Vol. B17 (1986) ACTA PHYSICA POLONICA No 7

QUANTUM THEORY OF GAUGE FIELDS WITH EXTERNAL SOURCES

By A. Cabo

Instituto de Matemática, Cibernética y Computación, La Habana*

and A. E. Shabad

P.N. Lebedev Physical Institute, Moscow

(Received January 2, 1986)

Quantum theory of the gauge field with nonvanishing classical source is formulated in a way free of the gauge-choosing ambiguity. The special example of constant non Abelian classical field is considered in more detail. The tree propagators and one-loop action are explicitly calculated for this case.

PACS numbers: 11.15.–q

1. Introduction

One may think that classical fields and/or currents in quantum gauge theory are of two-fold interest. The direct interest comes from problems where a part of physical system is treated as independent (heavy) source while the rest of it is quantal and subject to the influence of the classical field produced by the first part. This way of acting is of help in model building with QCD when, for instance, quarks are sometimes considered as heavy sources. On the other hand the classical, c-numerical fields and currents are inherent to the very formulation of the quantum gauge field theory already in vacuum as the functional argument of the generating functionals of the Green functions ($Z(J)$) or the irreducible vertex functions ($\Gamma(A)$).

The way the external field or current are introduced into the quantum gauge theory is burdened by the dependence on the gauge-fixing conditions, although the so-called background gauges [1, 2] provide that after the gauge condition is fixed the effective action $\Gamma(A)$ is invariant under the gauge transformation of its argument A. Still $\Gamma(A)$, and what is more, the spectrum of particles in external field is sensitive to the choice of the gauge

* Address: Instituto de Matemática, Cibernética y Computación, Academia de Ciencias de Cuba, Calle 0 No. 8, Vedado, La Habana, Cuba.

condition made. This is the consequence of the fact that the theory is independent of this choice only on the mass shell of asymptotic states in the vacuum, which are, however, modified by the presence of external field. The modified mass shell lacks the choice-of--gauge invariance. The only hitherto known exception to this situation is provided by the special case of external field which is exactly sourceless [3, 4], i.e. satisfies the equation $\delta\Gamma/\delta A = 0$. The simplest case of such field is the so-called Abelian covariantly constant classical field $A_\mu^a = A_\mu n^a$, where A_μ is a Minkowskian vector, $A_\mu = F_{\mu\nu}x_{\nu/2}$, $F_{\mu\nu} = \text{const}$ and the constant vector n^a lies in the isotopic space [5], which is sourceless already in the tree approximation $\delta S_0/\delta A_\mu^a = 0$, where S_0 is the classical (tree) action, as well as exactly. This fact is one of the reasons why the Abelian field was first to receive attention as external field in field theoretical calculations. The oneloop effective action was calculated in [6] and its gauge independence was explicitly demonstrated. A number of other results concerning the gauge field spectra and tachyonic instabilities are also obtained by now (see, for instance [7, 8] and references quoted therein). On the other hand another example of external field $A_i^a = \delta_i^a$, $A_4^a = 0$, $i = 1, 2, 3$; $a = 1, 2, 3$, which also gives rise to constant electric and magnetic field strengths is less studied, partially because it is not source-free and the results of calculations are drastically dependent on the gauge chosen. The recent attempt of calculating some of the spectra of gluon in external field of this class [9] is affected by the choice of the gauge. The other hitherto done calculations with these fields are those of Refs [10, 11] which deal with the fermion-loop contribution into effective action that do not suffer from the above ambiguity.

In the present paper we get rid of the restriction $\delta\Gamma/\delta A_\mu^a = 0$ and develop a general scheme of quantization which may be applied to the case of external fields with nonzero sources with the result that the quantum theory with the external field does not imply choosing the gauge condition and is hence unambiguous both on and off the mass shell. In particular the mass shell of asymptotic particles modified by the presence of the external field becomes unambiguous too. We deal with the Yang-Mills SU(2) theory but it seems certain that the main ideas may be extended to other gauge theories including gravitation.

Our starting point is the idea that adding external current, although it is not itself a dynamical variable, to a degenerate dynamical system changes considerably its Hamiltonian structure, namely it removes the degeneracy. Therefore it looks more consistent to quantize the gauge system with the external current included rather than to follow the usual way when the latter is added as (small or large) perturbation after the quantization has been performed and thus the starting position of the perturbation theory has been fixed. Quantum mechanical experience teaches as a lesson that when the perturbation removes degeneracy the initial functions should be prepared in accord with the perturbation to act. The quantization and construction of the perturbation theory following this philosophy is performed in the present paper.

In case when the external current has vanishing fourth component $J_4^a = 0$, to which type the current creating the above non-Abelian constant field belongs, the inclusion of it into the gauge field Hamiltonian does not affect the Poisson bracket relations between the generators of the infinitesimal gauge transformations (constraints) but makes the latter Poisson-non-commuting with the Hamiltonian. This is not unexpected since the external current violates the gauge invariance. The result of the Poisson-commuting of the Hamiltonian with the constraints is just the covariant derivative $\nabla_\mu^{ab} J_\mu^b(x)$ that should be equated to zero for self-consistency [12] thus producing the secondary constraints as conditions the field variables hidden in the covariant derivative ∇_μ^{ab} must obey during the functional integration. It is clear from the above discussion that the number of degrees of freedom remains the same as in the usual gauge theory. Now the quantization is straightforward. It effectively reduces to using the covariant derivative of the external current in place where usually the gauge condition stands [13], this time however without any arbitrariness left. This remark allows us to straightforwardly extend well established results of the quantum gauge theory to our case. These developments are presented in Section 2 for arbitrary external space-like current J_μ^a. By adding a new arbitrary small c-numerical current j_μ^a to the fixed current J_μ^a we build a generating functional $Z(J, j)$ whose j-derivatives at $j = 0$ give Green functions in the presence of the external current J. The diagrammatic technique of this Furry picture includes vertices with the external field propagators of the ghosts which serve the determinant bound for normalizing the δ-function of the secondary constraints $\delta(\nabla_\mu^{ab} J_\mu^b)$ and the propagator of the gauge field against the background of the external current. In Section 3 the special example of external current is studied. Namely, we take the current that is produced by applying the covariant derivative operator to the field strength of the non-Abelian constant classical gauge field discussed above. With this external field we calculate the vertices and the propagators needed for the perturbative expansions, explicitly. Special atention is paid to finding the Green function of the Yang-Mills field in the external non-Abelian constant field, to which end a 12×12 matrix has been diagonalized in Section 3. Some results of this section concerning the structure of the Green function and the gluonic modes go beyond the scope of the tree approximation considered and remain valid for the exact propagator of the gluon in the external field, with the inclusion of finite temperature, as well. It is notable that among the $3 \times 4 = 12$ eigenvectors of the operator, whose inversion is the Green function, only 6 correspond to propagating modes in the sense that the corresponding dispersion equations have solutions expressing the frequencies in terms of spatial momenta. (We shall sometimes refer to this situation by saying that some modes have and some have not the mass shell.) It is only on this modified mass shell that the propagating modes satisfy the initial and secondary constraints and this allows them to be the asymptotic states of the problem. This situation is in full correspondence with the statement made above on the basis of the Hamiltonian analysis that the number of degrees of freedom is the same as in the usual SU(2) gauge theory. For the latter case it is also known that only 6 three-dimensionally transversal (we refer to the Coulomb gauge) modes supply poles to the propagator while the others do not. The situation is analogous also to what happens in QED with external magnetic field, where it was noted by one of the present authors [14] that out of the three photonic eigenmodes, at least within the one-loop polarization operator, only two modes have the mass shell like in the case free of the magnetic field. Therefore the quantum electrodynamics with external magnetic field is another example of the situation when the principle of conservation of the number of degrees of freedom holds. (The counterexample is also known. It is the gauge field with external current which has solely the fourth component. Kiskis

[15] showed that in this case there are more degrees of freedom than in the source-free case. The philosophy of the quantization program claimed in [15] has much in common with ours. We shall argue in Section 2, however, that at least as far as Euclidean formulation is concerned the above Kiskis' statement may be avoided by using proper time Hamiltonian formalism.) We study in Section 3 also the spectrum of the propagating eigenmodes and classify the modes according to the eigenvalues of the symmetry operator of the external field problem.

In Section 4 the additional current j is put equal to zero and the effective action is defined as a Legendre transform of the vacuum-vacuum transition amplitude $Z(J)$ in the presence of external current J. We discuss in what sense $\Gamma(A)$ and $Z(J)$ may be used as generating functionals in the vacuum when differentiated with respect to their arguments.

It is shown that $Z(J)$ is covariant and $\Gamma(A)$ is invariant under the gauge transformations of their arguments.

For the constant non-Abelian classical field the effective one-loop action is calculated. It has an imaginary part responsible for instability of the external field against creation of positive helicity tachyons seen from the study of the spectrum made in Section 3 (cf. the instability of the Abelian classical field [7, 8]).

2. *General canonical formalism and path-integral quantization with external current*

Let

$$\mathscr{L} = -\tfrac{1}{4}\, G^a_{\mu\nu} G^a_{\mu\nu} + J^a_\mu V^a_\mu \tag{2.1}$$

be the Lagrangian density of the SU(2) Yang-Mills field V^a_μ interacting with the 4-point-dependent external current J^a_μ such that

$$J^a_4 = 0. \tag{2.2}$$

(The Lorentz-covariant generalization of the present canonical formalism will be also given later on in this section.) Here

$$G^a_{\mu\nu} = \partial_\mu V^a_\nu - \partial_\nu V^a_\mu + g\varepsilon^{abc} V^b_\mu V^c_\nu. \tag{2.3}$$

The covariant derivative is

$$\nabla^{ab}_\mu = \partial_\mu \delta^{ab} + g V^{ab}_\mu; \tag{2.4}$$

the indices run the values: $\mu = 1, 2, 3, 4$; $a, b, c = 1, 2, 3$ and quantities with two superscripts are defined as $V^{ab}_\mu = \varepsilon^{acb} V^c_\mu$ etc. The fourth components of the vectors are purely imaginary.

After introducing the three momenta (the latin subscripts span the 3-dimensional subspace $i, k = 1, 2, 3$)

$$iE^a_k = G^a_{k4} \tag{2.5}$$

canonically conjugate to the fields V^a_k via the Poisson brackets

$$\{E^a_k(x), V^b_i(y)\}_{x_0=y_0} = -\delta_{ik}\delta^{ab}\delta^3(x-y) \tag{2.6}$$

the system under consideration may be as usually equivalently presented by the Lagrangian density [16]

$$\mathscr{L} = iE_k^a \partial_4 V_k^a - h(E_k, V_k) + iV_4^a C^a(E_k, V_k), \tag{2.7}$$

where V_4^a play the role of the three Lagrange multipliers, C^a are the three constraints

$$C^a = \nabla_k^{ab} E_k^b \tag{2.8}$$

and the Hamiltonian density $h = h_0 + h_1$ consists of two parts: h_0 is the usual Hamiltonian

$$h_0(E_k, V_k) = \tfrac{1}{2}(E^2 + B^2) \tag{2.9}$$

and h_1 represents interaction with the external source

$$h_1 = -J_k^a V_k^a. \tag{2.10}$$

In (2.9)

$$B_k^a = \tfrac{1}{2}\varepsilon^{ijk} G_{ij}^a \tag{2.11}$$

and

$$B^2 = B_k^a B_k^a, \quad E^2 = E_k^a E_k^a. \tag{2.12}$$

The Poisson bracket relations between the constraints (2.8) retain being those for generators of infinitesimal gauge canonical transformations

$$\{C^a(x), C^b(y)\}_{x_4=y_4} = \varepsilon^{abc} C^c(x) \delta^3(x-y) \tag{2.13}$$

while the constraints (2.8) no longer Poisson commute with the Hamiltonian

$$\{\textstyle\int h(x)d^3x, C^a(y)\}_{x_4=y_4} = \{\int h_1(x)d^3x, C^a(y)\}_{x_4=y_4} = -\nabla_k^{ac} J_k^c(y). \tag{2.14}$$

Equation (2.14) tells that the gauge transformation generators C^a do not conserve to the extent the external current is present which violates the gauge symmetry. The consistency of the Hamiltonian description requires according to Dirac [12] that Eq. (2.14) should be treated as secondary constraints. To see this let us write the equation of motion for the constraints C^a:

$$\partial_4 C^a(x) = \{C^a(x), \textstyle\int h(y)d^3y\}_{x_4=y_4} + \int d^3y V_4^b(y)\{C^a(x), C(y)^b\}_{x_4=y_4}. \tag{2.15}$$

When taken together with the equations

$$C^a(E, V) = \nabla_k^{ab} E_k^b = 0, \tag{2.16}$$

which follow from the differentiation of (2.7) over the Lagrange multipliers V_4^a we see that with (2.13), (2.14) taken into account Eq. (2.15) leads to the necessary condition

$$\nabla_k^{ab} J_k^b = 0, \tag{2.17}$$

which in a way restores the gauge invariance violated by the interaction with external current. The secondary constraints thus appeared

$$\chi^a(V_i^c) = \nabla_k^{ab} J_k^b = (\partial_k \delta^{ab} + g\varepsilon^{adb} V_k^d) J_k^b(x) \tag{2.18}$$

depend only on canonical coordinates V_i^c (and not on the conjugate momenta E_i^c) which are to be found from Eq. (2.17) provided the current J_k^c is fixed as it is assumed. Note that in the case of Abelian gauge theory the functions (2.18) cannot serve as constraints and Eq. (2.17) degenerate merely to relations expressing the external current conservation.

