{14}$$

we obtain

$$\lambda^{(u)} \simeq 4, \quad 4.4, \quad 4.7, \quad 5.1, \quad 5.4, \tag{15}$$

respectively. The hypothetical next up quark (call it "h") is predicted at $m_h = m_3 \simeq 390$ GeV, 570 GeV, 780 GeV, 1000 GeV, 1300 GeV, respectively.

Finally, in the case of down quarks, where m_0, m_1, m_2 are equal to $m_d \simeq 0$, m_s, $m_b \simeq 5$ GeV, respectively, we have

$$\omega^{(d)} = m_s - m_d \simeq m_s, \quad \lambda^{(d)} = \left(\frac{m_b - m_s}{m_s - m_d}\right)^{1/2} \simeq \left(\frac{5 \text{ GeV}}{m_s} - 1\right)^{1/2}. \tag{16}$$

If we make the conjecture that $\lambda^{(u)} \simeq \lambda^{(d)}$, then from Eqs. (13) and (16) we have $m_t : m_c \simeq m_b : m_s$ and so taking m_t as given in Eq. (14) we obtain

$$\omega^{(d)} = m_s \simeq 0.3 \text{ GeV}, 0.25 \text{ GeV}, 0.21 \text{ GeV}, 0.19 \text{ GeV}, 0.17 \text{ GeV}, \tag{17}$$

respectively. The hypothetical next down quark (call it "f") is then predicted at $m_f = m_3 \simeq 84$ GeV, 100 GeV, 110 GeV, 130 GeV, 150 GeV.

Note that for $\lambda^{(u)} \simeq \lambda^{(d)} \simeq 3.5$ there would be $m_t \simeq 20$ GeV and $\omega^{(d)} \simeq m_s \simeq 0.38$ GeV, thus $\omega^{(u)} : \omega^{(d)} \simeq 4:1 \simeq Q^{(u)2} : Q^{(d)2}$ with $Q^{(u)} = 2/3$ and $Q^{(d)} = -1/3$ being the electric charges of the up and down quarks. Such a low value of m_t is, however, experimentally excluded. For a more realistic value $m_t \simeq 40$ GeV we have $\omega^{(u)} : \omega^{(d)} \simeq 8:1$ (when $\lambda^{(u)} \simeq \lambda^{(d)}$). The former ratio 4:1, if it were true, might be a signal that the internal Coulomb interaction of some sort is responsible for mass differences between up and down quarks. Though this ratio seems to be considerably higher, it is still natural to expect that such a Coulomb interaction is a kind of driving force [4] in producing mass differences between ups and downs, since the electric charge is the only visible quantum number which distinguishes up and down fermions.

The above discussion shows that the observed mass spectra of leptons and quarks can be reproduced by the mass operator of the form (8), separately for each of four fermion recurrences (ν_e, ν_μ, ν_τ), (e, μ, τ) and (u, c, t), (d, s, b). We will treat the operator M given in Eq. (8) as the zero-order approximation to the more accurate mass operator that, beside the term $N = a^+a$, includes linear terms in a and a^+. Such a perturbed Hermitian mass operator can be written as

$$M^{\mathrm{P}} = m_0+\omega[a^+a+g(a+a^+)+ig'(a-a^+)+g^2+g'^2], \tag{18}$$

where g and g' are dimensionless real "coupling constants". Here, the term $\omega(g^2+g'^2)$ is added in order to cancel the $g^2+g'^2$ correction to m_0 for $n = 0$ (cf. Eq. (22)). Note that $M \to M^{\mathrm{P}}$ when $a \to a+g-ig'$. In Eq. (18) only the ig' term is able to violate CP conservation, so one can try to assume that $|g| \gg |g'|$. Then, in the next-to-zero-order perturbative approach to the operator (18) the ig' and g'^2 terms can be neglected. Notice that replacing ig' by $g'\gamma_5$ we get from Eq. (18) a non-Hermitian mass operator conserving CP (which becomes a Hermitian operator when multiplied by $\beta = \gamma^0$). In this paper we shall use the Hermitian version (18).

