POSITION-DEPENDENT FRICTION IN QUANTUM MECHANICS

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The quantum description of motion of a particle subjected to position-dependent frictional forces is presented. The approximation applied to solve the problem is especially good in the field of heavy-ion physics. The two cases are taken into account: a motion without external forces and a motion in the harmonic oscillator potential. As an example, a frictional barrier penetration is considered.

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1. *Introduction*

A study of frictional phenomena in quantum mechanics has its roots in investigation of Langevin systems. A description of a motion of a Brownian particle requires the introduction of a frictional force, the term quite familiar in the field of the classical dynamics but somewhat embarrassing in quantum mechanics. Since the problem of a quantum description of a system subjected to dissipative as well as external and stochastic forces has no unambiguous solution, a number of methods developed in the last four decades is quite handsome (for review see [1–3]). In the Langevin equation describing a Brownian particle motion, a friction force is proportional, with a constant coefficient, to the velocity and in most of these methods no other possibilities have been taken into account. As an exception, Remaud and Hernandez [4] consider a time-dependent friction coefficient.

In many physical problems, concepts concerning a constant friction coefficient do not suffice. In particular, friction is the term widely used in descriptions (mainly in the classical manner) of heavy-ion collisions. Frictional forces depend in this case on position and their range is comparable with the nuclear radius. Hahn and Hasse [5] investigated quantum-mechanically heavy-ion collisions using the realistic frictional forces (with the position-dependent strength) but that work was purely numerical and required an immense computational effort.

The aim of this paper is to show what is to be expected if one allows a friction coefficient to vary with position. All considerations base on an approximation proved to be correct

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for heavy-ion collision problems. General formulae for the wave packet time-evolution, without external forces, on the basis of Kanai's model are presented in Sect. 2. In Sect. 3 they are applied to the case of viscous barrier penetration. Sect. 4 is devoted to the investigation of the wave packet motion in the field of the damped harmonic oscillator from Kanai's, as well as the Kostin model's point of view.

2. *Free particle in a viscous medium*

Let us assume that the classical dissipative force acting on a particle moving in an arbitrary external field $V(x)$ is proportional to the velocity of this particle: $F = -mf(t)v$, where m is the constant particle mass. $f(t)$ can be an arbitrary function of time. We restrict our considerations to one dimension. Following Kanai [6] (also [7, 8]), quantization of this system leads to the Schrödinger equation

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi(t) + V(x)\varphi^{-1}(t) \right] \psi(x,t), \tag{1}$$

where $\varphi(t)$ is given by

$$\varphi(t) = \exp\left(-\int f(t)dt\right). \tag{2}$$

The wave packet solution for the force-free motion ($V(x) = 0$) with the initial condition

$$\psi(x,0) = A \exp\left(-\frac{x^2}{2a^2} + ik_0 x\right), \tag{3}$$

where a is the initial width of the packet, $\hbar k_0$ — initial momentum and A — the normalization constant, is easy to obtain by the method of integral transformation. It has the form:

$$\psi(x,t) = \frac{A}{\sqrt{1 + \frac{\hbar\tau(t)}{ma^2} i}} \exp\left[- \frac{x^2 - 2ia^2 k_0 x + i \frac{\hbar a^2 k_0^2}{m} \tau(t)}{2a^2\left(1 + i \frac{\hbar}{ma^2} \tau(t)\right)} \right] \tag{4}$$

The function $\tau(t)$ is given by

$$\dot{\tau}(t) = \varphi(t). \tag{5}$$

Since the density distribution is in the form

$$|\psi(x,t)|^2 = \frac{|A|^2}{\sqrt{1 + \left(\frac{\hbar\tau(t)}{ma^2}\right)^2}} \exp\left[- \frac{\left(x - \frac{k_0\hbar}{m}\tau(t)\right)^2}{a^2\left[1 + \left(\frac{\hbar\tau(t)}{ma^2}\right)^2\right]} \right], \tag{6}$$

the centroid of the packet, moving like a classical particle, is given by

$$\langle x\rangle = \frac{k_0\hbar}{m}\dot{\tau}(t) \equiv \eta\tau(t). \tag{7}$$

In this way, we have solved the problem of a free particle moving in a viscous medium, determined by a time-dependent friction coefficient $f(t)$. However, our task is to deal with position-dependent form-factors $f(x)$. For this purpose we make the approximation

$$f(x) \to f(\langle x\rangle). \tag{8}$$

Now, the friction coefficient is merely a function of time. The approximation (8) means that the whole packet is subjected to the same value of frictional coefficient, determined by the classical motion of the centroid of the packet.

