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THE ELECTROMAGNETISM OF GENERALISED FIELD THEORY

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It is shown that a previous inclusion of electromagnetic material field in the generalised field theory using gauge structures is inconsistent with the theory itself. The derivation of the field equations is reviewed and the full equations of the electromagnetic field are found. The resulting theory bears some formal resemblance to the non-linear electrodynamics of Born and Infeld. A new interpretation is found for the Klotz-Russell skew-symmetric tensor.

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1. *Introduction*

Generalised Field Theory, described and summarised in references [1] and [2] and abbreviated as GFT throughout this work, claims to be a comprehensive theory of gravitation and electromagnetism.

The electromagnetic field tensor is identified in GFT by:

$$f_{\mu\nu} = kR_{[\mu\nu]}(\tilde{\Gamma}), \tag{1}$$

where k is a proportionality constant, Greek indices go from 0 to 3 and $R_{[\mu\nu]}(\tilde{\Gamma})$ is the skew symmetric part of the Ricci tensor constructed from the nonsymmetric affine connection $\tilde{\Gamma}$ whose components satisfy identically the four conditions.

$$\tilde{\Gamma}_\mu \equiv \tilde{\Gamma}^\sigma_{[\mu\sigma]} = 0. \tag{2}$$

The geometrical connection $\tilde{\Gamma}$ satisfies these conditions automatically if it is given by Schrödinger's relations

$$\tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \tfrac{2}{3}\,\delta^\lambda_\mu \Gamma_\nu \tag{3}$$

in terms of another connection, Γ which can be called "a physical connection". The field equations

$$R_{\mu\nu}(\Gamma) = 0$$

then include

$$R_{[\mu\nu]}(\tilde{\Gamma}) = \tfrac{2}{3}(\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}). \tag{4}$$

It is, of course, equation (4) which is the main reason for the identification (1). $R_{[\mu\nu]}(\tilde{\Gamma})$ is the curl of potential-like vector and therefore exactly a Maxwell-like tensor.

Nevertheless, the electromagnetic theory of GFT is incomplete and it is this problem that I shall attempt to resolve in the present work. As Einstein himself observed (Ref. [3]), a comprehensive theory of the (total) macrophysical field must contain something like Maxwell's equations. Indeed, this requirement is the only a priori clue as to what its own structure should be. On the other hand, we need not expect GFT to yield exactly the equations of Maxwell's theory. An equally a priori motivation is for seeking comprehensive account of the macrophysical fields at all was elimination from the fundamental field equations of source terms usually expressed by an energy momentum tensor $T_{\mu\nu}$. Such a tensor (Ref. [4]) must be defined and its components calculated but only after the field equations are solved in a given geometrical and physical situation. Now, unlike the general relativistic theory of the gravitational field, classical electromagnetism is given by two sets of equations

$$h^{\mu\nu}{}_{,\nu} = J^{\mu}, \quad f_{[\mu\nu,\lambda]} = 0. \tag{5}$$

Here $h^{\mu\nu} = -h^{\nu\mu}$ (and the current J^{μ}) must be tensor densities since otherwise the partial, comma derivative would have to be replaced by a covariant, semicolon operation. Such a replacement is usually called the minimum coupling hypothesis and if it is to be an additional assumption, however plausible, it is clearly to be avoided from the structure of a fundamental theory. The first and the second of the equations (5) are independent unless a relationship (e.g. the constitutive relations of homogeneous, isotropic, electrically active matter) between the density $\boldsymbol{h}$ and the tensor f is arbitrarily postulated. This again should be avoided. The first of the equations (5) resembles the field equations of General Relativity with the current J^{μ} corresponding to the energy-momentum tensor and the continuity equation

$$J^{\mu}{}_{,\mu} = 0$$

to the Bianchi identities or the conservation equations

$$T^{\mu\nu}{}_{;\nu} = 0.$$

The spirit of GFT then suggests that we cannot expect this equation to emerge from the structure of the theory and that it should serve merely as a definition of J^{μ} itself. Then however, the connection between the "material" tensor (density) $\boldsymbol{h}$ (related to the bivector $(\boldsymbol{D}, \boldsymbol{H})$ of classical electromagnetism) and the intensity tensor $\boldsymbol{f}$ (bivector $(\boldsymbol{E}, \boldsymbol{B})$ is irretrievably lost unless the theory contains some equations which determine the former.

The complete electromagnetic field theory of GFT resembles then necessarily the non-linear electromagnetism of Born and Infeld (e.g. Ref. [5]) in which likewise the current vector or vector density is defined a posteriori. All the same, this conclusion now follows from the general nature of GFT rather than from purely electromagnetic considerations.

It may not be out of place to mention at once a curious inconsistency of the Born-Infeld theory. In the spherically symmetric case, they obtained a nonsingular intensity field

$$E_r = \frac{e}{r_0^2}\left[1+\left(\frac{r}{r_0}\right)^4\right]^{-1/2} \tag{6}$$

but a singular, Coulomb induction

$$D_r = \frac{e}{r^2}. \tag{7}$$

Although the Born-Infeld theory leads, and this is its main attraction, to finite self--energy of a point charge at rest, through an arbitrary choice of the Lagrangian it cannot avoid the above singularity in the field itself. Since point charges are in any case little more that a mathematical fiction we may well ask whether the result concerning self-energy is really sufficient to establish its superiority over Maxwell's theory. In fact, the induction vector $\boldsymbol{D}$ is usually considered as referring to the field inside electrically active matter while the intensity vector subsists outside material sources or charges which give rise to the field. Since the spatial origin of the coordinate system is invariably chosen in the former region, say at the centre of a spherically symmetric source, we can say that this point is never reached by the field $\boldsymbol{E}$. Hence, it should not matter if $\boldsymbol{E}$ is singular there but it may matter if $\boldsymbol{D}$ is.

It seems therefore that from an abstract, field theoretical point of view, the relations (6) and (7) should be reversed and we shall find that this indeed is the conclusion of the GFT electromagnetic theory.

Returning to the main problem of what the latter should be, McInnes and I (Ref. [6]) proposed a solution based on the fibre bundle techniques. We shall consider in the next section, the reasons why the resulting theory is unsatisfactory as far as GFT is concerned. Roughly speaking, it implies necessarily that GFT itself is incomplete as a theory of the total macrophysical field. While it must be readily admitted that GFT is incomplete in the sense that it lays no claim to include essentially quantum mechanical, weak and strong interactions (see Ref. [7] for a discussion of the relationship of GFT and the electroweak, GUT etc. theories), its cosmological implications (see concluding Section) in particular make the above conclusion inadmissible. In Section 3 we shall review the variational derivation of the GFT field equations including the "metric hypothesis" that the symmetric components of the connection are the Christoffel brackets

$$\tilde{\Gamma}^{\lambda}_{(\mu\nu)} = \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}_a \tag{8}$$

constructed from a symmetric tensor $\boldsymbol{a}$ which is the metric tensor of the space-time. We shall use $\tilde{\Gamma}$ itself as (one) set of the variational parameters. Here, some repetition of previously published material is necessary to see that GFT does in fact contain a complete theory of the electromagnetic field and nothing else apart, of course, from the gravitational field. When symmetry of the field is restored, GFT collapses into General Relativity but it can be shown (Ref. [8]) that it contains also the New Theory of Gravitation of Moffat. Sections 4 and 5 are devoted to a detailed discussion of the problem of electromagnetism based on the static, spherically symmetric solution of the field equations which happens to be their only general solution discovered hitherto. In Section 6 we turn to a problem which will enable us to decide finally on the form of the electromagnetic field laws. Originally, the tensor f given by the equation (1) replaced in GFT an earlier (Refs. [9, 10]) expression,

$$w_{\mu\nu} = g^{\alpha\beta} g_{[\mu o \nu o];\alpha\beta} \tag{9}$$

for the electromagnetic intensity. In equation (9), g is the nonsymmetric total field tensor and the subscript "0" indicates that covariant derivatives are to be taken with respect to the symmetric part of the connection $\tilde{\Gamma}$. The Russell-Klotz tensor w is the curl of the potential Γ_μ only up to the second order in the expansion

$$g = \eta + \sum_{n=1}^{\infty} \varepsilon^n g^n,$$

where η the Minkowski tensor diag $(+1, -1, -1, -1)$ but this is enough for both f and w to lead to the same equations of motion for a charged test particle. Solution of the problem of motion of (Ref. [2]) was of course, the starting point of the investigation which led to GFT. w has been hitherto left out from the structure of the theory. However, we shall find that this tensor plays after all a part in the description of the electromagnetic field. Finally, in Section 7, we summarise the GFT laws of electromagnetism.