In the n representation defined in Eq. (2) the operator (18) is given by the matrix $M^{\mathrm{P}} = (M^{\mathrm{P}}_{n'n})$, where

$$M^{\mathrm{P}}_{n'n} = [m_n+(g^2+g'^2)\omega]\delta_{n'n}$$

$$+(g+ig')\omega\sqrt{N_n}\,\delta_{n',n-1}+(g-ig')\omega\sqrt{N_{n+1}}\,\delta_{n',n+1} \tag{19}$$

with $m_n = m_0+\omega N_n$, $N_{n+1} = \lambda^2 N_n+1$ and $N_n = (\lambda^{2n}-1)/(\lambda^2-1)$. Denoting the eigenvalues and eigenstates of M^{P} by m^{P}_n and $|n\rangle^{\mathrm{P}}$ ($n = 0, 1, 2, \ldots$), respectively, we can write

$$\langle n'|U^{-1}M^{\mathrm{P}}U|n\rangle = m^{\mathrm{P}}_n\delta_{n'n}, \quad |n\rangle^{\mathrm{P}} = U|n\rangle, \tag{20}$$

where U is a unitary operator.

If only three generations $n = 0, 1, 2$ are relevant, the mass matrix given by Eq. (19) takes the form

$$M^{\mathrm{P}} = \begin{pmatrix} m_0+(g^2+g'^2)\omega, & (g+ig')\omega, & 0 \\ (g-ig')\omega, & m_1+(g^2+g'^2)\omega, & (g+ig')\omega\sqrt{\lambda^2+1} \\ 0, & (g-ig')\omega\sqrt{\lambda^2+1}, & m_2+(g^2+g'^2)\omega \end{pmatrix}, \tag{21}$$

where $m_1 = m_0+\omega$ and $m_2 = m_0+\omega(\lambda^2+1)$. Note that the mass matrix (21) is not of the Fritsch type [5]. Note also that applying this mass matrix to up and down quarks the corresponding $M^{(\mathrm{u})\mathrm{P}}$ and $M^{(\mathrm{d})\mathrm{P}}$ are of the Stech type [6] if $\lambda^{(\mathrm{u})} = \lambda^{(\mathrm{d})}$, $g^{(\mathrm{u})} = g^{(\mathrm{d})}$ and $g^{(\mathrm{u})'} = 0$

(and $[m_u+(g^{(u)2}+g^{(u)'2})\omega^{(u)}]\mathbf{1}$ and $[m_d+(g^{(d)2}+g^{(d)'2})\omega^{(d)}]\mathbf{1}$ are treated as extra terms). When the terms with g' are neglected, the next-to-zero-order perturbative approach to the matrix (21) with respect to g gives

$$\left.\begin{aligned} m_0^P &= m_0-g^2\omega \\ m_1^P &= m_1-\frac{g^2\omega}{\lambda^2} \\ m_2^P &= m_2+\frac{g^2\omega(\lambda^2+1)}{\lambda^2} \end{aligned}\right\} +g^2\omega+O(g^4) \tag{22}$$

and

$$U = \begin{bmatrix} 1-\frac{1}{2}g^2, & g, & \dfrac{g^2}{\lambda^2\sqrt{\lambda^2+1}} \\ -g, & 1-\frac{1}{2}g^2\dfrac{\lambda^4+\lambda^2+1}{\lambda^4}, & g\dfrac{\sqrt{\lambda^2+1}}{\lambda^2} \\ \dfrac{g^2}{\sqrt{\lambda^2+1}}, & -g\dfrac{\sqrt{\lambda^2+1}}{\lambda^2}, & 1-\frac{1}{2}g^2\dfrac{\lambda^2+1}{\lambda^4} \end{bmatrix} +O(g^4 \text{ or } g^3). \tag{23}$$

If $O(g) = O(1/\lambda)$ then the omitted higher-order terms on the right-hand side of Eqs. (22) and (23) (and, in consequence, also of Eq. (24)) are still smaller by order $O(g^2)$ than the highest-order terms written down explicitly when $\sqrt{\lambda^2+1}$ and $1/\sqrt{\lambda^2+1}$ are replaced by $\lambda+1/(2\lambda)$ and $1/\lambda-1/(2\lambda^3)$, respectively (the analogical remark pertains also to Eq. (33) discussed later on).