The secondary constraints as depending upon coordinates alone Poisson commute with one another:

$$\{\chi^a(x), \chi^b(y)\}_{x_4=y_4} = 0. \tag{2.19}$$

Calculate now the Poisson brackets between the primary (2.8) and secondary (2.18) constraints:

$$\begin{aligned} M_J^{ba}(x, y) &= \{\chi^b(y), C^a(x)\}_{x_4=y_4} = J_k^{bc}(y)\nabla_k^{ca}(x)|_{x_4=y_4}\delta^3(x-y) \\ &= g(J_k^{ba}(y)\partial_k - J_k^f(y)V_k^f(x)\delta^{ba} + J_k^b(y)V_k^a(x))|_{x_4=y_4}\delta^3(x-y). \end{aligned} \tag{2.20}$$

We are interested whether the determinant of (2.20) is nonzero if (2.17) is fulfilled. We do not know the general proof of this fact. In Section 4 however, we shall calculate $\operatorname{Det} M^{ab}(x, y)$ for the case when J_k^a is constant and V_k^a in (2.20) coincides with the field A_k^a, $A_4^a = 0$ whose classical equations of motion have J_k^a as their source

$$\bar{\nabla}_\mu^{ab}\bar{G}_{\mu\nu}^b = -J_\nu^a \tag{2.21}$$

so that (2.17) is guaranteed. (In (2.21) the barred quantities are the same as (2.3), (2.4) but with V_μ^a replaced by A_μ^a.) This determinant is nonzero. The reversibility of (2.20) for other V_μ^a may be proved by expansion about the point $V_\mu^a = A_\mu^a$. This proof is in fact contained in the perturbation theory of Section 3. We believe that for large differences $V_\mu^a - A_\mu^a$ and for nonconstant J_μ^a the operator (2.20) may happen to be irreversible rather as exception than a rule. From now on we take the inequality

$$\operatorname{Det} M_J^{ab}(x, y) \neq 0 \tag{2.22}$$

for granted. Then Eqs. (2.17) and (2.16) are enough to exclude three pairs of canonical variables out of the nine pairs (V_k^a, E_k^a) involved initially in (2.7). Nine minus three makes six degrees of freedom — the same as in the usual gauge theory. We do not solve the problem of explicit exclusion of the redundant degrees of freedom from the Hamiltonian although this problem looks worth studying. Instead, we shall exclude them only implicitly by introducing the δ-function into the continual integral the way it is usually done in the quantum gauge theory. Indeed, we must point out unless this is already clear, that the secondary constraints (2.18) possess every property of a gauge fixing condition of the ordinary theory (we mean (2.13), (2.19), (2.22)), although in our case there is no freedom left in changing them. Therefore for the vacuum to vacuum transition amplitude in the presence of the external current J_μ^a subject to the condition (2.2) we have the following continual integral representation

$$\begin{aligned} Z(J) &= \langle 0|S|0\rangle \\ &= N^{-1}\int \exp\{\int d^4x[-\tfrac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a + J_\mu^a V_\mu^a]\} \\ &\quad [\prod_x \delta(\nabla_k^{ac}(x)J_k^c(x))] \cdot [\prod_{x_4} \operatorname{Det} M_J] \prod_x dV_\mu^a(x), \end{aligned} \tag{2.23}$$

where the operator $M_J^{ab}(x, y)$ is given by (2.20), and N is a normalization factor. It is defined as the same integral as (2.23) but with the term $J_\mu^a V_\mu^a$ omitted from the exponential. Then N does not depend on J_μ^a since any change in J as being at the most a transition to another gauge may be gauged out by appropriate change of the integration variable. (This statement along with another is explained more accurately in Section 4.) Therefore the normalization implies that $\langle 0|S(0)|0\rangle = 1$.

Representing $\det M_J$ as the continual integral over the Grassmanian ghost field variables $\bar{C}^a$, C^a

$$\prod_{x_4} \operatorname{Det} M_J = \int \exp\left\{\int \bar{C}^a(x) M_J^{ab}(x, y) C^b(y) d^4x d^4y\right\} \prod_x d\bar{C} dC \tag{2.24}$$

(we have introduced $\delta(x_4 - y_4)$ inside the definition of $M_J^{ab}(x, y)$ in the integrand in (2.24)) and using the Fourier representation of the continual δ-function

$$\delta(\nabla_k^{ac} J_k^c) = \int \exp\left\{\int \lambda^a(x) \nabla_k^{ac}(x) J_k^c(x) d^4x\right\} \prod_{x,a} d\lambda^a(x) \tag{2.25}$$

we may obtain another representation of (2.23) as an integral over the fields V_μ^a, $\bar{C}^a$, C and λ^a.

We are in a position to discuss now the covariant generalization of the above constraints. It is certain that the Lorentz-covariant constraint condition

$$\nabla_\mu^{ab} J_\mu^b = 0 \tag{2.17a}$$

and the corresponding determinant might be easily achieved by using the known trick of expanding the unity [16] (see Section 4). We find it more instructive to derive the covariant scheme in the Hamiltonian-like fashion [17]. Suppose the three 4-currents $J_\mu^a(x)$, $a = 1, 2, 3$ are all space-like in every spatial point x. Define the proper-time 4-vector $S_\mu(x)$ as satisfying the three equations

$$J_\mu^a(x) S_\mu(x) = 0. \tag{2.26}$$

The vector S_μ is arbitrary-length time-like vector. We may repeat the previous consideration of this section using the direction S_μ in place of the time. The latin subscripts above in this section now should be understood as referring to the space-like hyperplane orthogonal to $S_\mu(x)$ in the Minkowskian space, while the subscript 4 now denotes the projection onto S_μ. The covariant form of canonical quantization is therefore achieved if the scalar products $J_k^a V_k^a$, $\nabla_k^{ac} J_k^c$, $J_k^{bc} \nabla_k^{ca}$ are replaced by $J_\mu^a V_\mu^a$, $\nabla_\mu^{ac} J_\mu^c$, $J_\mu^{bc} \nabla_\mu^{ca}$, respectively, in (2.20, 23–25). In the rest of this paper we shall be working in that covariant way. For the Euclidean field theory the above procedure solves the problem of covariant quantization with external current. On the contrary in Minkowskian field theory, our solution directly serves only the case of space-like external current. If at least one of the four-vectors J_μ^a, $a = 1, 2, 3$ is time-like (and this is just the case considered by Kiskis [15]) the vector S_μ solving Eq. (2.26) is space-like. If one rejects the idea of using Hamiltonian formalism to describe evolution along a space-like direction our method does not apply to this case and one is left with the statement of Ref. [15] that there are only four constraints instead of the six constraints (2.8) and (2.19). In that case, however, we are facing the situation when a superluminal Lorentz transformation that changes a space-like external current into time-like may be responsible for changing the number of degrees of freedom. Also if we follow the statement of [15] literally we must conclude that the number of degrees of freedom increases after some quarks in the quantum system are nailed to become classical sources. Choosing the inadmissibility of this situation as a physical requirement on the quantization postulates one should accept our procedure both for space- and time-like external currents. We also believe that the same prescriptions of quantization may be derived using the generalized Hamiltonian description of Ref. [17].

In Section 4 we continue the general study of Eq. (2.23) as a generating functional of the Green functions and of its Legendre transform. Now we proceed with deriving the perturbation expansion in external field, the so-called Furry picture.

Let us define a generating functional $Z(J, j, \bar{\eta}, \eta)$ of the Green functions of the gauge fields V_μ^a and the ghosts $\bar{C}^a$, C^a in the presence of the external currents J_μ^a so that $Z(J, 0, 0, 0) = 1$. To this end we define the new integration variable

$$\varphi_\mu^a = V_\mu^a - A_\mu^a, \tag{2.27}$$

where A_μ^a is the classical field defined as a solution to the equation (2.21) and add to the action the interactions with new sources j_μ^a, η^a, $\bar{\eta}^a$

$$j_\mu^a \varphi_\mu^a + \bar{\eta}^a C^a + \bar{C}^a \eta^a, \tag{2.28}$$

η^a and $\bar{\eta}^a$ being Grassmanian. Then, taking into account that $\bar{\nabla}_\mu^{ac} J_\mu^c = 0$ we obtain

$$Z(J, j, \bar{\eta}, \eta) = \frac{1}{Z(J)} \cdot \exp \int \{-\tfrac{1}{4} \bar{G}_{\mu\nu}^a \bar{G}_{\mu\nu}^a + J_\mu^a A_\mu^a\} d^4x$$

$$\cdot \int \exp \{ \int [\varphi_\mu^a(x) D_{\mu\nu}^{-1ab}(x, y) \varphi_\nu^b(y) d^4y + j_\mu^a \varphi_\mu^a + g\lambda^b J_\mu^{ba} \varphi_\mu^a$$

$$+ \int \bar{C}^a(x) D_G^{-1ab}(x, y) C^b(y) d^4y + \bar{\eta} C + \bar{C}\eta - \mathscr{L}_{\text{int}}] d^4x\} \cdot \prod_x d\varphi d\bar{C} dC d\lambda, \tag{2.29}$$

where $\mathscr{L}_{\text{int}}$ collects the terms of higher than second power in the integration arguments

$$\mathscr{L}_{\text{int}} = -\left[\bar{C}^a g \varphi_\mu^{ac}(x) J_\mu^{cb} C^b(x) + \frac{g}{2} (\bar{\nabla}_\mu^{ab} \varphi_\nu^b) \varphi_\mu^{ac} \varphi_\nu^c + \frac{g^2}{4} \varphi_\mu^{ab} \varphi_\nu^b \varphi_\mu^{ac} \varphi_\nu^c \right] \tag{2.30}$$

and

$$D_{\mu\nu}^{-1ab}(x, y) = (\bar{\nabla}_\alpha^{ac} \bar{\nabla}_\alpha^{cb} \delta_{\mu\nu} + 2g\bar{G}_{\mu\nu}^{ab} - \bar{\nabla}_\mu^{ac} \bar{\nabla}_\nu^{cb}) \delta^4(x-y)$$

$$= -\tfrac{1}{4} \left. \frac{\delta \int G_{\alpha\beta}^a G_{\alpha\beta}^a d^4z}{\delta V_\mu^a(x) \delta V_\nu^a(y)} \right|_{V=A}, \tag{2.31}$$

$$D_G^{-1ab}(x, y) = g J_\mu^{ac}(x) \bar{\nabla}_\mu^{cb}(y) \delta^4(x-y) = M_J^{ab}(x, y)_{V=A}. \tag{2.32}$$

Remind that the bars over ∇ and G imply that A is substituted for V into them.