2. *Critique of the "nonsymmetron" electromagnetism*

It has been suggested in Ref. [6] that the field equations of nonlinear electromagnetic theory should be

$$\Gamma_{[\mu\nu,\lambda]} = 0, \tag{10}$$

and

$$W^{\lambda\mu}{}_{,\lambda} = 0,$$

where

$$\Gamma_{\mu\nu} = R_{[\mu\nu]}(\tilde{\Gamma}) + p g_{[\mu\nu]},$$

$$W^{\mu\nu} = \sqrt{-g}\,[a^{\mu\beta} g^{(\alpha\nu)} + a^{\nu\alpha} g^{(\mu\beta)} - a^{\mu\beta} a^{\nu\alpha}] F_{\alpha\beta},$$

$$p = \tfrac{1}{2}\,[R_G + \tfrac{1}{4}\, F_{\varrho\sigma} F^{\varrho\sigma}] \tag{11}$$

and

$$g = \det(g_{\mu\nu}).$$

Here the tensor $F_{\mu\nu}$ is the curl of gauge potentials (if the gauge group G is chosen to be the abelian U(1)) and $4R_G$ is the dimension of the gauge group.

Equations (10) are derived from a variational principle on the product space

$$M \times U(1),$$

where M represents the base manifold which is the space-time of macrophysics. The procedure of erecting a fibre bundle over M acted upon by a hypothetical (that is one whose structure must be postulated) gauge groups G (here, the electromagnetic U(1)), is legitimate enough when one seeks an extension of the fundamental theory which is supposed to describe the physics in M. Since, at present, we have no recognised method for determining the structure of G other than guesswork, it follows that the resulting theory effectively abandons the requirement or expectation that GFT should encompass a complete theory of the electromagnetic field. Unless this is explicitly demonstrated, we cannot accept the conclusion. In other words, GFT should be shown to be incomplete before the necessity of invoking fibre bundle methods becomes admissible. Logically, the situation is the same as seeking five, or more dimensional theories (in which there can be little doubt that there is enough freedom for including gravitation, electromagnetism and perhaps anything else we may feel like to include) before it is proved that a four-dimensional theory must fail. Since $R_{[\mu\nu]}$ too is the curl of a vector which can for no reason be the same as the vector defining $F_{\mu\nu}$, the nonsymmetron theory encounters the difficulty of explaining the physical nature of these multiple potentials. Moreover, the first of equations (10) implies the existence of yet another potential. When therefore the factor p is constant, as it clearly can be for some F fields, $g_{[\mu\nu]}$ itself becomes, from the first of the equations (11), a curl of a vector. This restriction of the field g on the base manifold (macrophysics) by a condition on the gauge field is again difficult to understand. A less serious, but different problem arises from the relation between the tensor F and the density W required by the fibre bundle method. The second of the equations (11) gives

$$W_{\varrho\sigma} = a_{\varrho\mu}a_{\sigma\nu}W^{\mu\nu} = \sqrt{-g}\,[\delta^{\beta}_{\varrho}g^{\alpha}_{\sigma}+\delta^{\alpha}_{\sigma}g^{\beta}_{\varrho}-\delta^{\beta}_{\varrho}\delta^{\alpha}_{\sigma}]F_{\alpha\beta}$$

and

$$a_{\varrho\mu}g_{(\sigma\nu)}W^{\mu\nu} = \sqrt{-g}\,[\delta^{\beta}_{\varrho}\delta^{\alpha}_{\sigma}+g^{\alpha}_{\sigma}g^{\beta}_{\varrho}-\delta^{\beta}_{\varrho}g^{\alpha}_{\sigma}]F_{\alpha\beta}$$

since both the metric a and the symmetric part of the field tensor g are necessarily nonsingular, the former by definition and the latter because GFT must collapse into General Relativity when it is the only field present. We have also written

$$g^{\alpha}_{\beta} = a_{\beta\sigma}g^{(\alpha\sigma)}.$$

Since, as we can easily verify,

$$g^{\varrho}_{\lambda}g^{\sigma}_{\varrho} = \delta^{\sigma}_{\lambda},$$

it follows that

$$g_{\lambda\mu}[a_{\sigma\nu}+g_{(\sigma\nu)}]W^{\mu\nu} = \sqrt{-g}\,[\delta^{\alpha}_{\sigma}+g^{\alpha}_{\sigma}]F_{\alpha\lambda}.$$

Hence, the relation between the tensor density W and the tensor F is invertible iff

$$\det\,[\delta^{\alpha}{}_{\sigma}+g^{\alpha}{}_{\sigma}] = 0.$$

On the other hand, we know that this determinant tends to zero in the weak field approximation and, maybe, for other fields as well. In itself, the above is not a destructive criticism but only a limitation on the possible solutions. However, it is more difficult to understand the "limiting" condition

$$R_G \to 0[F_{\mu\nu} \to 0]$$

needed to ensure the validity of the derivation of the equations of motion. It means, in effect, elimination of the gauge group and a return to the manifold M alone. If therefore the complete electromagnetic theory can be derived only with the help of gauge invariance, GFT itself cannot be complete. We run into a further difficulty by having to use the magnetic solution in constructing the nonsymmetron model of the charged particle. It is certainly true, as stated in Ref. [6] that the magnetic solution is an acceptable solution of the modified field equations (i.e. when the metric hypothesis is included in the lagrangian: see below). But the latter happens also to be a solution of Moffat's theory (Ref. [8]) which has been correctly associated with a pure gravitational field. We cannot have it both ways. Indeed, Moffat's theory fits neatly into GFT as it stood before the nonsymmetron article but not into its modified form. Because this correspondence explains also the negative result of Infeld (Ref. [11]) in connection with Einstein's original version (Ref. [12]) of the nonsymmetric unified field theory, it is clearly preferable to seek completion of electromagnetism within GFT. The gauge construction can then be reserved to account, if necessary, for more complicated fields.

3. Field equations of GFT and the variational principle

It is well known (Refs [13, 14]) that the Ricci tensor which is automatically Hermitian symmetric (Ref. [13]) or double transposition invariant (the condition which Einstein interpreted as representing charge configuration invariance of the physical laws of the macroscopic level of experience, thus giving the nonsymmetric theory a firm, phenomenological foundation; see also Ref. [2] for a discussion of the underlying concepts) has the form

$$\begin{aligned} R_{\mu\nu} = & -W^{\sigma}_{\mu\nu,\sigma}+[1+3\alpha_1+\alpha_2]\,[W^{\sigma}_{\mu\sigma,\nu}+W^{\sigma}_{\sigma\nu,\mu}]-\alpha_2[W^{\sigma}_{\sigma\mu,\nu}+W^{\sigma}_{\nu\sigma,\mu}] \\ & +W^{\sigma}_{\mu\varrho}W^{\varrho}_{\sigma\nu}-2(1+3\alpha_1)W^{\sigma}_{\mu\nu}W^{\varrho}_{(\sigma\varrho)}-3\alpha_1^2 W^{\sigma}_{\mu\sigma}W^{\varrho}_{\varrho\nu} \\ & -\alpha_1(1+3\alpha_1)\,[W^{\sigma}_{\mu\sigma}W^{\varrho}_{\mu\varrho}+W^{\sigma}_{\sigma\mu}W^{\varrho}_{\varrho\nu}]-3[\tfrac{1}{3}+\alpha_1]^2 W^{\sigma}_{\sigma\mu}W^{\varrho}_{\nu\varrho}. \end{aligned} \tag{12}$$

Here α_1 and α_2 are numerical parameters and the coefficients $W^{\lambda}_{\mu\nu}$, with respect to which the Ricci tensor is Hermitian, are related (Ref. [15]) to the components of the affine connection by

$$W^{\lambda}_{\mu\nu} = \tilde{\Gamma}^{\lambda}_{\mu\nu}+\frac{1}{2(9\alpha_1+6\alpha_2+2)}\,[\delta^{\lambda}_{\nu}\tilde{\Gamma}^{\sigma}_{(\mu\nu)}-\delta^{\lambda}_{\mu}\tilde{\Gamma}^{\sigma}_{(\nu\sigma)}]$$

$$-\frac{6\alpha_1+1}{2(15\alpha_1+4)}[\delta^\lambda_\nu \tilde{\Gamma}^\sigma_{(\mu\varrho)}+\delta^\lambda_\mu \tilde{\Gamma}^\sigma_{(\nu\sigma)}]$$

$$+\frac{2}{3(9\alpha_1+6\alpha_2+2)}[\delta^\lambda_\mu \Gamma_\nu-\delta^\lambda_\nu \Gamma_\mu]. \tag{13}$$

Hence, in particular (there is a minor misprint in Ref. [15])

$$W^\sigma_{(\mu\sigma)} = \frac{3}{2(15\alpha_1+4)}\tilde{\Gamma}^\sigma_{(\mu\sigma)},$$

$$W_\mu = \frac{1}{2(9\alpha_1+6\alpha_2+2)}[3\tilde{\Gamma}^\sigma_{(\mu\sigma)}-4\tilde{\Gamma}_\mu]. \tag{14}$$

We may note that the relation between the physical and the geometrical connections and W is

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu}-\tfrac{2}{3}\delta^\lambda_\mu\Gamma_\nu = W^\lambda_{\mu\nu}+\tfrac{1}{3}(6\alpha_1+1)\delta^\lambda_\nu W^\alpha_{(\mu\sigma)}$$

$$-\tfrac{1}{3}\delta^\lambda_\nu W_\mu-(3\alpha_1+1)\delta^\lambda_\mu W^\sigma_{(\nu\sigma)}+(3\alpha_1+3\alpha_2+1)\delta^\lambda_\mu W_\nu. \tag{15}$$