It is not apparent that the approximation which seems to be in contradiction with the spirit of quantum mechanics, is suitable for dealing with physical problems. It is possible that in some cases it may appear to be too rough. However, at least for description of heavy--ion collision it is excellent and well founded. Hahn and Hasse [5] have made a numerical analysis of heavy-ion collisions using a frictional form-factor in the approximation (8). Results (quantum trajectories, deflection functions and energy loss) have been compared with the more realistic case, in which the exact formula have been applied. An agreement turned out to be very satisfactory.

Relations (2), (5), (7) and (8) imply that the function $\tau(t)$ fulfils the equation

$$\ddot{\tau} + f(\eta\tau(t))\dot{\tau} = 0. \tag{9}$$

By these means, our problem resolves itself to the solution of Eq. (9). In the next paragraphs the exemplary solutions for typical physical problems are presented. It is easy to realize that Eq. (9) is identical with the classical equation of motion of a damped free particle. It is the immediate consequence of the Ehrenfest theorem.

Up to now, we discussed Kanai's method only. In Sect. 4 a few remarks will be made about the motion of a free particle from the Kostin model's point of view.

3. Examples and discussion

In this paragraph we are dealing with the physically most interesting case of friction confined to a finite region. At first, we take $f(x)$ in the form:

$$f(x) = \gamma[\theta(x) - \theta(x - x_0)], \quad x_0 \geqslant 0, \quad \gamma = \text{const}, \tag{10}$$

where $\theta(x)$ is Heaviside's function. In other words, friction is switched on at $x = 0$ and switched off at $x = x_0$. Our problem is to find such T that

$$f(t) = \gamma[\theta(t) - \theta(t - T(x_0))]. \tag{11}$$

From (2) gnd (5) we obtain

$$\tau(t) = \begin{cases} t, & t \leqslant 0, \\ \frac{1}{\gamma}(1-e^{-\gamma t}), & 0 < t < T, \\ e^{-\gamma t}\left(t-T-\frac{1}{\gamma}\right)+\frac{1}{\gamma}, & t \geqslant T, \end{cases} \tag{12}$$

where the integration constants have been chosen to satisfy the initial conditions:

$$\tau(0) = 0, \quad \dot{\tau}(0) = 1. \tag{13}$$

Since $x_0 = \langle x \rangle$ $(t = T) = \eta\tau(T)$, it is easy to calculate T:

$$T = \frac{x_0}{\eta} - \frac{1}{\gamma}\ln\left(1-\frac{\gamma x_0}{\eta}\right) \tag{14}$$

and to obtain

$$\tau(t) = \left[1-\frac{\gamma x_0}{\eta}\right]t+\text{const} \tag{15}$$

for $t > T$.

The barrier penetration by the packet results in slowing down of the centroid motion (in comparison to the friction-free case) and slower spreading of the width of the packet. However, if friction reaches the critical value ($\gamma x_0 = \eta$), the packet never leaves the viscous region in finite time. The packet eventually stops and ceases to spread [9]. When friction approaches the critical value, the uncertainty principle is violated. This difficulty has been already widely discussed [10–12].

Let us consider now a barrier in the form:

$$f(x) = \gamma e^{-\alpha|x|}. \tag{16}$$

Solution of the equation

$$\ddot{\tau}+\gamma e^{-\alpha\eta|\tau|}\dot{\tau} = 0, \tag{17}$$

with the initial conditions (13), on the positive half-axis is:

$$t = \frac{\alpha\eta}{\alpha\eta-\gamma}\left[\tau+\frac{1}{\alpha\eta}\ln\left(1+\frac{\gamma}{\alpha\eta}(e^{-\alpha\eta\tau}-1)\right)\right]. \tag{18}$$

For $t \gg 0$:

$$\tau = \left(1-\frac{\gamma}{\alpha\eta}\right)t+\text{const}. \tag{19}$$

Thus, for large time, the result is similar to (15). The final energy is

$$\langle E\rangle = \frac{\hbar^2 k_0^2}{2m}(\dot{\tau})^2 \underset{t\to\infty}{\rightarrow} \frac{\hbar^2 k_0^2}{2m}\left(1-\frac{\gamma}{\alpha\eta}\right)^2 \tag{20}$$

where $\hbar k_0$ stands for the average momentum at $t = 0$.