The generating functional (2.29) creates the diagrammatic technique of the Furry picture. The structure of the vertices is determined by the three- and four-linear terms collected in $\mathscr{L}_{\text{int}}$ (2.30). It may be specialized for time- and space-independent field A^a_μ and current J^a_μ as following. The three-gluon vertex for the φ-fields with the sets of quantum numbers (ν, c, r), (μ, d, q), (σ, b, p), where ν, μ, σ are Minkowski space indices, c, d, b are isotopic ones and r, q, p are 4-momenta, is

$$\Gamma^{bdc}_{\mu\nu\sigma} = \frac{g}{12}\{\delta_{\sigma\nu}[i(p_\mu - r_\mu)\varepsilon^{bdc} + A^k_\mu(\varepsilon^{akb}\varepsilon^{adc} + \varepsilon^{akc}\varepsilon^{adb})]$$

$$+\delta_{\mu\nu}[i(q_\sigma - r_\sigma)\varepsilon^{dbc} + A^k_\sigma(\varepsilon^{akd}\varepsilon^{abc} + \varepsilon^{akc}\varepsilon^{abd})]$$

$$+\delta_{\sigma\mu}[i(p_\nu - q_\nu)\varepsilon^{bcd} + A^k_\nu(\varepsilon^{akb}\varepsilon^{acd} + \varepsilon^{akd}\varepsilon^{acb})]\}\delta(q+p+r). \tag{2.33}$$

The gluon-ghost vertex is

$$\Gamma^{abc}_\mu = -g\varepsilon^{abd}J^{dc}_\mu\delta(p+q+r), \tag{2.34}$$

where (a, p) and (c, r) are the antighost and ghost isotropic index and momentum, respectively, while the quantum numbers (μ, b, q) belong to the field $\varphi^b_\mu(q)$. The four-gluon vertex is the same as in the theory without external field:

$$\Gamma_{\mu\sigma\nu\varrho} = \frac{g^2}{24}[\varepsilon^{sab}\varepsilon^{scd}(\delta_{\mu\sigma}\delta_{\nu\varrho} - \delta_{\mu\varrho}\delta_{\sigma\nu}) + \varepsilon^{sdb}\varepsilon^{sca}(\delta_{\mu\sigma}\delta_{\nu\varrho} - \delta_{\mu\nu}\delta_{\sigma\varrho})$$

$$+\varepsilon^{sad}\varepsilon^{scb}(\delta_{\mu\nu}\delta_{\sigma\varrho} - \delta_{\mu\varrho}\delta_{\sigma\nu})]\delta(p+q+r+s), \tag{2.35}$$

where the quantum numbers of gluons are grouped as (μ, a, p), (ν, b, q), (σ, d, r), (ϱ, c, s).

To find the vector (gluon) and ghost field propagators one needs to calculate the Gaussian integral into which Eq. (2.29) converts when $\mathscr{L}_{\text{int}}$ is dropped from it.

Performing the integrations subsequently over φ, λ, $\bar{C}$, C we have

$$Z_0(j, \bar{\eta}, \eta) = \exp\int[\tilde{j}^b_\nu(x)D^{bc}_{\nu\mu}(x, y)\tilde{j}^c(y) - \bar{\eta}^b(x)D^{bc}_G(x, y)\eta^c(y)]d^4x d^4y. \tag{2.36}$$

Here the propagators of the gluon $D^{ab}_{\mu\nu}(x, y)$ and of the ghost $D^{ab}_G(x, y)$ are defined as operators inverse to (2.31), (2.32), respectively

$$(\overline{\nabla}^{ac}_\alpha\overline{\nabla}^{cb}_\alpha\delta_{\mu\nu} + 2gG^{ab}_{\mu\nu} - \overline{\nabla}^{ac}_\mu\overline{\nabla}^{cb}_\nu)D^{bd}_{\nu\varrho}(x, y) = \delta^{ad}\delta_{\mu\varrho}\delta^4(x-y), \tag{2.37}$$

$$gJ^{ac}_\mu(x)\overline{\nabla}^{cb}_\mu D^{bd}_G(x, y) = \delta^{ad}\delta^4(x-y) \tag{2.38}$$

and $\tilde{j}^b_\nu$ stands for the projection of the c-numerical current j^b_ν on the covariantly transversal direction

$$\tilde{j}^b_\nu(x) = j^b_\nu(x) - gJ^{bc}_\nu(x)\int D^{cd}_G(x, y)\overline{\nabla}^{da}_\mu j^a_\mu(y)d^4y. \tag{2.39}$$

It satisfies the covariant conservation condition

$$\overline{\nabla}^{ab}_\nu\tilde{j}^b_\nu = 0. \tag{2.40}$$

The Furry perturbational diagrams are defined by the differentiation over j, $\bar{\eta}$, η of the functional Z_0 (2.36).

Some further general statements concerning the properties of asymptotic states of the quantum Yang-Mills field against the background of external field may be found in the last part of Section 3.

3. *Green functions of the Yang-Mills field and of ghosts in external constant non-Abelian field*

In this section we are dealing with the special non-Abelian classical field which in a special Lorentz frame and a special gauge has the form

$$A_i^a = \delta_i^a \left(\frac{A^2}{3}\right)^{1/2}, \quad A_4^a = 0, \tag{3.1}$$

where δ_i^a is a Kronecker symbol and

$$A^2 = A_\mu^a A_\mu^a. \tag{3.2}$$

Equation (2.21) with this field becomes

$$\bar{\nabla}_\mu^{ab} \bar{G}_{\mu\nu}^b = -J_\nu^a = -\tfrac{2}{3} g^2 A^2 A_\nu^a. \tag{3.3}$$

The current J_μ^a obviously belongs to the space-like class. With the field (3.1) we are going to solve the equations for the gluonic and ghost propagators (2.37) (2.38) and calculate the corresponding determinants, in other words to find explicitly all the quantities met in (2.36).

As a matter of fact we solve the Euclidean analogs of (2.37) and (2.38) and later reproduce the results related to the Minkowskian space by analytical continuation.

The differential operator (2.31) after making the Fourier transform becomes a 12×12 matrix whose inversion reduces to its diagonalization. To perform it in a covariant way let us first describe a convenient basis. Define an Euclidean unit length vector u_μ as

$$u_\mu = \tfrac{1}{6}\, \varepsilon_{\alpha\beta\gamma\mu} \varepsilon^{abc} A_\alpha^a A_\beta^b A_\gamma^c (\tfrac{1}{3} A^2)^{-3/2}, \tag{3.4}$$

where $\varepsilon_{\alpha\beta\gamma\mu}$ is the completely antisymmetrical unit tensor in Euclidean space-time. The vector u_μ has the meaning of Euclidean "4-velocity" of the frame in which the classical field has its fourth component zero. We call this frame special. Also $A_\mu^a u_\mu = 0$. Let p_μ be the four-momentum of the gluon, i.e. the Fourier variable conjugate to $(x_\mu - y_\mu)$. Call q_μ the vector

$$q_\mu = p_\mu \left(\frac{g^2 A^2}{3}\right)^{1/2} \tag{3.5}$$

and define the unit 4-vector l_μ, orthogonal to u_μ

$$l_\mu = (q_\mu - \kappa u_\mu)/K, \tag{3.6}$$

where

$$\kappa = q_\mu u_\mu, \quad K = 3(p_\mu A^a_\mu p_\nu A^a_\nu)^{1/2}/g^2A^2. \tag{3.7}$$

The square of u_μ (3.4) is unity. In the special system $u_4 = 1$, $u_i = 0$ and the invariants κ and K become

$$\kappa = p_4\left(\frac{g^2A^2}{3}\right)^{-1/2}, \quad K = (p_i^2)^{1/2}\left(\frac{g^2A^2}{3}\right)^{-1/2}. \tag{3.8}$$

In the 2-dimensional subspace orthogonal to u_μ and l_μ we may choose arbitrarily a basis of two unit vectors $n_\mu(1)$ and $n_\mu(2)$ such that $n_\mu(1)n_\mu(2) = n_\mu(1)u_\mu = n_\mu(2)u_\mu = n_\mu(1)l_\mu = n_\mu(2)l_\mu = 0$, $n^2_\mu(1) = n^2_\mu(2) = 1$. We shall be using, however, the combinations

$$n^\pm_\mu = \frac{1}{\sqrt{2}}(n_\mu(1) \pm i n_\mu(2)). \tag{3.9}$$

The arbitrariness in choosing these two basic vectors is due to the "combined rotational symmetry" of the problem and will be discussed below after the expression for the propagator is obtained. In the isotopic space we may define one unit vector dependent on the characteristic vectors of the problem

$$\beta^a = q_\mu A^a_\mu \Big/ \left[K\left(\frac{A^2}{3}\right)^{1/2}\right] \tag{3.10}$$

and two unit vectors orthogonal to it and between themselves which are dependent on the basic vectors $n^\pm_\mu$ chosen above

$$\eta^a_\pm = \left(\frac{A^2}{3}\right)^{-1/2}(n^\pm_\mu A^a_\mu), \tag{3.11}$$

$$\eta^a_+\eta^a_+ = \eta^a_-\eta^a_- = \eta^a_\pm\beta^a = 0, \eta^a_+\eta^a_- = 1. \tag{3.12}$$

By taking all possible products of the Euclidean vectors (3.4), (3.6), (3.9) by the isotopic vectors (3.10), (3.11) we define a basis of 12 orthonormalized vectors with the scalar product $(X^a_\mu)^*Y^a_\mu = (X, Y)$. The matrix representation of the Fourier-transformed operator $D^{-1ab}_{\mu\nu}$ (2.31) in this basis is a direct sum of 5 matrix blocks of the dimensions 3×3, 3×3, 1×1, 1×1, 4×4. The first two are

$$B^{(1)}_{ij}(\sigma) = \frac{g^2A^2}{3}\cdot\begin{bmatrix} -(q^2+1) & \sigma K+1 & \sigma\kappa \\ \sigma K+1 & -(q^2+1)+K^2 & (K-\sigma)\kappa \\ \sigma\kappa & (K-\sigma)\kappa & -(q^2+2)+2\sigma K+\kappa^2 \end{bmatrix}, \tag{3.13}$$

$\sigma = 1, -1$, $i, j = 1, 2, 3$.
The second two are

$$B^{(2)}(\sigma) = [-q^2+2\sigma K]\cdot\frac{g^2A^2}{3}, \quad \sigma = 1, -1 \tag{3.14}$$

and the fifth one is

$$B_{ij}^{(3)} = \frac{g^2A^2}{3}\cdot\begin{bmatrix} -(q^2+3+2K) & -1 & -(K+2) & -\kappa \\ -1 & -(q^2+3-2K) & K-2 & \kappa \\ -(K+2) & (K-2) & -(q^2+2-K^2) & K\kappa \\ -\kappa & \kappa & K\kappa & -(q^2+2-\kappa^2) \end{bmatrix}, \tag{3.15}$$

$i, j = 1, 2, 3, 4.$
Here $B_{ij}^{(1)}(1)$, $i, j = 1, 2, 3$ are the matrix elements of $D_{\mu\nu}^{-1ab}$ between the vectors

$$n_\mu^a(1, i) = (n_\mu^+\beta^a,\ l_\mu\eta_+^a,\ u_\mu\eta_+^a), \qquad i = 1, 2, 3; \tag{3.16}$$

$$B_{ij}^{(1)}(-1) = (n_\mu^a(-1, i))^* D_{\mu\nu}^{-1ab} n_\nu^b(-1, j),$$

where

$$n_\mu^a(-1, i) = (n_\mu^-\beta^a,\ l_\mu\eta_-^a,\ u_\mu\eta_-^a), \qquad i = 1, 2, 3; \tag{3.17}$$

$$B^{(2)}\begin{pmatrix}1\\2\end{pmatrix} = (n_\mu^\pm\eta_\pm^a)^* D_{\mu\nu}^{-1ab}(n_\nu^\pm n_\pm^b)$$

and finally

$$B_{ij}^{(3)} = (n_\mu^a(3, i))^* D_{\mu\nu}^{-1ab}(n_\nu^a(3, j)),$$

where

$$n_\mu^a(3, i) = (n_\mu^+\eta_-^a,\ n_\mu^-\eta_+^a,\ l_\mu\beta^a,\ u_\mu\beta^a). \tag{3.18}$$

All the other matrix elements are zero. Therefore the diagonalization and inversion of $D_{\mu\nu}^{-1ab}$ reduces to those of these matrix blocks. Let the twelve eigenvectors of $D_{\mu\nu}^{-1ab}$ expanded as linear combinations of the corresponding basic vectors be

$$\varphi_\mu^a(s, \sigma) = \sum_{i=1}^{3} a_i^s(\sigma) n_\mu^a(\sigma, i), \tag{3.19}$$

$$\psi_\mu^a(r) = \sum_{j=1}^{4} a_j^r n_\mu^a(3, j), \tag{3.20}$$

$$\xi_\mu^a(1) = n_\mu^+\eta_+^a, \tag{3.21}$$

$$\xi_\mu^a(-1) = n_\mu^-\eta_-^a, \tag{3.22}$$

where $n_\mu^a(\sigma, i)$, $n_\mu^a(3, j)$ are defined as (3.16–18). Here

$$D_{\mu\nu}^{-1ab}\varphi_\nu^b(s, \sigma) = \tau_\sigma^s\varphi_\mu^a(s, \sigma), \qquad \sigma = \pm 1, \qquad s = 1, 2, 3, \tag{3.23}$$

$$D_{\mu\nu}^{-1ab}\psi_\nu^b(r) = \tau^r\psi_\mu^a(r), \qquad r = 1, 2, 3, 4, \tag{3.24}$$

$$D_{\mu\nu}^{-1ab}\xi_\nu^b(\sigma) = \frac{g^2A^2}{3}(2\sigma K - q^2)\xi_\mu^a(\sigma), \qquad \sigma = \pm 1. \tag{3.25}$$