The condition that equation (15) should be soluble for W in terms of Γ is

$$\Delta \equiv (15\alpha_1+4)(9\alpha_1+6\alpha_2+2) = 0. \tag{16}$$

If the connection Γ were not given a priori (by the relation (3)), all that we could obtain from the equation (15) would be

$$\tilde{\Gamma}_\mu+\Gamma_\mu = \tfrac{1}{2}(15\alpha_1+4)W^\sigma_{(\mu\sigma)}-\tfrac{1}{2}(9\alpha_1+6\alpha_2+2)W_\mu. \tag{17}$$

We have mentioned in the Introduction that our aim is to use $\tilde{\Gamma}$ rather than W as one of the variational parameters, the others being the field g or the associated density $\hat{g} = \sqrt{-g}\,g$ [in components, $\hat{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}$], and the metric tensor a. Because of the equation (17), the procedure is not without some difficulties unless we actually know that

$$\tilde{\Gamma}_\mu \equiv 0. \tag{18}$$

Of course, equation (15) (or for that matter, equation (3) cannot be solved for W in terms of $\tilde{\Gamma}$ alone, that is, without the vector Γ_μ appearing as well (and this is the main reason for calling Γ a physical connection: vector Γ^μ is proportional to the electromagnetic four--potential). We overcome this difficulty because of a remarkable consequence of the transposition algebra.

To see what happens, let us now turn to the variational principle which leads to the generalised field equations. We assume that the latter follow from

$$\delta\int[\hat{g}^{\mu\nu}R_{\mu\nu}+\Lambda^{\mu\nu\lambda}a_{\mu\nu;\lambda}] = 0, \tag{19}$$

where the semicolon derivative is calculated with respect to $\tilde{\Gamma}^{\lambda}{}_{(\mu\nu)}$. It is because $a_{\mu\nu;\lambda}$ is explicitly introduced into the Lagrangian that we want to express variation of the Ricci tensor with respect to $\tilde{\Gamma}$ rather than with respect to W. For the moment, as in Ref. [15], we shall regard the coefficients (components of a tensor density)

$$\Lambda^{\mu\nu\lambda} = \Lambda^{\nu\mu\lambda}, \tag{20}$$

as Lagrange's multipliers. We write in the first instance,

$$\int [R_{\mu\nu}\delta\hat{g}^{\mu\nu} + \hat{g}^{\mu\nu}\delta R_{\mu\nu} - [\Lambda^{\mu\nu\lambda}{}_{,\lambda} + \Lambda^{\sigma\nu\lambda}\tilde{\Gamma}^{\mu}_{(\sigma\lambda)} + \Lambda^{\mu\sigma\lambda}\tilde{\Gamma}^{\nu}_{(\sigma\lambda)}]\delta a_{\mu\nu}$$
$$- a_{\sigma\nu}\Lambda^{\mu\nu\lambda}\delta\tilde{\Gamma}^{\varrho}_{(\mu\lambda)} - \alpha_{\mu\sigma}\Lambda^{\mu\nu\lambda}\delta\tilde{\Gamma}^{\sigma}_{(\nu\lambda)} + a_{\mu\nu;\lambda}\delta\Lambda^{\mu\nu\lambda}] = 0$$

or

$$\int [R_{\mu\nu}\delta\hat{g}^{\mu\nu} + N^{\mu\nu}_{\lambda}\delta W^{\lambda}_{\mu\nu} - \Lambda^{\mu\nu\lambda}{}_{;\lambda}\delta a_{\mu\nu}$$
$$- a_{\sigma\nu}\Lambda^{\mu\nu\lambda}\delta\tilde{\Gamma}^{\sigma}_{(\mu\lambda)} - a_{\mu\sigma}\Lambda^{\mu\nu\lambda}\delta\tilde{\Gamma}^{\sigma}_{(\nu\lambda)} + a_{\mu\nu;\lambda}\delta\Lambda^{\mu\nu\lambda}] = 0 \tag{21}$$

remembering that Λ is a tensor density and not a tensor. Here

$$N^{\mu\nu}_{\lambda} = \hat{g}^{\mu\nu}{}_{,\lambda} - [1+3\alpha_1+\alpha_2]\,[\hat{g}^{\mu\sigma}{}_{,\sigma}\delta^{\nu}_{\lambda} + \hat{g}^{\sigma\nu}{}_{,\sigma}\delta^{\mu}_{\lambda}]$$
$$+\alpha_2[\hat{g}^{\nu\sigma}{}_{,\sigma}\delta^{\mu}_{\lambda} + \hat{g}^{\sigma\mu}{}_{,\sigma}\delta^{\nu}_{\lambda}] + \hat{g}^{\mu\sigma}W^{\nu}_{\lambda\sigma} + \hat{g}^{\sigma\nu}W^{\mu}_{\sigma\lambda} - 2[1+3\alpha_1]\hat{g}^{\mu\nu}W^{\sigma}_{(\lambda\sigma)}$$
$$-[1+3\alpha_1]\hat{g}^{\alpha\beta}[W^{\mu}_{\alpha\beta}\delta^{\nu}_{\lambda} + W^{\nu}_{\alpha\beta}\delta^{\mu}_{\lambda}] - 3\alpha_1^2[\hat{g}^{\mu\sigma}W^{\alpha}_{\alpha\sigma}\delta^{\nu}_{\lambda} + \hat{g}^{\sigma\nu}W^{\alpha}_{\sigma\alpha}\delta^{\mu}_{\lambda}]$$
$$-2\alpha_1[1+3\alpha_1]\,[\hat{g}^{(\mu\sigma)}W^{\alpha}_{\sigma\alpha}\delta^{\nu}_{\lambda} + \hat{g}^{(\nu\sigma)}W^{\alpha}_{\alpha\sigma}\delta^{\mu}_{\lambda}]$$
$$-3[\tfrac{1}{3}+\alpha_1]^2\,[\hat{g}^{\nu\sigma}W^{\alpha}_{\sigma\alpha}\delta^{\mu}_{\lambda} + \hat{g}^{\sigma\mu}W^{\alpha}_{\alpha\sigma}\delta^{\nu}_{\lambda}]. \tag{22}$$

Using equation (13) we find that the variation in Γ_{μ} gives

$$N^{[\mu\sigma]}{}_{\sigma} = 0. \tag{23}$$

However, a straightforward calculation shows that the coefficients $N_{\lambda}{}^{\mu\nu}$ given by the equation (22) are independent of Γ_{μ}. Consequently, in carrying out the variation we can put

$$W^{\lambda} = \Gamma^{\lambda} - \frac{6\alpha_1+1}{2(15\alpha_1+4)}\,[\delta^{\lambda}_{\nu}\tilde{\Gamma}^{\sigma}_{(\mu\sigma)} + \delta^{\lambda}_{\mu}\tilde{\Gamma}^{\sigma}_{(\nu\sigma)}],$$

$$W^{\lambda}_{[\mu\nu]} = \tilde{\Gamma}^{\lambda}_{[\mu\nu]} + \frac{1}{2(9\alpha_1+6\alpha_2+2)}\,[\delta^{\lambda}_{\nu}\tilde{\Gamma}^{\sigma}_{(\mu\sigma)} - \delta^{\lambda}_{\mu}\tilde{\Gamma}^{\alpha}_{(\nu\sigma)}],$$

$$W^{\sigma}_{(\mu\sigma)} = \frac{3}{2(15\alpha_1+4)}\,\tilde{\Gamma}^{\sigma}_{(\mu\sigma)}, \qquad W_{\mu} = \frac{3}{2(9\alpha_1+6\alpha_2+2)}\,\tilde{\Gamma}^{\sigma}_{(\mu\sigma)} \tag{24}$$

and with $\tilde{\Gamma}$ as the variational parameter, readily obtain the field equations

$$R_{\mu\nu}(\Gamma) = 0, \quad N^{[\mu\nu]}{}_{\lambda} = 0, \quad a_{\mu\nu;\lambda} = 0, \quad \Lambda^{\mu\nu\lambda}{}_{;\lambda} = 0. \tag{25}$$

A comment here is required. The expression (12) of the Ricci tensor in terms of W is possible only if it is a priori constructed from the connection Γ with which W itself is in one to one relation. The name "physical" which we have given to it comes about only with our attempt to interpret the field equations themselves when the first, the vanishing of the Ricci tensor expressed in terms of Γ, becomes equivalent to the equations

$$R_{(\mu\nu)}(\tilde{\Gamma}) = 0, \quad R_{[\mu\nu]}(\tilde{\Gamma}) = \tfrac{2}{3}[\Gamma_{\mu,\nu}-\Gamma_{\nu,\mu}].$$