The explicit form (23) of the U matrix which diagonalizes the mass matrix (21) enables us to calculate the Kobayashi-Maskawa quark generation mixing matrix [7] which can be written as $V = U^{(u)-1}U^{(d)}$, where $U^{(u)}$ and $U^{(d)}$ are the U matrices for up and down quark recurrences u, c, t and d, s, b, respectively. A simple calculation gives $V = (V_{n'n})$ ($n' = 0, 1, 2$ $=$ u, c, t and $n = 0, 1, 2 =$ d, s, b) with

$$\begin{aligned} V_{00} &= 1-\tfrac{1}{2}V_{01}^2, \qquad V_{01} = g^{(d)}-g^{(u)} = -V_{10}, \\ V_{11} &= 1-\tfrac{1}{2}(V_{01}^2+V_{12}^2), \\ V_{12} &= g^{(d)}\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}}-g^{(u)}\frac{\sqrt{\lambda^{(u)2}+1}}{\lambda^{(u)2}} = -V_{21}, \\ V_{22} &= 1-\tfrac{1}{2}V_{12}^2, \\ V_{02} &= \frac{g^{(u)2}}{\sqrt{\lambda^{(u)2}+1}}-g^{(u)}g^{(d)}\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}}+\frac{g^{(d)2}}{\lambda^{(d)2}\sqrt{\lambda^{(d)2}+1}} = -V_{20}+V_{01}V_{12} \end{aligned} \tag{24}$$

where $V_{02} \to V_{20}$ if u $\rightleftarrows$ d. These formulae are valid up to $O(g^4$ or $g^3)$ terms. Comparing Eq. (24) with the usual form of the Kobayashi-Maskawa matrix [7, 8]

$$V = \begin{pmatrix} c_1, & -s_1c_3, & -s_1s_3 \\ s_1c_2, & c_1c_2c_3-s_2s_3e^{i\delta}, & c_1c_2s_3+s_2c_3e^{i\delta} \\ s_1s_2, & c_1s_2c_3+c_2s_3e^{i\delta}, & c_1s_2s_3-c_2c_3e^{i\delta} \end{pmatrix} \tag{25}$$

where $c_i = \cos\theta_i$ and $s_i = \sin\theta_i$ $(i = 1, 2, 3)$ (and $\delta = \pi$ because in our case the V matrix is real, while $V_{22} > 0$ with $c_i = 1+O(g^2)$), we deduce that

$$s_1 = V_{10}, \quad s_3-s_2 = V_{12}, \quad s_2 = V_{20}/V_{10}, \quad s_3 = V_{02}/V_{01} \tag{26}$$

with $V_{n'n}$ as given in Eq. (24). It is not difficult to see that if $\lambda^{(u)} = \lambda^{(d)}$ and in addition $|g^{(u)}-g^{(d)}| = 1/\lambda^{(u)}$ (what will appear to be the case at least in a good approximation), the Kobayashi-Maskawa matrix V given by Eq. (24) is of the Wolfenstein form [9].