The 12 eigenvalues τ_σ^s, τ^r and $(g^2A^2/3)(\pm 2K - q^2)$ are functions of the invariants κ, K, q^2. The coefficients of the expansions (3.19), (3.20) are themselves eigenvectors of the matrices (3.13), (3.15), respectively, with the same eigenvalues

$$\sum_{j=1}^{3} B_{ij}^{(1)}(\sigma)a_j^s(\sigma) = \tau_\sigma^s a_i^s(\sigma), \qquad \sigma = \pm 1, \qquad s = 1, 2, 3, \tag{3.26}$$

$$\sum_{j=1}^{4} B_{ij}^{(3)}a_j^r = \tau_r a_i^r, \qquad r = 1, 2, 3, 4. \tag{3.27}$$

The two $\sigma = \pm 1$ sets of three homogeneous equations (3.26) have each three solutions ($s = 1, 2, 3$)

$$a_1^s(\sigma) = \sigma\kappa(\lambda_\sigma^s - 2)/\mathscr{F}, \tag{3.28}$$

$$a_2^s(\sigma) = \sigma\kappa[(\sigma K - 1)\lambda_\sigma^s + 2]/\mathscr{F},$$

$$a_3^s(\sigma) = [(\lambda_\sigma^s)^2 - (K^2 - 2)\lambda_\sigma^s - 2\sigma K]/\mathscr{F},$$

where

$$\lambda_\sigma^s = \frac{3\tau_\sigma^s}{gA^2} + q^2 + 2 \tag{3.29}$$

are three (for each $\sigma = \pm 1$) solutions of the cubic equation

$$\lambda_\sigma^3 - (q^2 + 2 + 2\sigma K)\lambda_\sigma^2 + 2\sigma K(q^2 + 1)\lambda_\sigma + 4q^2 = 0. \tag{3.30}$$

The factor $\mathscr{F}$ in (3.28) is chosen so as to normalize the eigenvectors (3.28) as $\sum_{i=1}^{3} a_i^s(\sigma)a_i^s(\sigma) = 1$ which simultaneously implies the normalization of (3.19)

$$\sum_{\mu,a} (\varphi_\mu^a(s, \sigma))^* \varphi_\mu^a(s, \sigma) = 1. \tag{3.31}$$

The three solutions of (3.30) are real in the Euclidean region since they are related through (3.29) to the eigenvalues of the Hermitian matrix (3.13). They can be described as follows

$$\lambda_\sigma^s = \tfrac{1}{3}[q^2 + 2(\sigma K + 1)] + \varepsilon_s^1[-Q_\sigma + (Q_\sigma^2 + P_\sigma^3)^{1/2}]^{1/3} - \varepsilon_s^2[Q_\sigma + (Q_\sigma^2 + P_\sigma^3)^{1/2}]^{1/3} \tag{3.32}$$

where

$$\varepsilon_1^1 = \varepsilon_1^2 = 1,$$

$$\varepsilon_2^1 = (\varepsilon_2^2)^* = \varepsilon_3^2 = (\varepsilon_3^1)^* = -\tfrac{1}{2} + i\frac{\sqrt{3}}{2} \tag{3.33}$$

and

$$54Q_\sigma = -2q^6 + 6(\sigma K - 2)q^4 + 6(2K^2 + \sigma K + 14)q^2 - 16(\sigma K + 1)^3 + 36\sigma K(\sigma K + 1),$$

$$9P_\sigma = -q^4 + 2(\sigma K - 2)q^2 - 4(\sigma K + 1)^2 + 6\sigma K. \tag{3.34}$$

In the Euclidean region the discriminant $P_\sigma^3 + Q_\sigma^2$ is negative. Analogously, the set of four equations (3.27) has the four solutions ($r = 1, 2, 3, 4$)

$$a_1^r = (\lambda_r^2 + 2\lambda_r - 4K^2 - 8 + 4K)\kappa/\mathcal{G},$$

$$a_2^r = (-\lambda_r^2 - 2\lambda_r + 4K^2 + 8 + 4K)\kappa/\mathcal{G},$$

$$a_3^r = [-\lambda_r^2 - 4\lambda_r + 4(K^2+1)]\kappa K/\mathcal{G},$$

$$a_4^r = [-\lambda_r^3 + (K^2-2)\lambda_r^2 + 8(K^2+1)\lambda_r - 4K^2(K^2+3)]/\mathcal{G}, \tag{3.35}$$

where the normalizing factor $\mathcal{G}$ supplies the vectors a_j^r (3.35) and $\psi_\mu^a(r)$ (3.20) with unit lengths

$$\sum_{i=1}^{4} (a_i^r)^2 = \sum_{\mu,a} (\psi_\mu^a(r))^* \psi_\mu^a(r) = 1. \tag{3.36}$$

The quantities λ_r are four solutions of the quartic equation

$$\lambda^2 + (2-q^2)\lambda^3 - 4(q^2+2+K^2)\lambda^2$$

$$+4[(K^2+1)q^2 + 2K^2]\lambda + 16\kappa^2 = 0, \tag{3.37}$$

which are related to the eigenvalues of $B_{ij}^{(3)}$ and $D_{\mu\nu}^{-1ab}$ as (cf. (3.29))

$$\tau_r = \frac{g^2 A^2}{3}(\lambda_r - q^2 + 2). \tag{3.38}$$

The quantities λ_r as well as τ_r are real in Euclidean space since the matrix $B_{ij}^{(3)}$ (3.15) is Hermitian there. The solutions of the quartic equation (3.37) are given as follows

$$\lambda_r = \frac{q^2 - A - 2}{4} \pm \tfrac{1}{2}\left\{\left(\frac{q^2-A-2}{2}\right)^2 - 4\left[\frac{B(A+2-q^2) - 4(q^2(K^2+1) + 2K^2)}{A}\right]\right\}^{1/2}, \tag{3.39}$$

where

$$A = \pm\{8B + (2-q^2)^2 + 16(q^2+K^2+2)\}^{1/2},$$

$$B = -\tfrac{2}{3}(q^2+K^2+2) + \{-Q + (Q^2+P^3)^{1/2}\}^{1/3} + \{-Q - (Q^2+P^3)^{1/2}\}^{1/3} \tag{3.40}$$

and

$$54Q = 2(9K^2-10)q^6 - (36K^4 - 48K^2 + 318)q^4$$

$$-(168K^4 + 348K^2 + 1248)q^2 + (16K^6 + 564K^4 + 1776K^2 + 128)$$

$$9P = -(3K^2+7)q^4 - (8K^2+58)q^2 - 4(K^2+2)^2 + 60K^2. \tag{3.41}$$

Here all the arguments of the square roots are positive in Euclidean space.

Now we are in a position to write finally the diagonal momentum representation for the gluonic Green function in the external field (3.1), obeying equation (2.37), as the sum of propagators of 12 individual eigenmodes:

$$D^{ab}_{\mu\nu} = \left\{ \sum_{\sigma=\pm 1} \sum_{s=1}^{3} \frac{(\varphi^a_\mu(s,\sigma))^* \varphi^b_\nu(s,\sigma)}{\lambda^s_\sigma - q^2 - 2} \right.$$

$$\left. + \sum_{r=1}^{4} \frac{(\psi^a_\mu(r))^* \psi^b_\nu(r)}{\lambda_r - q^2 - 2} + \sum_{\sigma=\pm 1} \frac{(\xi^a_\mu(\sigma))^* \xi^b_\nu(\sigma)}{2\sigma K - q^2} \right\} \left(\frac{g^2 A^2}{3} \right)^{-1} . \quad (3.42)$$

Here the ten eigenvectors $\varphi^a_\mu(s, \sigma)$, $\psi^a_\mu(r)$ are defined through the substitution of (3.34), (3.41) into (3.28), (3.35) and then into (3.19), (3.20) wherein the basic vectors $n^a_\mu(\sigma, i)$, $n^a_\mu(3, j)$ are given by (3.16), (3.17), (3.18) in terms of the vectors defined by (3.6), (3.9), (3.10), (3.11). The eigenvectors $\xi^a_\mu(\sigma)$ are given by (3.21), (3.22). The denominators in (3.42) are the eigenvalues τ^s_σ, τ_r, $2\sigma K - q^2$. The quantities λ^s_σ, λ_r in them are determined by (3.34), (3.41) as functions of the scalars (3.7) built out of the gluon momentum and the external field.

Another, nondiagonal form of the gluon propagator is given in Appendix.

Note that among the eigenvalues of $D^{-1ab}_{\mu\nu}$ (2.31) none is identically zero. This corresponds to the fact discussed in the previous section that the degeneracy is removed from the Lagrangian already by introducing the external non-Abelian field: the operator (2.31) is free of zero-modes although it does not contain any subsidiary condition in it.

The operator (2.31) commutes with a rotation in the plane orthogonal to the 4-vectors u_μ (3.4) and l_μ (3.6) (in the special frame this is a rotation around the spatial part of the gluon momentum p) supplemented by the rotation by the same angle in the isotopic space around β^a (3.10). (We call this combined rotation about the momentum). This statement follows from the invariance of the external field (3.1) under the combined rotations around any direction and the fact that the gluon momentum is the only characteristic vector of the problem apart from the external field. In the isotopic space this vector specializes the direction β^a. It seems that there is no other operation commuting with (2.31). Therefore we may expect that the eigenvectors (3.19)–(3.22) of (2.31) are also eigenvectors of the combined rotation and may be labelled by its eigenvalues. This is really the case. The vectors $\varphi^a_\mu(s, \sigma)$ (3.19) acquire the factor $e^{-i\theta\sigma}$, under the action of the combined rotation around the momentum by the angle θ, since every i^{th} vector $n^a_\mu(\sigma, i)$ (3.16), (3.17) does. The vectors $\psi^a_\mu(r)$, (3.20) remain invariant under the combined rotation, whereas $\xi^a_\mu(\pm 1)$ (3.21), (3.22) acquire the factor $e^{\mp 2i\theta}$.

Call the infinitesimal combined rotation around p and β^a the projection of combined spin onto the direction of motion (combined helicity) and denote it as L. Then

$$L\varphi^a_\mu(s, \pm 1) = \mp \varphi^a_\mu(s, \pm 1),$$

$$L\xi^a_\mu(\pm 1) = \mp 2\xi^a_\mu(\pm 1),$$

$$L\psi^a_\mu(r) = 0.$$

The combined spin is a product of two spin-one representations of the two SO(3) groups one of which is for the usual and the other for isotopical 3-dimensional rotations. As the eigenvectors are multiplied by $e^{ip_\mu x_\mu}$ when going back to the configuration space we see that the effect of combined rotation around the momentum by the angle θ may be compensated by the space and/or time evolution. Therefore (virtual) eigenmodes corresponding to the eigenvectors $\xi^a_\mu(\pm 1)$ are left or right circularly polarized, i.e. their field vector rotates simultaneously and with the same angular speed in the plane orthogonal to p in the configuration space and in the plane orthogonal to β^a in the isotopic space. The eigenvectors $\varphi^a_\mu(\sigma, s)$ rotate twice as fast while $\psi^a_\mu(r)$ do not rotate. All the common eigenvectors (3.19)–(3.22) of (2.31) and of the combined helicity L have different eigenvalues of (2.31) for different eigenvalues of L. This means, in particular, that a sort of Faraday effect is present for which the inequalities $\tau^s_1 \neq \tau^s_{-1}$, $2K-q^2 \neq -2K-q^2$ are responsible. The Faraday effect comes from the Onsager relation $D^{-1ab}_{\mu\nu}(A) = D^{-1ba}_{\nu\mu}(-A)$ and the fact that (2.31) contains an odd part with respect to the external field (it must be also odd with respect to the momentum). Note, that this is not the case in quantum electrodynamics with external magnetic field [14] where the latter enters into the polarization operator only in even combinations as long as there is the charge symmetry and the Furry theorem holds. Therefore in QED with external field among the eigenmodes propagating along the latter (in QED this is the only direction for which the helicity conserves while in our present problem the combined helicity conserves for every direction of propagation) there are two transverse circularly polarized waves with *different* dispersion laws only when the charge nonsymmetric background like plasma of electrons and positrons with nonzero net charge is introduced [18].

Note that different eigenvalues of (2.31) may correspond to one eigenvalue of L.