In addition to equations (25) we also have

$$N_{\sigma}^{(\mu\nu)}-K[N_{\varrho}^{(\mu\varrho)}\delta_{\sigma}^{\nu}+N_{\varrho}^{(\nu\varrho)}\delta_{\sigma}^{\mu}]-\Lambda^{\alpha\nu\mu}a_{\alpha\sigma}-\Lambda^{\mu\alpha\nu}a_{\alpha\sigma} = 0$$

or

$$\Lambda^{\lambda\nu\mu}+\Lambda^{\mu\lambda\nu} = a^{\lambda\sigma}N_{\sigma}^{(\mu\nu)}-K[N_{\sigma}^{(\mu\sigma)}a^{\nu\lambda}+N_{\sigma}^{(\nu\sigma)}a^{\mu\lambda}], \tag{26}$$

where

$$K = \frac{6\alpha_1+1}{2(15\alpha_1+4)}.$$

The symmetry condition (20) enables us to solve equation (26) in the usual way to give finally

$$\Lambda^{\mu\nu\lambda} = \tfrac{1}{2}[a^{\mu\sigma}N_{\sigma}^{(\nu\lambda)}+a^{\nu\sigma}N_{\sigma}^{(\mu\lambda)}-a^{\lambda\sigma}N_{\sigma}^{(\mu\nu)}]-KN_{\sigma}^{(\lambda\sigma)}a^{\mu\nu}. \tag{27}$$

Using Einstein's notation with $\tilde{\Gamma}$ as the affine connection, we can readily show that

$$N_{\sigma}^{(\mu\nu)} = g^{(\mu\nu)}_{+-;\sigma}-[1+3\alpha_1][g^{\mu\alpha}_{+-;\alpha}\delta_{\sigma}^{\nu}+g^{\nu\alpha}_{+-;\alpha}\delta_{\sigma}^{\mu}],$$

$$N_{\sigma}^{(\mu\sigma)} = -(15\alpha_1+4)g^{\mu\sigma}_{+-;\sigma}. \tag{28}$$

Moreover, equation (23), which in any case is included in the second of the field equations (25), is equivalent to

$$\hat{g}^{[\mu\sigma]}{}_{,\sigma} = 0. \tag{29}$$

Thus

$$\hat{g}^{\mu\sigma}_{+-;\sigma} = \hat{g}^{(\mu\sigma)}_{+-;\sigma} = \hat{g}^{\mu\sigma}{}_{;\sigma}+\hat{g}^{\alpha\beta}\tilde{\Gamma}^{\mu}_{\alpha\beta},$$

and we can write the solution for $\Lambda^{\mu\nu\lambda}$ as

$$\Lambda^{\mu\nu\lambda} = \tfrac{1}{2}[a^{\mu\sigma}\hat{g}^{(\mu\lambda)}_{+-;\sigma}+a^{\nu\sigma}\hat{g}^{(\mu\lambda)}_{+-;\sigma}-a^{\lambda\sigma}\hat{g}^{(\mu\nu)}_{+-;\sigma}-a^{\mu\nu}\hat{g}^{(\lambda\sigma)}_{+-;\sigma}] \tag{30}$$

to which the much more complicated expression given in Ref. [15] is equivalent. The purpose of including multiplier terms in the variation (19) was to derive variationally the "metric hypothesis"

$$a_{\mu\nu;\lambda}[\tilde{\Gamma}^{\alpha}_{(\beta\lambda)}] = 0. \tag{31}$$

For this to follow, an "independent" variation in $\Lambda^{\mu\nu\lambda}$ has to be assumed. If now the expression (30) is inserted a priori into the variational principle it becomes necessary to postulate formal independence of $\Lambda^{\mu\nu\lambda}$ (now retained only as a convenient notation) and of $a_{\mu\nu}$. Alternatively we can let

$$\Lambda^{\mu\nu\lambda} = 0$$

after carrying out the variation itself. In either case, we obtain

$$\hat{g}^{(\mu\nu)}_{+-;\sigma} = 0.$$

Since it is also not difficult to show that the equation

$$N^{[\mu\nu]}_{\sigma} = 0$$

is equivalent to

$$\hat{g}^{[\mu\nu]}_{+-;\sigma} = 0,$$

we end up with the weak field equations of Einstein and Straus (with Γ_μ as the potential):

$$R_{(\mu\nu)}(\tilde{\Gamma}) = 0, \quad R_{[\mu\nu]}(\tilde{\Gamma}) = \tfrac{2}{3}\left[\Gamma_{\mu,\nu} - \Gamma_{\nu,\mu}\right], \quad \hat{g}^{\mu\nu}_{+-;\sigma} = 0 \tag{32}$$

together with the identities (29) and the metric definition (31). From the macrophysical point of view there are no other equations to be found unless one abandons the original variational principle. The metric hypothesis remains a superimposed definition, shown previously (Ref. [2]) to be the only one consistent with the principle of Hermitian symmetry. The full set of the GFT equations: (32), (29) and (31) is now the basis in which we have to find the complete laws of the electromagnetic field.

Let us note that the last of equations (32) and equation (31) imply that

$$\tilde{\Gamma}^{\alpha}_{(\alpha\sigma)} = (\ln\sqrt{-g})_{,\sigma} = (\ln\sqrt{-a})_{,\sigma}, \quad \text{i.e.} \sqrt{-g} \propto \sqrt{-a} \tag{33}$$

so that $\sqrt{-g}$ is necessarily proportional to the space-time volume. This identity enables us to resolve an otherwise considerable ambiguity in defining physical quantities such as a current density vector.

We can say that the relation (33) is all that remains in GFT of what I called a strong geometrisation principle of General Relativity in which components of the metric tensor are identified with gravitational potentials and the curvature of the space-time explicitly gives the gravitational properties of matter.

4. The problem of electromagnetism

The space-time in which the macrophysical electromagnetic and gravitational fields subsist and which correponds to the complete field equations of GFT, is a Riemannian (or rather, pseudo-Riemannian) V_4 with the metric

$$ds^2 = a_{\mu\nu}dx^\mu dx^\nu. \tag{33}$$

The physical fields themselves are represented by the sixteen components $g_{\mu\nu}$ of the non-symmetric tensor $\boldsymbol{g}$. This does not mean, say, that it should be possible to point to g_{11} and claim that it is a radial component (of gravitational potential tensor), or to $g_{[23]}$ that it is a component of magnetic induction vector or of electric intensity. Indeed, we know that $g_{[\mu\nu]}$ cannot be the electromagnetic (intensity) tensor because such identification neccessarily leads (Ref. [11]) to equations of motion of a supposedly charged test particle in an electric field but without a Lorentz force acting on it. All that we can say is that the field tensor $\boldsymbol{g}$ together with the vector Γ_μ represents the total macrophysical field. It is to be determined from solutions of the field equations but its components need not bear any direct resemblance to the known partial fields of physics. On the other hand, it GFT is to have any physical credibility, the partial fields themselves must be perceivable somewhere in the structure of the theory.

Thus, we must expect for example that, paraphrasing the words of Einstein: something like Maxwell equations should emerge from the comprehensive account. If however, there is to be any possibility of an empirical verification of the theory, and especially a local or laboratory scale verification of it, this "something" should not be exactly Maxwellian electromagnetism nor for that matter, exactly the gravitation of General Relativity.

We have already observed that both sets of Maxwell's equations should be contained in the field equations of GFT but, it is then self evident that material or the first set of the equation (5) can only represent a definition of the current density. They cannot, so to say, gratuitously drop out from the field equations. In other words, if the identification of the intensity tensor of the equation (1) is retained, and we can see no way how it could be altered without creating an unsurmountable problem of what the Maxwell-like field $R_{[\mu\nu]}(\tilde{\Gamma})$ might mean the complementary set of the electromagnetic field equations must be of the form

$$\hat{h}^{\mu\nu}{}_{,\nu} = 0,$$

where $\hat{\boldsymbol{h}}$ a tensor density. The electromagnetic field laws would then be

$$\hat{h}^{\mu\nu}{}_{,\nu} = 0, \quad f_{[\mu\nu,\lambda]} = 0, \quad f_{\mu\nu} = kR_{[\mu\nu]}(\tilde{\Gamma}),$$

with the current density vector j^μ defined suitably in terms of $\boldsymbol{h}$ or $\boldsymbol{f}$. It is then obvious that the electromagnetic theory of GFT is necessarily nonlinear and more of the Born-Infeld than of a Maxwell-like appearance.