Since experimentally $V_{01} = -s_1c_3 = \sin\theta_c \simeq \pm 0.23$, we have in Eq. (24) the input $|g^{(u)}-g^{(d)}| \simeq 0.23$. Then, in the case of $\lambda^{(u)} \simeq \lambda^{(d)}$ and $m_t \simeq 30$ GeV, 35 GeV, 40 GeV, respectively, we get from Eqs. (24) and (13)

$$\begin{gathered} |V_{01}| = |V_{10}| \simeq 0.23, \quad V_{00} \simeq 0.97, \quad V_{11} \simeq 0.97, \quad V_{22} \simeq 1.0, \\ |V_{12}| = |V_{21}| \simeq 0.054,\ 0.050,\ 0.046, \\ |V_{02}+V_{20}| \simeq 0.012,\ 0.011,\ 0.011 \end{gathered} \tag{27}$$

and

$$|V_{02}-V_{20}| \simeq 0, \tag{28}$$

the last relation being true if we tentatively put $|g^{(u)}| \simeq |g^{(d)}|$ (i.e., $g^{(u)} \simeq -g^{(d)} \simeq \mp 0.12$). It is so because with $\lambda^{(u)} \simeq \lambda^{(d)}$

$$\begin{gathered} V_{02}+V_{20} \simeq -(g^{(u)}-g^{(d)})V_{12}, \\ V_{02}-V_{20} \simeq -(g^{(u)}+g^{(d)})\left(1-2\frac{m_c}{m_t}\right)V_{12} \end{gathered} \tag{29}$$

due to Eqs. (24) and (13). Then

$$|V_{02}| \simeq |V_{20}| \simeq 0.0062,\ 0.0057,\ 0.0053, \tag{30}$$

respectively. Thus, taking $m_t \simeq 30$–40 GeV we obtain the estimation

$$|V| \simeq \begin{pmatrix} 0.97, & 0.23, & 0.0062\text{–}0.0053 \\ 0.23, & 0.97, & 0.054\text{–}0.046 \\ 0.0062\text{–}0.0053, & 0.054\text{–}0.046, & 1.0 \end{pmatrix}, \tag{31}$$

while from the experiment [8, 10, 11] we have

$$|V| = \begin{pmatrix} 0.9705 \text{ to } 0.9770, & 0.21 \text{ to } 0.24, & 0 \text{ to } 0.014 \\ 0.21 \text{ to } 0.24, & 0.971 \text{ to } 0.973, & 0.036 \text{ to } 0.070 \\ 0 \text{ to } 0.024, & 0.036 \text{ to } 0.069, & 0.997 \text{ to } 0.999 \end{pmatrix}. \tag{32}$$

Note that numerically $|g^{(u)}-g^{(d)}| \simeq 1/\lambda^{(u)}$ since $|g^{(u)}-g^{(d)}| \simeq 0.23$ and $1/\lambda^{(u)} \simeq 0.23-0.20$, the latter for $\lambda^{(u)} \simeq 4.4-5$ (corresponding to $m_t \simeq 30$–40 GeV). Thus, one may speculate that $g^{(u)} = I_3^{(u)}/\lambda^{(u)}$ and $g^{(d)} = I_3^{(d)}/\lambda^{(d)}$ with $I_3^{(u)} = -I_3^{(d)} = 1/2$, which gives $-\sin\theta_c = g^{(u)}-g^{(d)} = 1/\lambda^{(u)} \simeq 0.23-0.20$ if $\lambda^{(u)} = \lambda^{(d)} \simeq 4.4-5$ (corresponding to $m_t \simeq 30$–40 GeV).

We can see from our estimation (31) that in consistency with the experimental data (32) the quark generation mixing decreases with the increasing quantum number $n = 0, 1, 2$ labelling the quark generation. Such a consequence of the mass operator (18) is true also for any higher number of quark generations, $n_G = n_{max}+1 > 3$ (where $n = 0, 1, 2, \ldots, n_{max}$), because the mass separation $|m_{n\pm 1}-m_n|$ increases in the perturbative denominators for our perturbed operator $M^P = M+g\omega(a+a^+)+g^2\omega$ where $M = m_0+\omega a^+a$. Of course, this effective mass operator M^P, if taken at its face value, implies an infinite number of generations $n = 0, 1, 2, \ldots$.