The whole contents of the preceding paragraph is based solely on the symmetry of the problem and will remain therefore true if radiational corrections are taken into account in the equation (2.37) for the gluon Green function. If we take (3.42) as the representation for the exact gluon propagator in the external field (3.1) then the representations (3.19)–(3.22) for the eigenvectors are again valid but the coefficients $a^s_i(\sigma)$, a^r_j in them, as well as the eigenvalues τ^s_σ, τ^r, $2\sigma\kappa-q^2$ as being approximation-dependent quantities will be different, although subject to algebraical equations of the same power. Also the splitting of the inverse Green function operator into two 3×3, one 4×4 and two 1×1 blocks, like above, is also an exact fact because this operator as being combined-rotation-invariant may have nonvanishing matrix elements only if sandwiched between the states of the same combined helicity, i.e. selected within one and the same set among the sets (3.17)–(3.18), (3.21), (3.22). The situation does not change either if one places the same system into a hot gluonic thermostat at rest in the special frame. Then the vector of 4-velocity of the thermostat which should be used in forming the tensorial structure of the inverse Green function operator [18, 19] coincides with u_μ Eq. (3.4) and therefore cannot add anything new to this structure.

We proceed now with studying the spectra of gluonic eigenmodes in the external field (3.1). These should be found as poles in (3.42) from the twelve dispersion equations

$$\lambda^s_\sigma - q^2 - 2 = 0, \quad s = 1, 2, 3, \quad \sigma = \pm 1, \tag{3.43}$$

$$\lambda_r - q^2 - 2 = 0, \quad r = 1, 2, 3, 4, \tag{3.44}$$

$$2\sigma K - p^2 = 0, \quad \sigma = \pm 1. \tag{3.45}$$

Substituting (3.43), (3.44) into the equations for λ_σ^s, λ_r (3.30), (3.37) respectively, we find with the use of (3.7)

$$\kappa^2 = \frac{2\sigma K}{2-\sigma K} - K^2, \quad \sigma = \pm 1, \tag{3.46}$$

$$\kappa^2 = -3 \pm (4K^2+9)^{1/2} - K^2. \tag{3.47}$$

Remind that in the special frame $-\kappa^2$ and K^2 are the (normalized) Minkowskian frequency and spatial momentum squared. The dispersion equations (3.45)–(3.47) are at the same time the conditions of the vanishing of the determinants of the matrices (3.13)–(3.15) as well as of the denominators in the nondiagonal representation of the propagator given in Appendix. It is remarkable that we have found only one solution (3.46) for the three equations (3.43). It corresponds to $s = 1$ in (3.32). For the other two values $s = 2,3$ (3.43) does not have solutions. Analogously, the two values (3.47) are solutions of Eq. (3.44) for λ_r in (3.39) with coinciding signs in (3.39) and (3.40). For the other two expressions (3.39) the dispersion equation (3.44) has no solutions.

We conclude that out of the 12 eigen modes only 6 have the mass shell and provide poles for the corresponding terms of the propagator (3.42). We call them propagating modes. For the other six modes the dispersion equations (3.43)–(3.45) have no solutions and their corresponding propagators, among the 12 terms of (3.42), have no poles. We call them confined modes. They have the features of the so-called quark-virton states of Ref. [20]. Contrary to [20], however, their propagators are not entire functions of momenta characteristic of Efimov's nonlocal theory [21]. Instead, the denominators in the propagators of the confined modes may have branching points in the complex plane of frequency squared (this is clear from Eqs. (3.32), (3.39) but no zeros anywhere on the first sheet. Unfortunately, however, the Green functions for some of the propagating modes have forbidden complex poles in the first sheet, corresponding to the instability of external field so common in the non-Abelian theory. We shall discuss this point later.

Note that if one added a "gauge fixing term" to the Yang-Mills action (e.g. in such a way as to create a factor $(1-\alpha)$ in front of $\overline{\nabla}_\mu^{ac}\overline{\nabla}_\nu^{cb}$ in the operator (2.31)) all the 12 modes would become propagating [23]. It may be demonstrated that when α tends to zero six solutions of the dispersion equations (3.43), (3.44) giving κ as a function of K escape to infinitely remote region (cf. [21]). (We must stress once again however that no such term associated with the background gauge, should be added. The consistent quantization procedure of Section 2 implies uniquely that $\alpha = 0$. This is our principal finding which provides the unambiguity of the Green functions and the spectra). The number of propagating modes found is in accord with the fact derived in the previous section from the canonical formalism that the number of polarizational degrees of freedom in the case with external space-like current is the same as without, i.e. equals six. Four of the propagating modes have their combined helicities equal to $L = \pm 2$ (3.45), $L = \pm 1$ (3.46) and two have $L = 0$ (3.47).

The propagating modes, taken along with the background external field, must obey the constraint equation (2.17a) at least on the mass shell because it must be fulfilled for the in and out states. The fulfillment of the constraint equation is provided by the following consideration whose generality goes beyond the example of the external field (3.1) under consideration. From the gauge invariance of the Yang-Mills action

$$S_0 = -\tfrac{1}{4}\int G^a_{\mu\nu}G^a_{\mu\nu}d^4x \tag{3.48}$$

it follows that

$$0 = \frac{\delta}{\delta V^b_\nu(y)}\left(\nabla^{ac}_\mu(x)\frac{\delta S_0}{\delta V^c_\mu(x)}\right) = \nabla^{ac}_\mu(x)\frac{\delta^2 S_0}{\delta V^c_\mu(x)\delta V^b_\nu(y)} + g\varepsilon^{abc}\frac{\delta S_0}{\delta V^c_\mu(x)}\delta^4(x-y). \tag{3.49}$$

Restricting (3.49) onto the external field $V^a_\mu = A^a_\mu$ and using (2.31), (2.21) we get

$$\bar{\nabla}^{ac}_\mu(x)D^{-1cb}_{\mu\nu}(x, y) = -gJ^{ab}_\nu(x)\delta\ (x-y). \tag{3.50}$$

Now, acting by $\bar{\nabla}^a_\mu$ on the both sides of the equations (3.23)–(3.25) (taken in the configuration space) we have

$$gJ^{ab}_\mu\mathscr{V}^b_\mu(x, i) = -\tau_i\bar{\nabla}^{ab}_\mu\mathscr{V}^b_\mu(x, i), \tag{3.51}$$

where $\mathscr{V}$ stands for any of the eigenvectors of $D^{-1ab}_{\mu\nu}$ $\mathscr{V}^a_\mu(i) = (\varphi^a_\mu(s, \sigma), \psi^a_\mu(r), \xi^a_\mu(\sigma))$ and τ_i for any of its eigenvalues τ^s_σ, τ_r, $(2\sigma K-q^2)$. With the use of the identity $\bar{\nabla}^{ab}_\mu J^b_\mu = 0$ Eq. (3.51) may be rewritten as

$$\nabla^{ab}_\mu(i)J^b_\mu = -\tau_i\bar{\nabla}^{ab}_\mu\mathscr{V}^b_\mu(x, i), \tag{3.52}$$

where the covariant derivative $\nabla^{ab}_\mu(i)$ contains the field of the mode $\mathscr{V}^a_\mu(x, i)$

$$\nabla^{ab}_\mu(i) \equiv \delta^{ab}\partial_\mu + g\varepsilon^{acb}(A^c_\mu + \mathscr{V}^c_\mu(x, i)). \tag{3.53}$$

We see that the constraint equations (2.17a) are satisfied either if

$$\bar{\nabla}^{ab}_\mu\mathscr{V}^b_\mu(x, i) = 0 \tag{3.54}$$

or if

$$\tau_i = 0. \tag{3.55}$$

The eigenvectors $\xi(\pm 1)$ (3.21), (3.22) satisfy the (momentum representation counterpart of) Eq. (3.54) (and certainly annihilate the l-h sides of Eqs. (3.51), (3.52) as well). Therefore the two propagating eigenmodes with combined helicities $L = \pm 2$ obey the constraints "kinematically", i.e. both on and off the mass shell. On the contrary, the other four propagating modes satisfy the constraint equations (2.17a) only on the mass shell $\tau_i = 0$. To make sure of this, note, first, that the vectors $u_\mu\beta^a$, $u_\mu\eta^a_\pm$, $l_\mu\beta^a$ which participate in the decompositions (3.19), (3.20) for the eigenvectors $\varphi^a_\mu(s, \sigma)$ and $\psi^a_\mu(r)$ do not contribute into the left-hand sides of (3.51) (3.52) when these eigenvectors are substituted there for

$\mathscr{V}^a(x, i)$. The rest of the vectors (3.16)–(3.18), however, do contribute and it may be shown that the l-h sides of (3.51), (3.52) disappear only if $d_1^s(\sigma) = d_2^s(\sigma)$ in (3.19) and $d_1^r = d_2^r$ in (3.20). These relations with $s = 1$ and $r = 1, 2$ are fulfilled on the mass shells (3.46), (3.47), respectively, and might be used as alternative to the dispersion equations.

Analogously, the constraint equations (2.8), when linearized about the external field (3.1) take the form (in the special frame)

$$\partial_k^2 \mathscr{V}_4^a - \partial_4 \partial_k \mathscr{V}_k^a - g A_k^{ab} \partial_4 \mathscr{V}_k^b + 2g A_k^{ab} \partial_k \mathscr{V}^b + g^2 A_k^{ab} A_k^{bc} \mathscr{V}_4^c = 0. \tag{3.56}$$

Again this is fulfilled kinematically for the modes with the combined helicity $L = \pm 2$ and only on the mass shell for the other propagating modes. To see this one must exploit the relations $\kappa K a_2^1 = (K^2 - 2K + 2) a_3^1$ for the coefficients (3.28) and the relations $a_1^{1,2} = a_2^{1,2}$, $K\kappa a_3^{1,2} = (K^2+2) a_4^{1,2}$ for the coefficients (3.35) valid via the relations (3.46), (3.47), respectively.

We conclude that all the asymptotic states in external field satisfy the constraints (2.17), (2.8) as they should.

Let us discuss now the spectra (3.45)–(3.47). It is seen from (3.8) that in the special frame $\kappa = ip_0 \left(\frac{g^2 A^2}{3}\right)^{1/2}$, where p_0 is the frequency of the gluon. The solutions (3.45)–(3.47) of the dispersion equations express the frequency in function of the spatial momentum length $p = \sqrt{p_i p_i}$ as follows

$$p_0^2 = p^2 - 2\sigma p \left(\frac{g^2 A^2}{3}\right)^{1/2}, \quad \sigma = \pm 1, \tag{3.57}$$

$$p_0^2 = p^2 - \frac{2\sigma p \left(\frac{g^2 A^2}{3}\right)}{2\left(\frac{g^2 A^2}{3}\right)^{1/2} - p\sigma}, \quad \sigma = \pm 1, \tag{3.58}$$

$$p_0^2 = p^2 + g^2 A^2 \mp (3g^2 A^2 + 4p^2)^{1/2} \left(\frac{g^2 A^2}{3}\right)^{1/2}. \tag{3.59}$$

The spectrum (3.57) belongs to the combined helicity ± 2 propagating wave with the 4-polarizations $\xi_\mu^a(\pm 1)$ (3.21), (3.22). The spectrum (3.58) belongs to the combined helicity ± 1 propagating wave with the 4-polarization $\varphi_\mu^a(1, \pm 1)$ (3.19). Equation (3.59) is the spectrum of the two combined helicity 0 waves ψ_μ^a (1,2) (3.20). All the spectra (3.57)–(3.59) but one present zero rest masses $p_0^2(p = 0) = 0$. The exception is (3.59) with the lower sign. The rest mass for it is $p_0^2(0) = 2g^2 A^2$. We guess that the massive longitudinally polarized branch of the spectrum belongs to the irreducible representation with total combined spin equal to zero, while the pentiplet of the other massless branches form an irreducible representation of combined spin equal to 2. Within the same irreducible representation the rest masses must be degenerate since at $p = 0$ the whole space of the problem is combined-rotationally isotropic, while the terms, odd in the field disappear from $D_{\mu\nu}^{-1ab}$ (2.32) making the problem left and right symmetric, too. Among the dispersion curves (3.57), (3.58) there are ones for which the frequency squared is negative for some portions of the p axis. This happens for (3.57) and (3.58) if $\sigma = 1$ in the interval $0 < p/(g^2A^2/3)^{1/2} < 2$. The imaginary frequency solutions of dispersion equations as lying on the first sheet of the complex plane are tachyons and indicate the instability of external field configuration, the same as it is known for the Abelian external field (see [22], [7], [8] and references therein). A difference with the latter case is in that the imaginary rest mass of the tachyons (3.57), (3.58) is zero. It is notable that the combined-helicity-zero mode (3.59) is not tachyonic. This is quite natural since this mode has the same symmetry as the external field (3.1) itself: they are both invariant under the combined rotation. On the contrary the combined-helicity ± 2 and ± 1 modes (3.57), (3.58) are not invariant and the tachyonic instability of the positive helicity modes when developed will lead the external field configuration out of the class invariant under the combined rotations. We are facing here the instability with respect to phase transition down to a state of lesser symmetry. A program of finding this perhaps stable field configuration to which the initial external field must develop via the tachyonic instability may be thought of along the same lines as that of Refs. [22, 7]. It is also plausible that the initial symmetry is restored and the field configuration done stable by heating [8].