At first sight, the identities (29) are an alluring candidate for the missing set of the electromagnetic equations. We shall find presently however, that their unqualified adoption as field laws leads to difficulties. There is another reason why the classical electromagnetism of Maxwell does not fit the comprehensive field theory. The standard Maxwell theory without the constitutive relations between material and intensity vectors is given by the equations

$$h^{\mu\nu}{}_{,\nu} = j^\mu \text{ [assuming } h^{\mu\nu} = -h^{\nu\mu}], \quad f_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu},$$

where, with latin indices going from 1 to 3, the field vectors are identified as

$$B_k = f_{ij},\ E_k = f_{k0},\ h^{0k} = D_k,\ h^{ij} = H_k;\ i, j, l \text{ cyclic } 1, 2, 3;$$

and the potential and current are

$$\phi_\mu = (-\phi, A), j^\mu = (\varrho, j).$$

If partial derivatives are replaced by covariant (minimum coupling hypothesis) the first set becomes

$$h^{\mu\nu}{}_{;\nu} = j^\mu \quad \text{or} \quad \hat{h}^{\mu\nu}{}_{,\nu} = \hat{j}^\mu$$

with

$$\hat{h}^{\mu\nu} = \sqrt{-g}\, h^{\mu\nu}, \quad \hat{j}^\mu = \sqrt{-g}\, j^\mu.$$

All this, of course, is elementary. It is nevertheless important to observe that before constitutive relations between the tensors $\boldsymbol{h}$ and $\boldsymbol{j}$ are imposed, and this can only be done a priori, that is without reference to a specific kind of electromagnetically active matter, by a hypothesis, the first set of Maxwell's equations is little more than a definition of the current satisfying the continuity equations $j^\sigma{}_{;\sigma} = 0$ or $\hat{j}^\sigma{}_{,\sigma} = 0$. Similarly, if $\hat{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} f_{\alpha\beta}$, where $\varepsilon^{\mu\nu\alpha\beta}$ are the Levi-Civita premutation symbols, the second set of Maxwell's equations can be written as $\hat{F}^{\mu\nu}{}_{;\nu} = 0$.

Of course,

$$\hat{F}^{\mu\nu} = \tfrac{1}{2}(-g)^{-1/2}\varepsilon^{\mu\nu\alpha\beta} j_{\alpha\beta}$$

is a tensor, the converse relation being

$$f_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta}\hat{F}^{\alpha\beta}.$$

Let us note also the following. The identification of the field vectors above assumes Cartesian coordinates. If we go over to polar coordinates:

$$[x^0, x^1, x^2, x^3] \to [x^{0'}, x^{1'}, x^{2'}, x^{3'}] = [x^0, r, \theta, \phi],$$

the tensor transformation laws

$$f'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x^{\alpha'}}\frac{\partial x^\nu}{\partial x^{\beta'}} f_{\mu\nu}, h'^{\alpha\beta} = \frac{\partial x^{\alpha'}}{\partial x^\mu}\frac{\partial x^{\beta'}}{\partial x^\nu} h^{\mu\nu}$$

give

$$f'_{01} = -\sin\theta\cos\phi E_1 + \sin\theta\sin\phi E_2 + \cos\theta E_3) = -E_r,$$

and

$$h^{123} = \frac{1}{r^2\sin\theta}[\sin\theta\cos\phi h^{23} + \sin\theta\sin\phi h^{31} + \cos\theta h^{12}] = \frac{1}{r^2\sin\theta}H_r.$$

Let us now recall (Ref. [2]) the only known general solution of the field equations (29), (31), (32) namely the spherically symmetric, static, solution:

$$a_{\mu\nu} = \text{diag}\,[\sigma, -\alpha, -r^2, -r^2\sin^2\theta], \tag{34}$$

with

$$\sigma = 1+c\sqrt{\frac{r_0^2}{r^2}-1}, \qquad \alpha\sigma = \left(1-\frac{r^2}{r_0^2}\right),$$

and c and r_0 constant, while

$$g_{\mu\nu} = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & f\sin\theta \\ 0 & 0 & -f\sin\theta & -\beta\sin^2\theta \end{pmatrix}, \tag{35}$$

with

$$\beta^2+f^2 = r^4 \quad \text{and} \quad f = \frac{2r^4}{r_0^2}\sqrt{\frac{r_0^2}{r^2}-1}.$$

Since the only surviving skew component of the Ricci tensor in this case is:

$$R_{[23]} = c\sin\theta, \tag{36}$$

it follows that the constant c cannot vanish without the field equations collapsing into the strong field equations of Einstein (Ref. [12]) related by Moffat to the theory of a pure gravitational field and yielding no Lorentz force in the equations of motion.

We see that for this solution $\sqrt{-g} = \sqrt{-a}$. If now we lower tensor indices with the metric tensor $\boldsymbol{a}$ as required by GFT,

$$\hat{F}_{\mu\nu} = a_{\mu\varrho}a_{\nu\sigma}\hat{F}^{\varrho\sigma}$$

then, tentatively identifying $\check{f}_{\mu\nu}$ as the electromagnetic intensity, so that

$$\hat{f}_{\mu\nu} \propto R_{[\mu\nu]}(\tilde{\Gamma}) \tag{37}$$

(this amendment of the identification (1) appears to be necessary since we expect the solution to correspond to a radial field and so the "2–3" component must be converted into the "0–1" one), we obtain

$$\hat{F}_{23} = (-g)^{-1/2}a_{22}a_{33}f_{01}.$$

Hence, in polar coordinates

$$F_{23} = kc\sin\theta = f_{01}r^2\sin\theta\sqrt{1-\frac{r^2}{r_0^2}}$$

or

$$f_{01} = -E_r = \frac{kc}{r^2\sqrt{1-\frac{r^2}{r_0^2}}}, \tag{38}$$

where k is the proportionality constant in the equation (37). I have claimed previously (Ref. [2]) that the static, spherically symmetric solution of the field equations leads exactly to the Coulomb law so that identification of the electromagnetic vectors must be changed a priori with $f_{ij} = E$ etc. suggested long ago by Einstein himself. Exactness of the inverse square law was in fact one of the main reasons for the cosmological interpretation crucial to GFT (a Coulomb law with an apparent cut off at $r = r_0 < +\infty$ makes sense only if r_0 is the finite radius of the observable Universe!). Do we have to change this conclusion?

The answer is no, at least not if instead of r we take the radial coordinate to be

$$\bar{r} = \frac{r}{\sqrt{1-\frac{r^2}{r_0^2}}} \tag{39}$$

previously called the cosmological or global coordinate. Then

$$\bar{f}_{01} = \frac{dr}{d\bar{r}} f_{01} = \frac{kc}{\bar{r}^2} = -E_{\bar{r}} \tag{40}$$

and the exact Coulomb law is recovered except that the cut off now occurs at infinity. However, the identification (37) is questionable for another reason which we shall consider in the next section.

5. The electromagnetic problem continued

We are now faced with the following problem. If $R_{[\mu\nu]}(\tilde{\Gamma})$ is directly proportional to the electromagnetic intensity tensor and the theory collapses into General Relativity when the field $g_{\mu\nu}$ becomes symmetric (equations

$$g_{\mu\nu,\lambda} - \tilde{\Gamma}^{\sigma}_{\mu\lambda} g_{\sigma\nu} - \tilde{\Gamma}^{\sigma}_{\lambda\nu} g_{\mu\sigma} = 0,$$

then immediately imply that $\tilde{\Gamma}^{\lambda}{}_{\mu\nu}$ is likewise symmetric: actually, we can still have Moffat-like theory because Γ need not be symmetric as well unless $\Gamma_{\mu} = 0$), and if GFT is to include the full electromagnetic field, the identities

$$\hat{g}^{[\mu\nu]}{}_{,\nu} = 0$$

clearly refer, at least in some sense, to the latter. No other conclusion can be reasonably drawn.

For the static, spherically symmetric solution (34), (35) we have

$$\sqrt{-a} = \sqrt{-g} = \sqrt{\alpha\sigma(f^2+\beta^2)}\sin\theta \tag{41}$$

and

$$g^{\mu\nu} = \begin{bmatrix} 1/\sigma & & & \\ & -1/\alpha & & \\ & & -\frac{\beta}{f^2+\beta^2} & -\frac{f\operatorname{cosec}\theta}{f^2+\beta^2} \\ & & -\frac{f\operatorname{cosec}\theta}{f^2+\beta^2} & -\frac{\beta\operatorname{cosec}^2\theta}{f^2+\beta^2} \end{bmatrix} \tag{42}$$

whence

$$\hat{g}^{[23]} = -\frac{r}{r_0} \text{ and } g^{[23]} = -\frac{2}{r_0 r}\sqrt{1-\frac{r^2}{r_0^2}}. \tag{43}$$

For the "global" radial coordinate (39)

$$g^{[23]}(r) = g^{[23]}(r) = -\frac{2}{r_0 r}\operatorname{cosec}\theta. \tag{44}$$

Hence the related radial component of the (electromagnetic) vector is given by

$$V_r = g^{[23]}r^2\sin\theta = -\frac{2r_0 r}{r_0^2+r^2} \tag{45}$$

and vanishes both at the origin and at infinity.

Now, the problem of motion requires the static, spherically symmetric solution to correspond to an electric and not to a magnetic field. Since we have also tentatively related the tensor $\hat{F}$ to the intensity field, there appears to be no choice but to regard

$$V_r \equiv D_r, \tag{46}$$

i.e. the radial component of the electric induction vector. However, we find that this identification leads to a conceptual complication with reference to the total charge in the GFT universe.