4. *The damped harmonic oscillator*

In the Kanai's method, the Hamiltonian for the damped harmonic oscillator has the form:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\varphi(t)+\frac{m}{2}\Omega^2 x^2 \varphi^{-1}(t), \tag{21}$$

where Ω is the classical frequency and $\varphi(t)$ is given by (2). We shall seek the centroid $\langle x\rangle$ and the square width χ of the distribution

$$|\psi|^2 = (2\pi\chi)^{-1/2}\exp[-(x-\langle x\rangle)^2/2\chi]. \tag{22}$$

Due to the approximation (8), from the Ehrenfest theorem we obtain:

$$\langle\ddot{x}\rangle + f(\langle x\rangle)\langle\dot{x}\rangle + \Omega^2\langle x\rangle = 0, \tag{23}$$

as an analogue to the Eq. (9). The moments of the packet can be calculated [13] from the equation for the time-evolution of an arbitrary operator A:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \frac{i}{\hbar}[H, A]. \tag{24}$$

The function $\chi(t)$ is the second moment. Expression (24) leads to the equation (obtained by Remaud and Hernandez [4] for the time-dependent friction):

$$\ddot{u} + f(\langle x\rangle)\dot{u} + \Omega^2 u = \frac{1}{m^2}\exp[-2\int f(\langle x\rangle)dt]u^{-3}, \tag{25}$$

where

$$u(t) = \sqrt{\frac{2\chi}{\hbar}}.$$

Equations (23) and (25) determine the density (22) completely. However, in general, it is not easy to solve Eq. (25) analytically. We shall do this for the case of rectangular barrier (10).

Eq. (23) for the barrier (10), with the initial conditions

$$\langle x\rangle(t=0) = 0$$

$$\langle\dot{x}\rangle(t=0) = p_0/m, \tag{26}$$

has the following solution

$$\langle x\rangle = \begin{cases} \dfrac{p_0}{m\Omega}\sin \Omega t, & t \leqslant 0, \\ \dfrac{p_0}{m\omega}\, e^{-\gamma t/2}\sin \omega t, & 0 < t < T, \\ C_1 \cos \Omega t + C_2 \sin \Omega t & t \geqslant T. \end{cases} \tag{27}$$

where

$$\omega^2 = \Omega^2 - \gamma^2/4$$

and

$$C_1 = \frac{p_0}{m\omega}\, e^{-\gamma T/2}\left(\sin \omega T \cos \Omega T - \frac{\omega}{\Omega}\sin \Omega T \cos \omega T + \frac{\gamma}{2\Omega}\sin \omega T \sin \Omega T\right)$$

$$C_2 = \frac{p_0}{m\omega}\, e^{-\gamma T/2}\left(\sin \omega T \sin \Omega T - \frac{\gamma}{2\Omega}\sin \omega T \cos \Omega T + \frac{\omega}{\Omega}\cos \omega T \cos \Omega T\right).$$

T is given by

$$x_0 = \frac{p_0}{m\omega}\, e^{-\gamma T/2}\sin \omega T.$$

The function $\langle x\rangle (t)$ and its first derivative are continuous for all t.

Eq. (25) has been solved by Remaud and Hernandez [4] for the constant friction coefficient. In our case we have

$$\ddot{u} + \Omega^2 u = \frac{1}{m^2}\, u^{-3}, \qquad t < 0, t > T \tag{28a}$$

$$\ddot{u} + \gamma \dot{u} + \Omega^2 u = \frac{1}{m^2}\, e^{-2\gamma t} u^{-3}, \qquad 0 < t < T. \tag{28b}$$

Assuming that for $t < 0$ we can restrict ourselves to a constant solution, we have:

$$u = \frac{1}{\sqrt{m\Omega}}, \qquad t \leqslant 0. \tag{29}$$

Defining the new function $W(t)$ by

$$u(t) = \frac{1}{\sqrt{m}}\, W(t) \exp\left(-\tfrac{1}{2}\gamma t\right) \tag{30}$$

one can transform (28b) to the form:

$$\ddot{W} + \omega^2 W = W^{-3}. \tag{31}$$

The general solution is:

$$W^2 = \omega^{-1}[(1+A^2+B^2)^{1/2}+A\cos 2\omega t+B\sin 2\omega t]. \tag{32}$$

Constants A and B can be determined from the continuity conditions at $t = 0$. Finally, we have

$$u(t) = \sqrt{\frac{\Omega}{m}}\,\omega^{-1}\sqrt{1-\frac{\gamma}{2\Omega}\sin(2\omega t+\theta_1)}\exp(-\tfrac{1}{2}\gamma t), \quad 0 < t < T, \tag{33}$$

where

$$\theta_1 = \operatorname{arctg}\left(-\frac{\gamma}{2\omega}\right).$$

Thus, switching on the dissipative mechanism generates oscillations. They are damped and the width of the packet tends to zero if friction persists.