The present problem, however, presents an extra source of instability due to the non-hermiticity of matrices (3.13), (3.15) after going to Minkowskian matrics where κ becomes imaginary. Vanishing of the discriminants of equations (3.30), (3.37) gives rise to branching points in the propagator which cannot be assigned to reasonable absorption processes. This problem was discussed earlier in connection with QED at finite temperature and external magnetic field [18].

We conclude this section by writing the solution of the equation (2.38) for the ghost propagator with (3.1) as the external field. In momentum representation the operator (2.32) has the form

$$D_G^{-1ab} = 2\left(\frac{g^2A^2}{3}\right)^2 (2\delta^{ab} - i\varepsilon^{abc}\beta^c K). \tag{3.60}$$

Its inverse is

$$D_G^{ab} = \frac{1}{4}\left(\frac{g^2A^2}{3}\right)^{-2}\left(\delta^{ab} - \frac{K^2}{4-K^2}\beta^{ac}\beta^{cb} + \frac{2iK\beta^{ab}}{4-K^2}\right). \tag{3.61}$$

None of the three ghost modes have the mass shell.

4. *Generating functional and effective action*

In this section we abandon considering the perturbation expansion within the Furry picture. Instead we shall study some general properties of the vacuum-to-vacuum transition amplitude (2.23) viewed upon as a generating functional of Green functions both with and without external field and of its Legendre transform. The latter functional called effective action will be shown to depend only on gauge-invariant field combinations and calculated within quasi-classical (one-loop) approximation.

The first remark about the generating functional (2.23)

$$Z(J) = \langle 0|S(J)|0\rangle \tag{4.1}$$

is that since it may be looked at as a usual generating functional of the gluon Green functions with the constraint (2.17a) considered as a special gauge-fixing condition it is clear that all the $J = 0$ mass-shell results obtained with the use of these Green functions remain unchanged as being gauge-independent.

Define the effective action $\Gamma(A)$ as Legendre transform of (4.1)

$$\Gamma(A) = \ln Z(J) - \int J^a_\mu A^a_\mu dx^4, \tag{4.2}$$

where J should be substituted as a function of A to be found from the relation

$$\frac{\delta \ln Z}{\delta J^a_\mu(x)} = A^a_\mu(x). \tag{4.3}$$

The remarkable property of A^a_μ (4.3) is that it coincides with the continual average of the field V^a_μ:

$$A^a_\mu(x) = \langle V^a_\mu(x)\rangle$$

$$\equiv \frac{1}{Z(J)} \int V^a_\mu(x) \exp\left\{S_0 + \int J^a_\mu V^a_\mu d^4x\right\} \delta(\nabla^{ac}_\mu J^c_\mu) \operatorname{Det} M_J \prod_x dV^a_\mu(x). \tag{4.4}$$

Here S_0 is the Yang-Mills action (3.48).

To understand this suffices it to note that the functional $Z(J)$ remains unchanged when the current J^a_μ is varied in the integrand of (2.23) in the postexponential factors alone. Really, the variation of J^a_μ in the postexponential factors changes the gauge condition (or nothing at all) and this change (if any) may be compensated by appropriate gauge transformation of the field V^a_μ considered as a change of the variable in the functional integral. The Yang-Mills action (3.48) in (2.23) is gauge invariant and remains unaffected by this change of variables. The term $\int J^a_\mu V^a_\mu d^4x$ *does not change either* since its infinitesimal gauge transformation calculated as the Poisson brackets (2.14) disappears on the space of the functions V^a_μ restricted by the δ-function in (2.23). Therefore the variational derivative (4.3) acts as a matter of fact only on the exponential in the integrand of (2.23) and this results in the statement (4.4). Let us derive this conclusion in a formal way which will also serve to more formal understanding of the J-independence of the normalizing factor N in (2.23) stated in Section 2.

As it is usual in such cases [16] consider the representation of the unity as an integral over the gauge group with the invariant measure $d\omega$

$$1 = \operatorname{Det} M'_J \int \delta(\nabla^{ab}_\mu(\omega) J^b_\mu) d\omega. \tag{4.5}$$

Here $\nabla_\mu^{ab}(\omega)$ is the covariant derivative (2.4) with the gauge-transformed field V^ω instead of V, $\nabla_\mu^{ab}(1) \equiv \nabla_\mu^{ab}$. Supposing that there is one and only one root of the δ-function in (4.5) $\omega = I$ it is straightforward to show that M_J' coincides with the operator (2.20).

$$M_J' \equiv gJ_\mu^{ac}\nabla_\mu^{cb}(x) = g\nabla_\mu^{ac}(x)J_\mu^{cb} \equiv M_J, \tag{4.6}$$

where (2.17a) is meant to be fulfilled and has led to the vanishing of the commutator between ∇ and J in (4.6). We shall omit the prime over M_J' therefore.

Let us take now the infinitesimally varied current

$$J_\mu'^a(x) = J_\mu^a(x) + \delta J_\mu^a(x) \tag{4.7}$$

in place of J in the postexponential factors in (2.23), substitute the unity (4.5) into the integrand (2.23) and make the change of variables $V \to V^{\omega^{-1}}$. Then we obtain using the gauge invariance of S_0, Det M_J, Det $M_{J'}$

$$\frac{1}{N}\iint_\omega \exp\left\{S_0 + \int J_\mu^a(V^{\omega^{-1}})_\mu^a d^4x\right\} \delta(\nabla_\mu^{ac}(\omega^{-1})J_\mu'^c) \text{ Det } M_{J'}$$

$$\cdot \text{ Det } M_J \delta(\nabla_\mu^{ac}J_\mu^c) d\omega \prod_x dV_\mu^a. \tag{4.8}$$

To find the matrix value of the gauge-group transformation ω_0 as the root of the first δ-function in (4.8) whose difference $\tau^a\delta u^a$ from unity is initiated by δJ_μ^a it is necessary to look for it in the infinitesimal form

$$\omega_0^{-1} = I + \tau^a\delta u^a(x), \tag{4.9}$$

$$(V^{\omega_0^{-1}})_\mu^a = V_\mu^a + \nabla_\mu^{ac}\delta u^c, \tag{4.10}$$

where τ^a are the Pauli matrices. Then ω_0^{-1} or δu must be found from the equation

$$\nabla_\mu^{ac}(\omega_0^{-1})J_\mu'^c(x) = -gJ_\mu^{ab}\nabla_\mu^{bd}\delta u^d(x) + \nabla_\mu^{ab}\delta J_\mu^b(x) = 0, \tag{4.11}$$

where we have taken into account that $\nabla_\mu^{ab}J_\mu^b = 0$ as dictated by the second δ-function in (4.8) and omitted the second order terms.

The inhomogeneous equation (4.11) allows one to express δu^a in terms of δJ_μ^a using the operator inverse to (4.6) which exists due to (2.22). This gives the gauge transformation induced by arbitrary small variation of the current. Note that the "transversal" variation of the current $\delta^T J_\mu^a$ subject to the condition $\nabla_\mu^{ab}\delta^T J_\mu^b = 0$, wherein the field V_μ^a represents all the fields obeying the condition $\nabla_\mu^{ab}J_\mu^b = 0$ does not change the postexponential factors nor induce any gauge transformation therefore. The variation $\delta^T J_\mu^a = J_\mu^a\delta\alpha$, where α is a number, is an example of transversal variation. The most general transversal variation has the form of Eq. (2.39) with the bar dropped from over ∇ in (2.39) and in the equation (2.38) for D_G.

The term in the action responsible for the interaction with the current becomes after the substitution of (4.10) and integration by parts equal to its primary value

$$\int J_\mu^a(V^{\omega^{-1}})_\mu^a d^4x = \int J_\mu^a V_\mu^a d^4x + \int J_\mu^a \nabla_\mu^{ab}\delta u^b d^4x = \int J_\mu^a V_\mu^a d^4x. \tag{4.12}$$

Now, the integration over $d\omega$ turns (4.8) into (2.23) again. We see that the arbitrary variation of current (4.7) has not changed the postexponential factors in (2.23). This explains why the differentiation of them has not contributed into (4.4) and also into the normalization factor N.

Consider now the second variational derivative of Z with respect to the current. To this end let us first take the continual average of equation (4.11) with the same weight and the same factor $(Z(J))^{-1}$ as in (4.4) and then solve it for $\delta u^a(x)/\delta J^b_\nu(z)$ using the ghost Green function defined as in (2.38) but this time with the average field (4.3) meant to be used as A in $\overline{\nabla}$.

$$\left\langle \frac{\delta u^a(x)}{\delta J^b_\nu(z)} \right\rangle = -\int D^{ab}_{\mathrm{G}}(x, x')\overline{\nabla}^{bd}_\nu(x')\delta^4(x'-z)d^4x'. \tag{4.13}$$

Now, when differentiating (4.4) with respect to $J^b_\nu(z)$ the contribution of the derivative of the postexponential factors is obtained if one inserts

$$\frac{\delta}{\delta J^b_\nu(z)}(V^{\omega_0^{-1}}(x))^a_\mu = \nabla^{ac}_\mu(x)\frac{\delta u^c(x)}{\delta J^b_\nu(z)} \tag{4.14}$$

into the integrand of (4.8). After the functional integration of (4.14) we may use (4.13). In this way the following equation is obtained which shows how the vacuum-to-vacuum transition amplitude (4.1), (2.23) generates the two-point gluon Green functions which appear as the functional average $\langle VV\rangle$ defined like in (4.4).

$$\frac{\delta^2 \ln Z(J)}{\delta J^b_\nu(z)\delta J^a_\mu(x)} = \frac{\delta A^a_\mu(x)}{\delta J^b_\nu(z)} \equiv \frac{\delta A^b_\nu(z)}{\delta J^a_\mu(x)}$$

$$= \langle V^a_\mu(x)V^b_\nu(z)\rangle - A^a_\mu(x)A^b_\nu(z) - \overline{\nabla}^{ad}_\mu(x)\int D^{dc}_{\mathrm{G}}(x, x')\overline{\nabla}^{cb}_\nu(x')\delta^4(x'-z)d^4x'. \tag{4.15}$$

If Eq. (4.15) is projected onto any transversal vector $\tilde{j}^b_\nu(z)$ or $\tilde{j}^a_\mu(x)$ of the form (2.39) the last term in Eq. (4.15) disappears and it acquires the form characteristic of the case when the gauge condition is independent of the external field or current. This is the direct consequence of the fact mentioned above that the transversal variation of the current does not affect the postexponential factors in (4.4). As it is implied by (2.36) only this projection is important for the total gluon Green function as long as the perturbational Furry picture is concerned. So, to this extent, the last term in (4.15) is not essential.

The subspace supplementing the subspace spanned by $\tilde{j}^a_\mu(x)$ is formed by the vectors J^{ab}_μ. It follows from (2.39) that $\tilde{J}^{ab}_\mu = 0$. Multiplying Eq. (4.15) by $J^{fa}_\mu(x)$ (or $J^{fb}_\nu(z)$) we get the covariantly longitudinal projection of it:

$$\frac{\delta^2 \ln Z(J)}{\delta J^b_\nu(z)\delta J^a_\mu(x)} J^{fa}_\mu(x) = J^{fa}_\mu(x)\,(\langle V^a_\mu(x)V^b_\nu(z)\rangle$$

$$-A^a_\mu(x)A^b_\nu(z)) - \overline{\nabla}^{fb}_\nu(x)\delta^4(x-z). \tag{4.16}$$

In the tree approximation $\langle VV\rangle - AA = 0$ and (4.15) turns into equation

$$J^{ac}_{\mu}(x)\,\frac{\delta A^{c}_{\mu}(x)}{\delta J^{b}_{\nu}(z)} = J^{ac}_{\mu}D^{cb}_{\mu\nu}(x, z) = \overline{\nabla}^{ab}_{\nu}(x)\delta^{4}(x-z), \tag{4.16a}$$

which follows from (3.50).