Let first

$$g^*_{[\alpha\beta]} = \tfrac{1}{2}\sqrt{-a}\,\varepsilon_{\alpha\beta\mu\nu}g^{[\mu\nu]}. \tag{47}$$

Raising indices with the metric $a_{\mu\nu}$:

$$g^{*[\varrho\sigma]} = a^{\varrho\alpha}a^{\sigma\beta}g^*_{\alpha\beta},$$

we define, as indeed seems to be the only possibility open to us, the current by

$$j^\sigma = \frac{1}{\sqrt{-a}}[\sqrt{-a}\,g^{*[\sigma\alpha]}]_{,\alpha}. \tag{48}$$

If this is so, the charge density becomes

$$\frac{1}{\sqrt{-a}}\frac{d}{dr}\left(\frac{dr}{dr}\sqrt{-a}\,g^{*[01]}\right)$$

and the total charge q (or charge excess) in the universe is

$$q \propto 4\pi\int_0^\infty \frac{d}{dr}\left(\sqrt{-a}\,\frac{dr}{dr}\,g^{*[01]}\right)dr = \left.\frac{8\pi r_0^3 r^3}{[r_0^2+r^2]^2}\right|_0^\infty = 0. \tag{49}$$

In other words, the inevitable conclusion from the above identification is that globally, the universe is electrically neutral. In itself, this would not be fatal result. Previously (Ref. [16]), the apparent charge excess (which, because of local consideration, had to be negative) was regarded as responsible for the expansion of the universe.

In a way, the present result seems to confirm the subsequent claim of my then coauthor (Ref. [17]) that the static solution (35) or rather the metric (34) is a neutral solution anyway. Vlachynsky obtained his interpretation by transforming the GFT metric into a Vaidya solution of General Relativity usually regarded as representing the gravitation field of a rotating black hole immersed in a Schwarzschild background. However, Vlachynsky's interpretation is based on a misunderstanding. In particular, it ignores two crucial results of GFT, namely the solution of the problem of motion of an indisputably charged particle and the appearance of the Coulomb law in the structure of the theory. The point is that the meaning of the Vaidya solution depends on the standard interpretation of General Relativity which is rejected in GFT. Hence to say that a particular solution (requiring non empty field equations) is, for example, a black hole has no more right to being accepted than that this solution should mean something totally different in a different theory.

All the same, the GFT world could still be neutral. The electric charge density at a given distance from the origin does not vanish and globally its positive and negative values could balance out. Local charge could be responsible for the observed expansion but the neatness of the original interpretation would be lost irretrievably. Before resolving this question, let us digress somewhat and review again both the identification of $R_{[\mu\nu]}(\tilde{\Gamma})$ and of the old, Russell-Klotz tensor.

6. *The Ricci and the Russell-Klotz tensors*

Let us first review the reasons why $R_{[\mu\nu]}(\tilde{\Gamma})$ is a suitable, electromagnetic field tensor.

The geometric affine connection $\tilde{\Gamma}$ is given in GFT in terms of the field tensor g and its first derivatives by the equations

$$g_{\underset{+-}{\mu\nu};\lambda} \equiv g_{\mu\nu,\lambda}-\tilde{\Gamma}^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\tilde{\Gamma}^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0 \tag{50}$$

(which are easily obtained from the variationally derived $\hat{g}^{\underset{+-}{\mu\nu}}{}_{;\lambda} = 0$). Similarly, the Ricci tensor constructed from $\tilde{\Gamma}$ is

$$R_{\mu\nu}(\tilde{\Gamma}) = -\tilde{\Gamma}^{\sigma}_{\mu\nu,\sigma}+\tilde{\Gamma}^{\sigma}_{(\mu\sigma),\nu}+\tilde{\Gamma}^{\sigma}_{\mu\varrho}\tilde{\Gamma}^{\varrho}_{\sigma\nu}-\tilde{\Gamma}^{\sigma}_{\mu\nu}\tilde{\Gamma}^{\varrho}_{(\sigma\varrho)}. \tag{51}$$

Since equation (50) implies that $\tilde{\Gamma}^{\sigma}_{(\mu\varrho)} = \{\ln\sqrt{-g}\}_{,\mu}$ so that $\tilde{\Gamma}^{\sigma}_{(\mu\varrho),\nu} = \tilde{\Gamma}^{\sigma}_{(\nu\sigma),\mu}$ it is easy to see that

$$R_{[\mu\nu]}(\tilde{\Gamma}) = -\tilde{\Gamma}^{\sigma}_{\underset{00}{[\mu\nu]};\sigma}. \tag{52}$$

It is an immediate consequence of Schrödinger's definition (3) of the "physical" connection Γ which incidentally (definition (3) was chosen to achieve just that) guarantees $\tilde{\Gamma}_{\mu} \equiv 0$, Eq. (2), and is responsible for the symmetric components in equation (51), that $R_{[\mu\nu]}(\tilde{\Gamma})$

should be given also by the equation (4). Suppose now that we expand the field $\boldsymbol{g}$ in a power series of a "small" parameter ε:

por (31)
$$g_{\mu\nu} = \eta_{\mu\nu}+\varepsilon h_{\mu\nu}+\varepsilon^2 y_{\mu\nu}+O(\varepsilon^3), \tag{53}$$

and

$$g^{\mu\nu} = \eta^{\mu\nu}+\varepsilon H^{\mu\nu}+\varepsilon^2 Y^{\mu\nu}+O(\varepsilon^3). \tag{54}$$

Assumption that the field tensor is $O(\varepsilon^0)$ the Minkowski tensor $\boldsymbol{\eta}$, is forced by the requirement that GFT should contain General Relativity. The above expansions then correspond to the "weak field approximation" in the latter. The same reason leads us to the further assumption, namely that $h_{\mu\nu} = h_{\nu\mu}$ or that $\boldsymbol{g}$ is symmetric $O(\varepsilon^1)$. From a physical point of view, this is equivalent to the empirical fact that an interaction between electromagnetic and gravitational fields is locally difficult to observe (General Relativity is an excellent heo ry of pure gravity) and has not been discovered hitherto.

It now follows readily from

$$g^{\mu\sigma}g_{\nu\sigma} = g^{\sigma\mu}g_{\sigma\nu} = \delta^{\mu}{}_{\nu},$$

that

$$H^{\mu\nu} = -\eta^{\mu\varrho}\eta^{\nu\sigma}h_{\varrho\sigma} \equiv -h^{\mu\nu}, \tag{55}$$

say, and

$$Y^{\mu\nu} = -\eta^{\mu\sigma}\eta^{\nu\varrho}y_{\varrho\sigma}+\eta^{\mu\varrho}h^{\mu\sigma}h_{\varrho\sigma} \equiv -y^{\nu\mu}+h^{\mu\sigma}h_{\sigma}{}^{\nu}, \tag{56}$$

tensor indices being raised and lowered with the Minkowski tensor $\boldsymbol{\eta}$. Then $Y^{(\mu\nu)} = -y^{(\mu\nu)} +h^{\mu\sigma}h_{\sigma}{}^{\nu}$ and $Y^{[\mu\nu]} = y^{[\mu\nu]}$.

Therefore

$$g^{\mu\nu} = \eta^{\mu\nu}-\varepsilon h^{\mu\nu}+\varepsilon^2[-y^{(\mu\nu)}+h^{\mu\sigma}h_{\sigma}{}^{\nu}+y^{[\mu\nu]}]+O[\varepsilon^3]. \tag{57}$$

Because also

$$\tilde{\Gamma}^{\sigma}_{\mu\nu} = \varepsilon\overset{1}{\tilde{\Gamma}}{}^{\sigma}_{\mu\nu}+\varepsilon^2\overset{2}{\tilde{\Gamma}}{}^{\sigma}_{\mu\nu}+O[\varepsilon^3], \tag{58}$$

it follows that

$$\overset{1}{\tilde{\Gamma}}{}^{\sigma}_{\nu\mu} = \overset{1}{\tilde{\Gamma}}{}^{\sigma}_{\mu\nu} = \tfrac{1}{2}\eta^{\sigma\lambda}[h_{\lambda\nu,\mu}+h_{\mu\lambda,\nu}-h_{\mu\nu,\lambda}], \tag{59}$$

$$\begin{aligned}\overset{2}{\tilde{\Gamma}}{}^{\sigma}_{(\mu\nu)} &= \tfrac{1}{2}\eta^{\sigma\lambda}[y_{(\lambda\nu),\mu}+y_{(\mu\nu),\nu}-y_{(\mu\nu),\lambda}]-\tfrac{1}{2}h^{\sigma\lambda}[h_{\lambda\nu,\mu}+h_{\mu\lambda,\nu}-h_{\mu\nu,\lambda}]\\ &= \tfrac{1}{2}\eta^{\sigma\lambda}[y_{(\lambda\nu),\mu}+y_{(\mu\lambda),\nu}-y_{(\mu\nu),\lambda}]-h^{\sigma}{}_{\lambda}\overset{1}{\tilde{\Gamma}}{}^{\lambda}_{(\mu\nu)},\end{aligned} \tag{60}$$

and, what interests us most,

$$\overset{2}{\tilde{\Gamma}}{}^{\sigma}_{[\mu\nu]} = \tfrac{1}{2}\eta^{\sigma\lambda}[y_{[\lambda\nu],\mu}+y_{[\mu\lambda],\nu}+y_{[\mu\nu],\lambda}]. \tag{61}$$

The identity (2) immediately implies that

$$\overset{2}{\tilde{\Gamma}}_{\mu} = \eta^{\sigma\lambda} y_{[\mu\lambda],\sigma} \equiv 0 \tag{62}$$

and then

$$\overset{2}{R}_{[\mu\nu]}(\tilde{\Gamma}) = -\tfrac{1}{2}\eta^{\sigma\lambda} y_{[\mu\nu],\sigma\lambda} = \tfrac{1}{2}\Box\, y_{[\mu\nu]}, \tag{63}$$

where $\Box$ is the D'Alembertian operator.