In our preliminary calculations of the Kobayashi-Maskawa matrix we neglected in the mass operator (18) the terms with ig' responsible for CP violation. In a more accurate approach including CP-nonconservation effects these terms must be taken into account. Then, if we assume tentatively that $O(g) = O(g')$, we get the Kobayashi-Maskawa matrix in the form corresponding directly to the convenient parametrization proposed by Gronau and Schechter [12]. But two complex "coupling constants" $G^{(u)} = g^{(u)}+ig^{(u)\prime}$ and $G^{(d)} = g^{(d)}+ig^{(d)\prime}$ make our discussion more involved. In this case, in Eq. (22) the constant $GG^* = g^2+g'^2$ substitutes the constant g^2, while Eq. (24) is replaced by

$$V_{00} = 1-\tfrac{1}{2}|V_{01}|^2-i\,\mathrm{Im}\,(G^{(u)*}G^{(d)}), \qquad V_{01} = G^{(d)}-G^{(u)} = -V_{10}^*,$$

$$V_{11} = 1-\tfrac{1}{2}|V_{01}|^2-\tfrac{1}{2}|V_{12}|^2+i\,\mathrm{Im}\,(G^{(u)*}G^{(d)})\left(1-\frac{\sqrt{\lambda^{(u)2}+1}}{\lambda^{(u)2}}\,\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}}\right),$$

$$V_{12} = G^{(d)}\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}}-G^{(u)}\frac{\sqrt{\lambda^{(u)2}+1}}{\lambda^{(u)2}} - V_{21}^*, \tag{33}$$

$$V_{22} = 1-\tfrac{1}{2}|V_{12}|^2+i\,\mathrm{Im}\,(G^{(u)*}G^{(d)})\,\frac{\sqrt{\lambda^{(u)2}+1}}{\lambda^{(u)2}}\,\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}},$$

$$V_{02} = \frac{G^{(u)2}}{\sqrt{\lambda^{(u)2}+1}}-G^{(u)}G^{(d)}\frac{\sqrt{\lambda^{(d)2}+1}}{\lambda^{(d)2}}+\frac{G^{(d)2}}{\lambda^{(d)2}\sqrt{\lambda^{(d)2}+1}} = -V_{20}^*+V_{01}V_{12}$$

where $V_{02} \to V_{20}^*$ if u $\leftrightarrows$ d. Here, $|G^{(u)}-G^{(d)}| \simeq 0.23$. The phase which is invariant with respect to the rephasing transformations of the quark up-down fields [12] is in our case

$$\varphi = \arg V_{01}+\arg V_{12}-\arg V_{02}. \tag{34}$$

Its nonzero value is responsible for CP violation [11]. Note that if the ratio $G^{(u)}:G^{(d)}$ is real then $\varphi = 0$ as it can be seen from Eq. (33). If $\lambda^{(u)} = \lambda^{(d)}$ and tentatively $G^{(u)} = -G^{(d)*}$

then $\varphi = \arctan[(g^{(d)'}/g^{(d)})(\lambda^{(u)2}-1)/(\lambda^{(u)2}+1)]$, thus $|\varphi| \simeq 42°$–$43°$ when $\lambda^{(u)} \simeq 4.4$–5 and $|g^{(d)'}| \simeq |g^{(d)}|$ (in this case we never obtain $|\varphi| = 90°$ since $g^{(u)} = -g^{(d)} \neq 0$ as here $|\sin\theta_c| = |g^{(u)} - g^{(d)}| \simeq 0.23$). In particular, one may speculate that $G^{(u)} = (I_3^{(u)} + i\xi Y^{(u)}/2)/\lambda^{(u)}$ and $G^{(d)} = (I_3^{(d)} + i\xi Y^{(d)}/2)/\lambda^{(d)}$ with $I_3^{(u)} = -I_3^{(d)} = 1/2$ and $Y^{(u)} = Y^{(d)} = 1/3$, which gives $-\sin\theta_c = G^{(u)} - G^{(d)} = 1/\lambda^{(u)} \simeq 0.23 - 0.20$ if $\lambda^{(u)} = \lambda^{(d)} \simeq 4.4$–$5$ (corresponding to $m_t \simeq 30$–40 GeV) and $-\varphi = \arctan[(\xi/3)(\lambda^{(u)2}-1)/(\lambda^{(u)2}+1)] \simeq 17°$–$18°$, $42°$–$43°$, $70°$ when the parameter $\xi = 1, 3, 9$, respectively. Eventually, if our mass matrices (21) for up and down quarks are of the Stech type [6] i.e., if $\lambda^{(u)} = \lambda^{(d)}$, $g^{(u)} = g^{(d)}$ and $g^{(u)'} = 0$, then $\varphi = -\arctan[(g^{(d)}/g^{(d)'})(\lambda^{(u)2}-1)]$, thus $|\varphi| \simeq 87°$–$88°$ when $\lambda^{(u)} \simeq 4.4$–5 and $|g^{(d)}| \simeq g|^{(d)'}|$ (in this case, keeping in V_{02}, Eq. (33), only the leading terms in $1/\lambda^{(u)}$ one gets $|\varphi| = 90$).