In the third region ($t > T$), the solution is more complicated:

$$u(t) = \frac{1}{\sqrt{m\Omega}}\left[\frac{A_1^2+B_1^2+1}{2B_1}+\frac{B^2}{\sqrt{A^2+B^2}}\sin(2\Omega t+\theta_2)\right]^{1/2}, \quad t > T, \tag{34}$$

where

$$A = -A_1\sin 2\Omega T-\frac{\cos 2\Omega T}{2B_1}(A_1^2-B_1^2+1),$$

$$B = A_1\cos 2\Omega T-\frac{\sin 2\Omega T}{2B_1}(A_1^2-B_1^2+1),$$

$$A_1 = -\frac{\gamma}{2\omega^2}\left(\omega\cos(2\omega T+\theta_1)-\frac{\gamma}{2}\sin(2\omega T+\theta_1)+\Omega\right),$$

$$B_1 = \frac{\Omega^2}{\omega^2}\left[1-\frac{\gamma}{2\Omega}\sin(2\omega T+\theta_1)\right]e^{-\gamma T},$$

$$\theta_2 = \operatorname{arctg}(A/B).$$

It must be emphasized that after leaving the frictional region by the packet, oscillations can never die down, regardless of the size of this region and the strength of the frictional force. The larger the value of γT the larger the amplitude of the final oscillations.

The question arises whether other models predict the similar phenomenon. In Hasse's method (e.g. [14, 13]), with the constant friction coefficient, oscillations are present but they seem to behave unphysically, as it has been reported by Remaud and Hernandez [4]. We shall discuss now Kostin's approach [15, 16].

Kostin's method takes friction into account by including an additional, nonlinear term into the Hamiltonian:

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \tfrac{1}{2}m\Omega^2x^2 - \frac{i\hbar}{2}f(\langle x\rangle)\left(\ln\frac{\psi}{\psi^*} - \left\langle \ln\frac{\psi}{\psi^*}\right\rangle\right). \tag{35}$$

The wave packet solution for the constant friction has been found by Kan and Griffin [16]. Kostin's model has an important superiority over Kanai's one because it gives the correct uncertainty relation. Our problem of the barrier penetration appears to be very simple in this formalism.

As above, we seek the density distribution (22). Eq. (23) with solution (27) is still valid. The width of the packet can be obtained from

$$\ddot{u}+\Omega^2 u = m^{-2}u^{-3}, \qquad t < 0, t > T,$$

$$\ddot{u}+\gamma\dot{u}+\Omega^2 u = m^{-2}u^{-3}, \qquad 0 < t < T. \tag{36}$$

The constant solution

$$u = \frac{1}{\sqrt{m\Omega}}$$

is stable and fulfils both equations (36). Thus, the presence of the frictional force does not affect the shape of the packet at all, in contradiction to the Kanai's model predictions.

Now, let us return for a moment to the case of a free motion, discussed in Sections 2 and 3. Kostin's method has the packet solution in which the width is given by the equation:

$$\ddot{u}+f(\langle x\rangle)\dot{u} = m^{-2}u^{-3}. \tag{37}$$

Unfortunately, even for a constant friction coefficient Eq. (37) has no analytical solution. Hasse [1] has shown numerically that $u(t)$ behaves similarly to the width in Kanai's method. For small times both results are almost identical. Thus, if friction is limited to a small region, Kanai's and Kostin's models are in full agreement.

5. *Summary and conclusions*

We have discussed the behaviour of a particle subjected to a dissipative force, whose strength depends on the position operator. The problem has been treated in an approximate way: a position dependence has been transfered on a time dependence by taking the average value of the position operator. The two cases have been considered: a free particle, without external field, and a particle in a quadratic field.

The problem of a free motion is especially simple in Kanai's approach. One second order differential equation (9) determines the wave packet evolution completely. By means of this equation the problem of frictional barrier penetration has been solved. As a result of such a penetration, the packet travels and spreads similarly to the friction-free case but with the diminished speed. Qualitatively, this outcome does not depend on the particular shape and hight of the barrier.

It has been argued that results of Kostin's method are very similar though more difficult to obtain.

A study of the case of the harmonic oscillator requires an investigation of the system (23) and (25). For the rectangular barrier (10) the Kostin model gives a very simple picture. The packet does not change its shape during the pass through the barrier. The width remains fixed everywhere. Only the classical motion of the centroid of the packet is affected by dissipative forces. Results of Kanai's model are entirely different. It predicts that, as the final outcome, the width of the packet will oscillate with a constant frequency and an amplitude depends on the frictional barrier parameters. The fact that this amplitude (as well as an equilibrium position) grows exponentially with γT, is especially strange. In Hasse's model, in turn, oscillations have a different character: they are not damped in the frictional region. This comparison reminds us that the quantum theory of dissipative phenomena is far from being complete.

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