It is this result which guarantees (Ref. [10]) that the tensor $R_{[\mu\nu]}(\tilde{\Gamma})$ should lead to the correct equations of motion of a charged test particle. However, the same result holds also for the Russell-Klotz tensor

$$\omega_{\mu\nu} = g^{\alpha\beta} g_{\underset{00}{[\mu\nu]};\alpha\beta} \tag{9}$$

and for an infinite class of tensors which satisfy the equations of motion condition. The whole class need not be explicitly considered because relation of its individual members with the electromagnetic field becomes progressively more tenuous. They may represent in the order of approximation higher than second, conceptually "possible" particles while electromagnetic field itself is regarded as the simplest differential vector field. That it occurs in Nature is then a direct consequence of Ockham's simplicity postulate. On the other hand, the form of ω appears to throw light on the interpretation of the skew part of the field tensor although its meaning has not been hitherto adequately explained in GFT once, on account of the equation (4), $R_{[\mu\nu]}(\tilde{\Gamma})$ was adopted as the intensity tensor ω was, so to say, left hanging in the air together with the original insight (Ref. [9]) into the interpretation of the non-symmetric theory. We shall now show that far from being a curiosity, the presence of ω is essential to the resolution of the problem of the missing set of Maxwell's (or Maxwell-like) equations.

In this Section, we shall confine ourselves to the calculation of ω for the static spherically symmetric case of the GFT field equations given by equations (34) and (35). Somewhat unexpectedly, but entirely in comformity with the ideas of GFT, we shall be led to a further modification of the tensor ω itself.

The metric of space-time (a pseudo-Riemannian manifold) corresponding to the above solution is expressed in the global coordinates by

$$ds^2 = ydf^2 - y^{-1}\omega^2 dr^2 - r^2\omega d\Omega^2, \tag{64}$$

where

$$y = 1 - \frac{2m}{r} = 1 + \frac{cr_0}{r}, \qquad \omega = \frac{r_0^2}{r_0^2 + r^2},$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2,$$

and we have dropped the distinguishing bar over r. Since we do not require a scalar contraction of ω in the present work, the notation ω for the factor in equation (64) which distinguishes it from a Schwarzschild metriccann ot lead to any ambiguity. The solution for the field tensor g now becomes

$$g_{00} = y; \quad g_{11} = -y^{-1}\omega^2, \quad g_{22} = g_{23}\,\mathrm{cosec}^2\,\theta$$
$$= -\frac{r_0^2 r^2[r_0^2 - r^2]}{[r_0^2 + r^2]^2}, \quad g_{23} = -g_{32} = \frac{2r_0^3 r^3 \sin\theta}{[r_0^2 + r^2]^2}. \tag{65}$$

The nonzero Christoffel brackets, which are also the symmetric components of the geometrical affine connection $\hat{\Gamma}$, are

$$\left\{\begin{matrix}0\\10\end{matrix}\right\} = \tfrac{1}{2}\frac{y'}{y},$$

$$\left\{\begin{matrix}1\\00\end{matrix}\right\} = \tfrac{1}{2}\frac{yy'}{\omega^2}, \quad \left\{\begin{matrix}1\\11\end{matrix}\right\} = \tfrac{1}{2}\frac{y}{\omega^2}\left(\frac{\omega^2}{y}\right)',$$

$$\left\{\begin{matrix}1\\22\end{matrix}\right\} = \left\{\begin{matrix}1\\33\end{matrix}\right\}\mathrm{cosec}^2\,\theta = -\tfrac{1}{2}\frac{y}{\omega^2}[\omega r^2]',$$

$$\left\{\begin{matrix}2\\12\end{matrix}\right\} = \left\{\begin{matrix}3\\13\end{matrix}\right\} = \tfrac{1}{2}\frac{[\omega r^2]'}{\omega r^2}, \quad \left\{\begin{matrix}2\\33\end{matrix}\right\} = -\sin\theta\cos\theta,$$

$$\left\{\begin{matrix}3\\23\end{matrix}\right\} = \cot\theta, \tag{66}$$

where dashes denote derivation with respect to r.

A rather tedious but straightforward calculation now shows that the only relevant, nonzero second derivatives of $g_{[23]}$ are

$$g_{[23];00} = -\left\{\begin{matrix}1\\00\end{matrix}\right\} g_{[23];1}; \quad g_{[23];11} = [g_{[23];1}]_{,1} - 2\left\{\begin{matrix}2\\12\end{matrix}\right\} g_{[23];1} - \left\{\begin{matrix}1\\11\end{matrix}\right\} g_{[23];1};$$

$$g_{[23];22} = -\left\{\begin{matrix}1\\22\end{matrix}\right\} g_{[23];2} - \left\{\begin{matrix}1\\22\end{matrix}\right\} g_{[23];1};$$

$$g_{[23];33} = -\left\{\begin{matrix}1\\33\end{matrix}\right\}[g_{[21];3} + g_{[23];1}];$$

and

$$g_{[23];1} = g_{[23];1} - 2\left\{\begin{matrix}1\\12\end{matrix}\right\} g_{[23]}; \quad g_{[31];2} = \left\{\begin{matrix}2\\12\end{matrix}\right\} g_{[23]}.$$

It is interesting to note that both

$$g_{[23];23} = 0 = g_{[23];32}. \tag{67}$$

(In the above, the covariant derivatives are with respect to the Christoffel brackets.) We also have

$$g^{\alpha\beta} \underset{00}{g}_{[\mu\nu];\alpha\beta} = -g^{\alpha\beta} \underset{00}{g}_{[\nu\mu];\alpha\beta} \tag{68}$$

which has been used already instead of the original definition

$$\omega_{\mu\nu} = g^{\alpha\beta} \underset{+-}{g}_{[\mu\nu];\underset{(+)}{\alpha\beta}}$$

of the tensor $\boldsymbol{\omega}$ in Ref. [9], which was not skew symmetric. A non-skew electromagnetic field tensor would be too much of a departure from Maxwellian theory to contemplate. Equations (67) show also that in the case of the solution (64) and (65) it does not matter whether we write

$$\omega_{\mu\nu} = \underset{1}{\omega}_{\mu\nu} = g^{\alpha\beta} \underset{00}{g}_{[\mu\nu];\alpha\beta} \text{ or } \omega_{\mu\nu} = \underset{2}{\omega}_{\mu\nu} = a^{\alpha\beta} \underset{00}{g}_{[\mu\nu];\alpha\beta}\,. \tag{69}$$

Close to the origin, the tensors $\underset{1}{\omega}_{\mu\nu}$ and $\underset{2}{\omega}_{\mu\nu}$ are indistinguishable. Indeed elementary calculation shows that

$$\underset{1}{\omega}_{23} = \frac{4r_0r^2[r+cr_0]\,[3r_0^2+r^2]\sin\theta}{[r_0^2+r^2]^3},$$

$$\underset{2}{\omega}_{23} = \frac{12r_0r^2[r+cr_0]}{[r_0^2+r^2]^2}\sin\theta. \tag{70}$$

Since either of these is $V_r r^2 \sin\theta$, where V_r is the radial component of the electromagnetic field vector, we find in both cases that for $r \to 0$,

$$V_r = \frac{12c}{r_0^2} \tag{71}$$

and $V_r \to 0$ as $r \to \infty$.

Let us postpone to the next section the decision whether $\boldsymbol{V}$ is to be identified with the electric induction $\boldsymbol{D}$ or the electric intensity $\boldsymbol{E}$ vector. However, we can settle at once the choice between the two tensors (69). In GFT tensor indices are raised and lowered with the help of the metric tensor $\boldsymbol{a}$ and not with the field tensor $\boldsymbol{g}$. Since either form (69) of $\boldsymbol{\omega}$ gives the same equations of motion result, it is clearly preferable and more natural to choose $\underset{2}{\boldsymbol{\omega}}$ as field tensor. Thus, we finally adopt for the latter

$$\omega = a^{\alpha\beta} \underset{00}{g}_{[\mu\nu];\alpha\beta}. \tag{72}$$

So far, of course, we have only a hint that it represents in fact the electromagnetic field or part thereof.

7. *The electromagnetic field of GFT*

In the electrodynamics of Born and Infeld, nonlinearity is expressed by a relation between the material and the intensity field tensors, $(\boldsymbol{D}, \boldsymbol{H})$ and $(\boldsymbol{E}, \boldsymbol{B})$ in the 3-dimensional notation, and follows from a postulated (nonlinear, and in fact irrational Lagrangian). The electromagnetic theory demanded by GFT is likewise nonlinear but in a somewhat different sense.