Finally, we would like to point out that if $\lambda^{(u)} = \lambda^{(d)}$ and $m_u = 0 = m_d$ our mass matrix (21) implies the relation

$$\frac{M^{(u)P}}{m_t} - \frac{M^{(d)P}}{m_b} = -\frac{1}{\lambda^{(u)2}+1}\begin{pmatrix} 0, & V_{01}, & 0 \\ V_{01}^*, & 0, & V_{01}\sqrt{\lambda^{(u)2}+1} \\ 0, & V_{01}^*\sqrt{\lambda^{(u)2}+1}, & 0 \end{pmatrix} + \frac{|G^{(u)}|^2 - |G^{(d)}|^2}{\lambda^{(u)2}+1}\mathbf{1}, \tag{35}$$

where $V_{01} = G^{(d)} - G^{(u)}$ and $\mathbf{1}$ is a 3×3 unit matrix. Thus, since numerically $|V_{01}| \simeq 1/\lambda^{(u)}$ (if $m_t \simeq 30$–40 GeV), the right-hand side of Eq. (35) is $0(|V_{01}|^2)$ with $|V_{01}|^2 = \sin^2\theta_c \simeq 0.053$, what was observed recently by Frampton and Jarlskog [13]. They also have shown that in the standard model, where the right-handed fermions are singlets of the electroweak group, the mass matrix may be taken to be Hermitian without loss of generality.

In conclusion, we can claim that as far as three quark generations are concerned the mass operator of the up or down quark recurrence can be represented in a satisfactory way by the algebraic Hermitian operator (18),

$$M^P = m_0 + \omega(a^+a + Ga + G^*a^+ + GG^*) = m_0 + \omega(a^+ + G)(a + G^*), \tag{36}$$

where $aa^+ - \lambda^2 a^+ a = 1$ and possibly $G \equiv g + ig' = C/\lambda$ with $|C^{(u)} - C^{(d)}| = 1$ and $\lambda^{(u)} = \lambda^{(d)} \lesssim 5$, while $\omega^{(u)} \simeq m_c \simeq 1.5$ GeV and $\omega^{(d)} \simeq m_s$. If the relationship $G = C/\lambda$ holds, we get $|\sin\theta_c| \simeq 0.23 - 0.20$ for $\lambda^{(u)} = \lambda^{(d)} \simeq 4.4$–$5$. More information on the complex constants $C^{(u)}$ and $C^{(d)}$ is to be gained from PC nonconservation effects (tentatively $C^{(u)} = -C^{(d)*}$). For $\lambda^{(u)} = \lambda^{(d)} \simeq 4.4$–$5.1$ we obtain $m_t \simeq 30$–40 GeV and $m_s \simeq 0.25$–-0.19 GeV when $m_b \simeq 5$ GeV and predict the hypothetical up and down quarks of the fourth generation at 570–1000 GeV and 100–130 GeV, respectively. In the case of lepton recurrences, the hypothetical charged lepton of the fourth generation is predicted at 28.5 GeV.

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