Equation (4.16) (also (4.16a) within its scope of validity) shows that under the assumption that the term $(\langle VV\rangle - AA)J$ in (4.16) disappears when $J \to 0$ the longitudinal projection of the Green function $\delta A^{a}_{\mu}(x)/\delta J^{b}_{\nu}(z)$ is singular in this limit since the contact term $\overline{\nabla}^{ab}_{\mu}\delta^{4}(x-z)$ in (4.16), (4.16a) remains nonzero. The possibility of appearing singularities at the origin $J = 0$ or $A = 0$ would be a manifestation of the infrared difficulties of the Yang-Mills theory within the present context. They look the price to be paid for the explicit gauge invariance (to be proved somewhat below) of the effective action. (For the case of sourceless field A the gauge invariant effective action is known [6, 22] to contain logarithmic nonanalyticity in A. We shall see below in this section that this is also present in our case.) The origin behaviour is important for studying the spontaneous symmetry breaking. For instance, Eq. (4.4) allows that A do not disappear in the limit $J \to 0$. Indeed, if J tends to zero remaining "transversal", e.g. if one takes a numerical factor α in front of J in (4.4) and makes $\alpha \to 0$ then the postexponential factors in the integrand of (4.4) do not depend on α, the limiting integrand "remembers" the direction of the vector J. The average field A looks therefore nonzero and the spontaneous symmetry breakdown is likely to take place fixed by the arbitrary direction J.

We now concentrate on the proof of the gauge invariance of the functional $\Gamma(A)$ (4.2) and gauge covariance of $Z(J)$ (4.1), (2.23).

Consider the transformation of the current

$$J^{a}_{\mu}\tau^{a} \to (J^{\omega})^{a}_{\mu}\tau^{a} \doteq \omega J^{a}_{\mu}\tau^{a}\omega^{-1} \tag{4.17}$$

performed by the gauge matrix ω. According to (4.4) this causes the general gauge transformation of the average field (4.4)

$$A^{a}_{\mu}\tau^{a} \to (A^{\omega})^{a}_{\mu}\tau^{a} = \omega A^{a}_{\mu}\tau^{a}\omega^{-1} + (\partial_{\mu}\omega)\omega^{-1}. \tag{4.18}$$

To see this let us take J^{ω} in place of J in (4.4) and perform the change of the integration variables $V \to V^{\omega}$. The postexponential factors remain unchanged (this is where the gauge condition (2.17a) works specifically), while the exponential acquires the factor $\exp\{\int \mathrm{Tr}[(\partial_{\mu}(\omega)J^{a}_{\mu}\tau^{a}\omega^{-1}]d^{4}x\}$ which is independent of the integration variable and is cancelled by the analogous factor acquired by Z^{-1} in (4.4). Hence the alteration in (4.4) caused by the transformation (4.17) reduces to the transformation $V^{a}_{\mu}(x) \to (V^{\omega})^{a}_{\mu}$ in the integrand which leads to the gauge transformation (4.18) of the whole integral (4.4) since ω is independent of the integration variable and may be taken outside of the integral while Z^{-1} cancels the integral appearing as a factor in the inhomogeneous part of the gauge transformation. To prove the gauge invariance of $\Gamma(A)$ it remains to show that the right-

-hand side of (4.2) is left invariant under (4.17). This is the case:

$$\Gamma(A^\omega) = \ln Z(J^\omega) - \int (J^\omega)^a_\mu (A^\omega)^a_\mu dx^4$$
$$= \ln \{Z(J) \exp [\int \mathrm{Tr}\, ((\partial_\mu \omega) J^a_\mu \tau^a \omega^{-1}) d^4x]\} - \int (J^\omega)^a_\mu (A^\omega)^a_\mu dx^4$$
$$= \ln Z(J) - \int J^a_\mu A^a_\mu dx^4. \tag{4.19}$$

Simultaneously we have seen that under the transformation (4.17) the functional $Z(J)$ is not invariant but is an eigenfunctional:

$$Z(J^\omega) = \exp \{ \int \mathrm{Tr}\, [(\partial_\mu \omega) J^a_\mu \tau^a \omega^{-1}] d^4x\} Z(J). \tag{4.20}$$

The statement proved above about the gauge invariance of the effective action Γ implies that it depends upon gauge invariant combinations of the field A like the strength tensor squared and its derivatives. This may serve as a source of the Ward identities, one of them is just

$$\bar{\nabla}^{ab}_\mu \frac{\delta \Gamma}{\delta A^b_\mu} = 0, \tag{4.21}$$

where the bar means that the classical field A is taken inside the covariant derivative. If taken together with the usual relation $\delta\Gamma/\delta A^a_\mu = -J^a_\mu$ which follows from (4.2), (4.3) the equation (4.21) implies also that

$$\bar{\nabla}^{ab}_\mu J^b_\mu = 0. \tag{4.22}$$

We now proceed with calculation of the effective action $\Gamma(A)$ (4.2) within the one--loop approximation. This reduces, as usually, to calculating $Z(J)$ within the same approximation and to substituting J^a_μ as expressed in terms of its classical field subject to (2.21) into it. The tree plus one-loop expression Z_1 for $Z(J)$ (4.1) may be obtained by Gaussian integration in (2.29) with $\mathscr{L}_{\text{int}} = j = \eta = \bar{\eta} = 0$ and the normalizing factor Z^{-1} replaced by N^{-1} in it.

$$Z_1(J) = N_0^{-1} \exp \left\{ \int d^4x (-\tfrac{1}{4} \bar{G}^a_{\mu\nu} \bar{G}^a_{\mu\nu} + J^a_\mu A^a_\mu) \right\} \frac{|\mathrm{Det}\, D_G^{-1}|^{1/2}}{|\mathrm{Det}\, D^{-1}|^{1/2}}. \tag{4.23}$$

Note, that the ghost determinant appears to the power 1/2 unlike what the determinant of M did in (2.23) since the λ-integration has produced $|\mathrm{Det}\, D_G^{-1}|^{-1/2}$. The normalizing factor N_0^{-1} deserves a special discussion. The normalizing factor defined in (2.23) as independent of the current J cannot be expanded in powers of the Planck constant at least as far as the integration over V is performed first and the stationary phase point $V^a_\mu = \lambda^a = 0$ is dealt with, since the operator $\dfrac{\delta^2 S_0}{\delta V^a_\mu \delta V^b_\nu}$ is degenerate in this point. Therefore, for the present purposes of the loop-expansion another definition of N^{-1} in (2.23) should be adopted: N is the $J \to 0$ limit of the same integral as in (2.23). Beyond the stationary phase expansion the new and old definitions are equivalent. With the new definition of the normalizing factor its tree plus one-loop approximation N_0 provides that (4.23) is normalized as $Z_1(0) = 1$. Now the substitution of (4.23) into (4.2) gives for the tree S_0 plus one-loop S_1 effective action

$$\Gamma(A) = S_0(A)+\ln\left(\frac{1}{N_0}\frac{|\text{Det } D_G^{-1}|^{1/2}}{|\text{Det } D^{-1}|^{1/2}}\right) = S_0(A)+S_1(A), \tag{4.24}$$

where the normalization N_0^{-1} implies that $S_1(0) = 0$.

Our task now is to calculate the logarithm in the r-h side of (4.24) for A given as (3.1). After calculating the determinants of the matrices (3.13)–(3.15) and of the ghost matrix (3.60) which is equal to

$$16\left(\frac{g^2A^2}{3}\right)^6 (4-K^2) \tag{4.25}$$

we find for $S_1(A)$ the expression in which the contributions of all the six modes having the mass shell are clearly distinguishable while those of the other modes have been cancelled out not without help of the ghost determinant (4.25)

$$S_1(A) = -\frac{VT}{2(2\pi)^4}\int \ln\left\{\frac{1}{N_0}\prod_{\sigma_1,\sigma_2,\sigma_3}\left[\left(p^2+\frac{2va^2}{v-2\sigma_1 a}\right)\right.\right.$$
$$\left.\left.\times(p^2+3a^2+\sigma_2 a\sqrt{4v^2+9a^2})(p^2-2\sigma_3 va)\right]\right\} d^4p, \tag{4.26}$$

$\sigma_1,\ \sigma_2,\ \sigma_3 = \pm 1$.

The integration is performed over the Euclidean space-time, VT is the four-volume and the notations are used

$$a = (g^2A^2/3)^{1/2}, \quad v = aK, \tag{4.27}$$

where K is given by (3.7) or (3.8). In the special frame v is the length of the spatial momentum $v = (p_i^2)^{1/2}$. The integral (4.26) is calculated by differentiating over a^2, using the dimensional regularization and then integrating back over a^2 within the limits $(0, a^2)$. Near the value of the dimension $n = 4$ the effective Lagrangian defined as $\mathscr{L}_1 VT = S_1$ has the structure

$$\mathscr{L}_1 = Y\left(\frac{g^2A^2}{3}\right)^2 \ln\left(\frac{g^2A^2}{3}\right)+\frac{R(g^2A^2)^2}{n-4}+X(g^2A^2)^2. \tag{4.28}$$

The diverging part $(n-4)^{-1}$ is absorbed through the renormalization of the charge $g_r = gZ^{-1}$ and the field $A_r = ZA$ into the tree Lagrangian $\mathscr{L}_0$ which for our constant field (3.1) is

$$\mathscr{L}_0 = -\tfrac{1}{6}\, g^2A^4. \tag{4.29}$$

The renormalized charge g_r is fixed in the arbitrary normalization point $A_r = A_0$ by the condition

$$\left.\frac{d^4\,\text{Re}\,\mathscr{L}}{dA^4}\right|_{A_0} = -4g_r^2 \tag{4.30}$$

while the condition

$$\left.\frac{d^2\mathscr{L}}{dA^2}\right|_{A=0} = 0 \tag{4.31}$$

is fulfilled for (4.28) owing to the explicit gauge invariance of the dimensional regularization used (a term A^2 could not be expressed in terms of the field strength tensor $G^a_{\mu\nu}$ without using the nonanalytical operation of taking the square root: $g(A^a_\mu)^2 = \frac{3}{2}\sqrt{(G^a_{\mu\nu})^2}$). The condition (4.30) makes the calculation of the Re X and the real constant R unnecessary unless we are interested in the renormalization factor Z. The constant Y is calculated to be

$$Y = \frac{28}{32\pi^2} - \frac{5}{32\pi^2} - \frac{5}{32\pi^2} = \frac{9}{16\pi^2}. \tag{4.32}$$

Here the sequence of the terms corresponds to the subsequent contributions of the bracketed factors in (4.26) from left to right, in other words, the two combined-helicity ± 1 massless modes taken together contribute as 28, the massive and massless modes of combined helicity 0 contribute as -5 and so do the two massless combined helicity ± 2 modes. Finally:

$$\operatorname{Re}\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 = -\tfrac{1}{6}\, g_r^2 A_r^4 + \frac{g_r^4 A_r^4}{16\pi^2} \ln \frac{A_r^2}{A_0^2} \tag{4.33}$$

or, in terms of the renormalized field strength $(G^a_{\mu\nu})^2 = 2(B^a_i)^2$

$$\operatorname{Re}\mathscr{L} = -\tfrac{1}{4}(G^a_{\mu\nu})^2 \left(1 + \frac{25 g_r^2}{16\pi^2}\right) + \frac{3 g_r^2}{64\pi^2}(G^a_{\mu\nu})^2 \ln \frac{(G^a_{\mu\nu})^2}{(G^a_{0\mu\nu})^2}. \tag{4.34}$$

The imaginary part of the constant X is finite and may be calculated either within the dimensional regularization scheme or directly as a contribution of the tachyonic modes integrated over v within the interval $(0, 2a)$ and over p_4 within the regions of negativity of the arguments of the first and third logarithms in (4.25) ($\sigma_1 = \sigma_2 = 1$)

$$\operatorname{Im}\mathscr{L}_1 = -\frac{g_r^4 A_r^4}{(2\pi)^2}\int_0^2 K^2 dK \left\{\left(\frac{2K}{2-K} - K^2\right)^{1/2} + (2K - K^2)^{1/2}\right\}$$

$$\cong -\frac{(10.23 + 1.92)}{(2\pi)^2}\, g_r^4 A_r^4 = -\tfrac{3}{2}\frac{g_r^2}{(2\pi)^2}(G^a_{\mu\nu})^2\,(12.15). \tag{4.35}$$

The first term in (4.35) represents the contribution of the combined helicity 1 tachyonic mode and the second of the combined helicity 2.

The convex properties of the curve Re $\mathscr{L}(G^2)$ (4.34) are like those of the Heisenberg-Euler effective Lagrangian in quantum electrodynamics and not like in case [6, 23] of Abelian classical field in non-Abelian quantum field theory.