Since the problem of the equations of motion requires that one of the field tensors $f'_{\mu\nu} \propto R_{[\mu\nu]}(\tilde{\Gamma})$, with dash donoting merely that it should be calculated in polar coordinates, we can easily restore the exact Coulomb law into the theory. In fact, for the static spherically symmetric solution of the GFT field equations, the only nonzero component of $\boldsymbol{f}'$ is $f'_{23} \propto c \sin\theta$.

If therefore, we agree to write

$$B_k = f_{0k}, \quad f_{ij} = E_k, \; i, j, k = \text{cyclic } 1, 2, 3; \tag{73}$$

we get for one set of the electromagnetic field equations

$$\mathfrak{R}^{[\mu\nu]}{}_{,\nu} = 0, \tag{74}$$

where

$$\mathfrak{R}^{[\mu\nu]} = \sqrt{-a}\, a^{\mu\alpha} a^{\nu\beta} f_{\alpha\beta};$$

suppressing a possible proportionality constant $\mathfrak{R}^{[\mu\nu]}$, of course, is a tensor density. Then $E_r \propto 1/r^2$ as required. We have seen also that equation (29), $\hat{\mathfrak{g}}^{[\mu\nu]}{}_{,\nu} = 0$, is, within the structure of GFT, the only possible choice for the missing set of the electromagnetic field iaws, analogous to the first set of Maxwell's equations though without the current. On the other hand, we have rejected the simple identification of $\hat{\mathfrak{g}}^{[\mu\nu]}$ as the tensor density corresponding to the material tensor because of the difficulty concerning the residual charge in the universe and its possible influence on the expansion of space time.

This leaves little option but to write (again without a proportionality constant which can always be inserted)

$$D_k = \omega_{ij} \quad \text{and} \quad H_k = \omega_{0k}, \tag{75}$$

with $\omega_{\mu\nu}$ given by the Russell-Klotz expression

$$\omega_{\mu\nu} = a^{\alpha\beta} \underset{00}{g}_{[\mu\nu];\alpha\beta}. \tag{72}$$

Considerations of the last Sections now show that corresponding to the static, spherically symmetric solution, we have

$$D_r \propto \frac{r + r_0 c}{[r_0^2 + r^2]^2}\,.$$

Equations (29) can now be seen to look somewhat like Maxwell's equations expressed in terms of the Hertz potential

$$H = \operatorname{curl} \frac{\partial Z}{\partial t}, \qquad \varrho = \operatorname{div} Z,$$

except that, of course, Hertz's solution refers to the wave region of the field whereas our law must be regarded as holding quite generally. This is possible if, in accordance with the program of GFT, we require current density to be introduced by definition in exactly the same way in which an energy-momentum tensor

$$T_{\mu\nu} \overset{\mathrm{Df}}{=} -G_{\mu\nu}[\tilde{\Gamma}^{\alpha}_{(\beta\gamma)}], \tag{76}$$

where $G_{\mu\nu}$ is the Einstein tensor constructed from thesy mmetric part of the geometrical affine connection. It is shown in Ref. [2] that the energy momentum tensor then necessarily involves a nonzero cosmological constant. An analogy with the definition (76) now suggests that we should write

$$\tilde{s}^{\mu\nu} = \frac{1}{2\sqrt{-a}} \varepsilon^{\mu\nu\alpha\beta} \omega_{\alpha\beta}$$

and define the current by

$$j^{\mu} = \overset{\mu\nu}{\underset{\sim}{s}}{}_{;\nu}.$$

Since this is a definition, no minimum coupling hypothesis (MCH) which we certainly do not wish to impose in GFT, is involved but again we obtain a zero charge on the universe. To resolve this problem, we must go back to Maxwellian electrodynamics in the curved space-time of General Relativity determined by equation (76) rather than by the metric hypothesis. There a curious situation arises (Ref. [18]) with respect to the alternative identifications $B_k = f_{ij}$ or $D_k = \omega_{ij}$ of the electromagnetic field vectors. In the standard first case, the second set of Maxwell's equations is automatically covariant but the current definition, with

$$(\varrho, j) \tag{77}$$

being a four-vector requires at first sight MCH. This can be remedied by the equivalent insertion of $\sqrt{-a}$. In the second case adopted above the situation is reversed and it is the second set of Maxwell's equations which seems to call for MCH while the first is automatically covariant providing both the current and the field are tensor densities. In General Relativity of course, it does not matter which choice we care to adopt. If, however, we want to avoid MCH rigorously in GFT then we must identify this four component quantity (77) with a vector density and not with a vector. Thus, the required current vector density becomes

$$J^{\lambda} = k s^{\lambda\sigma}{}_{,\sigma}, \tag{78}$$

where

$$s^{\lambda\sigma} = \tfrac{1}{2}\,\varepsilon^{\lambda\sigma\alpha\beta}\omega_{\alpha\beta} = \sqrt{-a}\,\omega^{\lambda\sigma}, \tag{79}$$

and

$$J^{\lambda} = (\varrho, j), \tag{80}$$

and k (in (78)) is a constant of proportionality. In particular, we now have $s^{01} = \sqrt{-a}\,\omega^{01} = \omega_{23}$ and $\varrho = s^{01}{}_{,1} = \frac{12k}{r_0^3}$. For the metric (64) (that is in the universe) the total charge then becomes

$$q = \int \varrho\sqrt{-a}\,drd\theta d\phi = 48kr_0\pi\int\limits_0^\infty \frac{r^2dr}{[r_0^2+r^2]^2} = 12\pi^2k = e, \tag{81}$$

if

$$k = \frac{e}{12\pi^2}\,.$$

Thus finally, the laws of the electromagnetic field in the generalised field theory are given by

$$\hat{g}^{[\mu\nu]}{}_{,\nu} = 0, \tag{29}$$

$$R^{[\mu\nu]}{}_{,\nu} = 0, \tag{74}$$

$$J^{\lambda} = \frac{e}{12\pi^{-2}}\,s^{\lambda\sigma}{}_{,\sigma}, \tag{78}$$

with the field tensors proportional to $R_{[\mu\nu]}(\tilde{\Gamma})$ and, under the identification (75), to

$$\omega_{\mu\nu} = a^{\alpha\beta}\underset{00}{g}_{[\mu\nu];\alpha\beta}. \tag{72}$$

8. Conclusions

Generalised field theory is a geometrical account of the total, though strictly macrophysical field consisting of gravitation and electromagnetism and exhibiting the global interaction between them. We shall discuss elsewhere (Ref. [7]) its relation to the current unification theories which are more quantum mechanically oriented. What we have sought and found in the present work was an answer to the question whether GFT was sufficiently comprehensive to include the full electromagnetic field equation. That it does, turned out to be the consequence of insisting on not importing into its structure of extraneous and unnecessary assumptions such as the minimum coupling hypothesis or gauge fields, and of retaining the cosmological interpretation with distinguished GFT from the original nonsymmetric theory of Einstein and Straus (Ref. [13]). The result is an electromagnetic theory similar in form to the nonlinear electrodynamics of Born and Infeld. However, its interpretation in terms of Maxwell's field vectors (viz. equations (29) and (72)) is considerably more complicated and this is the price for what might be called the logical purity of the argument employed in determining its structure.

We have seen in Section 2 that the previous attempt (Ref. [6]) to include all electromagnetic equations was inconsistent with the comprehensiveness requirement of GFT. To the discussion described therein, we can add an observation that it is difficult to see how equations (10) and (11) should reduce to Maxwell's electrodynamics in a first approximation. Since the latter can be expected to correspond to a symmetrisation of the field, the reduction is self evident in the present account. Equation (29) which characterises GFT electromagnetism then disappears, a nonsymmetric connection still allows the definition of the intensity tensor even if not by $R_{[\mu\nu]}$, but of course, the material tensor has to be inserted, so to say, by hand. There is no room in GFT for the constitutive relations which are also extraneous to the fundamental theory.

An interesting consequence of excluding additional assumptions is the need to regard Maxwell's four-quantity (ϱ, j) as a vector density rather than a vector. The distinction is of course trivial in a flat space-time and immaterial in General Relativity but, as we have seen it does matter in GFT.

Apart from the introduction of the current vector density (equation (78)) which is a question of definition, we have shown in Section 3 that the electromagnetic field equations follow readily from a variational principle. This cannot be said about the full field equation of GFT without employing either a very artificial Lagrangian or allowing the equations themselves to become complicated beyond any hope of leading to tractable solutions. On the other hand, retention of the "metric hypothesis" (equation (8)) as a restriction on the domain of possible solutions is hardly reprehensible. From the point of view of geometrisation of physics, it is a law which tells us how the geometry of space-time is to be determined. It is shown in Ref. [2] however, that it is not a hypothesis but a theorem which follows from the basic representation of physics in the theory. It is then questionable whether its inclusion in the variational principle is necessary at all.

The discussion of the present work is based exclusively on the static, spherically symmetric solution of the field equations. I shall consider consequences of other symmetries in a separate publication. In particular, cylindrically symmetric fields can be expected to provide us with a GFT model of such things as the Ohm's law and perhaps lead to a result making possible a laboratory verification of the theory.

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