The imaginary part (4.35) cannot be removed by renormalization unless we admit the complex bare coupling constant and determines the probability (per unit time and volume) of the decay of the external field (3.1) via developing the tachyons of positive combined helicity.

5. Conclusion

In this paper we weie dealing mainly with the case when the external current and its field are nonzero. It would be also of interest to learn how to use the above procedure of uniquely prescribed gauge to build gauge-unambiguous perturbation theory for the vacuum case $J \to 0$. In doing so we shall meet the singularities in propagators at $J \to 0$, which property we have handled successfully in calculating the effective action (4.24).

Also of interest it would be to modify the Furry picture developed above to cover the unstable classical fields to which class our example (3.1) just belongs. The point is that in the case of unstable fields the condition $Z(J, 0, 0) = 1$ traditionally used to normalize the generating functional of the Furry diagrams does not go since the probability for the external-field-containing vacuum to remain the vacuum is less than unity. For this case the procedure [24] developed in quantum electrodynamics with external field creating electron-positron pairs should be applied.

The authors are deeply indebted to E. S. Fradkin and I. A. Batalin for very helpful and encouraging interest in the work. The authors also acknowledge having profited from discussions with R. E. Kallosh, I. V. Tyutin, M. A. Vasiliev, and B. L. Voronov.

APPENDIX

Here we present a nondiagonal expression for the gluon propagator different from (3.42) obtained by inversing the matrix blocks (3.13), (3.14), (3.15). It is independent of the vectors $n_\mu^\pm$, $\eta_\pm^a$ which depend on an arbitrary choice and does not include solving the cubic and quartic equations.

To exclude the arbitrary vectors we used the relations (to avoid misunderstanding we must stress that they cannot be excluded from the eigenvectors (3.19)–(3.22), only from the tensors)

$$v_\mu^a = n_\mu^+ \eta_-^a = \tfrac{1}{2}(\tfrac{1}{3}A^2)^{-1/2}(\delta^{ab} - i\beta^{ab})A_\mu^b - l_\mu \beta^a/2, \tag{A.1}$$

$$\tilde{v}_\mu^a = n_\mu^+ \eta_-^a = \tfrac{1}{2}(\tfrac{1}{3}A^2)^{-1/2}(\delta^{ab} + i\beta^{ab})A_\mu^b - l_\mu \beta^a/2, \tag{A.2}$$

$$\omega_{\mu\nu} = \tilde{\omega}_{\nu\mu} = n_\mu^- n_\nu^+ = \tfrac{1}{2}(\tfrac{1}{3}A^2)^{-1}A_\mu^a(\delta^{ab} - i\beta^{ab})A_\nu^b - \tfrac{1}{2}l_\mu l_\nu, \tag{A.3}$$

$$\gamma^{ab} = \tilde{\gamma}^{ba} = \eta_+^a \eta_-^b = \tfrac{1}{2}(\delta^{ab} + i\beta^{ab} - \beta^a\beta^b). \tag{A.4}$$

The solution of equation (2.37) for the propagator is given in the momentum space as

$$D_{\mu\nu}^{ab} = \sum_{l=1}^{5} D_{\mu\nu}^{ab}(l), \tag{A.5}$$

where l labels the matrices obtained by inversion of the blocks as follows: $l = 1, 2$ denotes the inversion of $B^{(2)}(\pm 1)$ (3.14), $l = 3, 4$ denotes the inversion of $B_{ij}^{(1)}(\pm 1)$ (3.13) and $l = 5$ corresponds to the inversion of $B_{ij}^{(3)}$ (3.15), respectively.

$$\begin{aligned} D_{\mu\nu}^{ab}(l) = {} & (a_1^l \beta^a \beta^b + a_2^l \gamma^{ab} + a_3^l \tilde{\gamma}^{ab})\tilde{\omega}_{\mu\nu} + (a_4^l \beta^a \beta^b + a_5^l \tilde{\gamma}^{ab} + a_6^l \gamma^{ab})\omega_{\mu\nu} \\ & + (a_7^l \gamma^{ab} + a_8^l \tilde{\gamma}^{ab} + a_9^l \beta^a \beta^b) l_\mu l_\nu + (a_{10}^l \gamma^{ab} + a_{11}^l \tilde{\gamma}^{ab} + a_{12}^l \beta^a \beta^b) u_\mu u_\nu \\ & - (a_{13}^l \beta^a \beta^b + a_{14}^l \gamma^{ab} + a_{15}^l \tilde{\gamma}^{ab})(u_\mu l_\nu + l_\mu u_\nu) \\ & - a_{16}^l (v_\mu^a v_\nu^b + \tilde{v}_\mu^a \tilde{v}_\nu^b) - a_{17}^l (l_\mu \beta^b \tilde{v}_\nu^a + l_\nu \beta^a v_\mu^b) \\ & + a_{18}^l (u_\mu \beta^b \tilde{v}_\nu^a + u_\nu \beta^a v_\mu^b) - a_{19}^l (l_\mu \beta^b v_\nu^a + l_\nu \beta^a \tilde{v}_\mu^b) \\ & + a_{20}^l (u_\mu \beta^b v_\nu^a + u_\nu \beta^a \tilde{v}_\mu^b) + a_{21}^l (v_\mu^a l_\nu \beta^b + l_\mu \beta^a \tilde{v}_\nu^b) \\ & - a_{22}^l (v_\mu^a u_\nu \beta^b + u_\mu \beta^a \tilde{v}_\nu^b) - a_{23}^l (\tilde{v}_\mu^a l_\nu \beta^b + l_\mu \beta^a v_\nu^b) - a_{24}^l (\tilde{v}_\mu^a u_\nu \beta^b + u_\mu \beta^a v_\nu^b). \end{aligned} \tag{A.6}$$

Now we list the sets of 24 coefficients a_n^l for each $l = 1, 2, 3, 4, 5$

a. $l = 1$:

$$a_2^1 = (-q^2 + 2K)^{-1} \left(\frac{g^2 A^2}{3}\right)^{-1}, \quad a_n^1 = 0 \quad \text{if} \quad n \neq 2,$$

$l = 2$:

$$a_5^2 = -(q^2 + 2K)^{-1} \left(\frac{g^2 A^2}{3}\right)^{-1}, \quad a_n^2 = 0 \quad \text{if} \quad n \neq 5. \tag{A.7}$$

b. $l = 3$:

$$\begin{aligned} a_1^3 &= [q^2 + 2(1-K)]/\Delta_3, \\ a_7^3 &= [(1-K)^2 (q^2+2) + 2K]/\Delta_3, \\ a_{14}^3 &= -\kappa[(K-1)q^2 + 2K]/\Delta_3, \\ a_{17}^3 &= [(1-K)q^2 - 2]/\Delta_3, \\ a_{18}^3 &= \kappa q^2/\Delta_3, \\ a_{10}^3 &= [(q^2+2)^2 - (K^2+2)q^2 + 2) - 2K]/\Delta_3, \\ a_n^3 &= 0 \quad \text{if} \quad n \neq 1, 7, 10, 14, 17, 18, \quad \Delta_3 = [(K-2)q^2 + 4K]\left(\frac{g^2 A^2}{3}\right), \end{aligned} \tag{A.8}$$

$l = 4$:

$$\begin{aligned} a_4^4 &= [q^2 + 2(1+K)]/\Delta_4, \\ a_8^4 &= [(1+K)^2 (q^2+2) - 3K]/\Delta_4, \end{aligned}$$

$$a^4_{15} = -\kappa[(1+K)q^2+2K]/\Delta_4,$$

$$a^4_{19} = [(1+K)q^2-2]/\Delta_4,$$

$$a^4_{20} = -\kappa q^2/\Delta_4,$$

$$a^4_{11} = [(q^2+2)^2-(K^2+2)(q^2+2)+2K]/\Delta_4,$$

$$a^4_n = 0 \quad \text{if} \quad n \neq 4, 8, 11, 15, 19, 20, \quad \Delta_4 = -[(K+2)q^2+4K]\left(\frac{g^2A^2}{3}\right),$$

c. $l = 5$:

$$\begin{aligned}
a^5_3 &= [-\lambda^2-4\lambda+4(K^2+2)]/\Delta_5,\\
a^5_6 &= [-\lambda^2-4\lambda+4(K^2+2)]/\Delta_5,\\
a^5_9 &= [-K^2\lambda^2-4(1+K^2)\lambda-4(1-K^2)(K^2+2)]/\Delta_5,\\
a^5_{12} &= [-\lambda^3+(K^2-2)\lambda^2+8(K^2+1)\lambda-4K^2(K^2+3)]/\Delta_5,\\
a^5_{13} &= K\kappa[\lambda^2+4\lambda-4(K^2+1)]/\Delta_5,\\
a^5_{16} &= [-\lambda^2+4(K^2+2)]/\Delta_5,\\
a^5_{21} &= [K\lambda^2+2K\lambda-4(K-1)(K^2+2)]/\Delta_5,\\
a^5_{22} &= -\kappa[\lambda^2+2\lambda-4(K^2-K+2)]/\Delta_5,\\
a^5_{23} &= [K\lambda^2+2K\lambda-4(K+1)(K^2+2)]/\Delta_5,\\
a^5_{24} &= -\kappa[\lambda^2+2\lambda-4(K^2+K+2)]/\Delta_5,\\
a^5_n &= 0 \quad \text{if} \quad n \neq 3, 6, 9, 12, 13, 16, 21, 22, 23, 24, \quad \lambda = q^2+2;\\
\Delta_5 &= 4[\lambda^2+2\lambda-4(K^2+2)]\left(\frac{g^2A^2}{3}\right).
\end{aligned} \tag{A.9}$$

REFERENCES

[1] B. S. DeWitt, *Phys. Rev.* **162**, 1195, 1239 (1967).
[2] R. Kallosh, *Nucl. Phys.* **B78**, 293 (1974).
[3] R. Fukuda, T. Kugo, Preprint RIFP-237 (1975).
[4] I. A. Batalin, G. K. Savvidy, preprint EFI-365(23)-79.
[5] Only SU(2) as the gauge group is considered in the present paper.
[6] I. A. Batalin, S. G. Matinyan, G. K. Savvidy, *Yad. Fiz. (Sov. J. Nucl. Phys.)* **26**, 407 (1977).
[7] J. Ambjorn, N. K. Nielsen, P. Olesen, *Nucl. Phys.* **B152**, 75 (1979).
[8] A. Cabo, O. K. Kalashnikov, A. E. Shabad, *Nucl. Phys.* **B185**, 473 (1981).
[9] G. K. Savvidy, Preprint EFI-350(8)-79.
[10] L. S. Brown, W. I. Weissberger, *Nucl. Phys.* **B157**, 285 (1979).

[11] T. Saito, K. Shigemoto, *Prog. Theor. Phys.* **63**, 256, 993 (1980).
[12] P. A. M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva University, New York 1964.
[13] The dependence of the gauge condition on the current prescribed in the unique way meets the above philosophy that "the starting position of perturbation expansion is prepared in accord with the degeneracy-removing perturbation".
[14] A. E. Shabad, *Ann. Phys.* (N.Y.) **90**, 166 (1975).
[15] J. Kiskis, *Phys. Rev.* **D21**, 1074 (1980).
[16] For facts already well known in the theory of gauge fields we are referring to the book by A. A. Slavnov and L. D. Faddeev, *Vvedenie v kvantovuyu teoriyu kalibrovochnikh poley*, Nauka, Moscow 1978; *Gauge Fields. Introduction to Quantum Theory*, Adison-Wesley, New York 1980.
[17] E. S. Fradkin, T. E. Fradkina, *Phys. Lett.* **72B**, 343 (1978).
[18] H. Pérez Rojas, A. E. Shabad, *Ann. Phys.* (N.Y.) **121**, 432 (1979).
[19] E. S. Fradkin, in *Quantum Field Theory and Statistics*, Proc. of P.N. Lebedev Phys. Inst. v. 28, p. 7, Nauka, Moscow 1965 (Translated into English: Consultants Bureau, Plenum, New York 1967).
[20] A. L. Dubničkova, G. A. Efimov, M. A. Ivanov, *Fortschr. Phys.* **27**, 403 (1979).
[21] G. V. Efimov, *Ann. Phys.* (N.Y.) **71**, 466 (1972); G. V. Efimov, *Nonlocal Interactions of Quantized Fields*, Nauka, Moscow 1977.
[22] N. K. Nielsen, P. Olesen, *Nucl. Phys.* **B144**, 376 (1978); *Phys. Lett.* **79B**, 304 (1978).
[23] G. K. Savvidy, *Phys. Lett.* **71B**, 133 (1979).
[24] E. S. Fradkin, D. M. Gitman, preprint KFKI-1979-83, Budapest.