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DIRAC PARTICLE IN FOUR SPECIES

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We establish a formal correspondence between the Duffin-Kemmer-Petiau equation for a spin-0 or spin-1 particle and the Dirac equation for a spin-1/2 particle existing in four species. We hint at the possibility that any elementary Dirac particle appears necessarily in four such species (which might be identified with four fermionic generations).

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The well-known Duffin-Kemmer-Petiau equation for a free spin-0 or spin-1 particle can be written in the form [1]

$$[\tfrac{1}{2}(\gamma_1^\mu+\gamma_2^\mu)p_\mu-M]\psi = 0, \tag{1}$$

where $(\gamma_i^\mu) = (\beta_i, \beta_i\vec{\alpha}_i)$, $i = 1, 2$, are two mutually *commuting* sets of Dirac matrices,

$$\{\gamma_i^\mu, \gamma_i^\nu\} = 2g^{\mu\nu}, \quad [\gamma_1^\mu, \gamma_2^\nu] = 0, \tag{2}$$

so that

$$\gamma_1^\mu = \gamma^\mu\otimes\mathbf{1}, \quad \gamma_2^\mu = \mathbf{1}\otimes\gamma^\mu \tag{3}$$

with $(\gamma^\mu) = (\beta, \beta\vec{\alpha})$ and $\mathbf{1}$ denoting the usual 4×4 Dirac matrices and the 4×4 unit matrix, respectively. In fact, the Duffin-Kemmer-Petiau matrices can be represented as $\beta^\mu = \frac{1}{2}(\gamma_1^\mu+\gamma_2^\mu)$ [1]. Note that such a representation of β^μ leading to Eq. (1) may enable one to interpret the Duffin-Kemmer-Petiau particle as a formal limit of a tight system of two Dirac particles carrying equal momenta $\vec{p}_1 = \vec{p}_2 = \frac{1}{2}\vec{p}$ and equal effective masses $m_1 = m_2 = \frac{1}{2}M$.

In the present note we ask what happens with Eq. (1) if $(\gamma_i^\mu) = (\beta_i, \beta_i\vec{\alpha}_i)$, $i = 1, 2$, instead of being commuting, become now two mutually *anticommuting* sets of Dirac matrices,

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2g^{\mu\nu}\delta_{ij}, \tag{4}$$

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so that they can be represented minimally as the 16×16 matrices

$$\gamma_1^\mu = \gamma^\mu \otimes \mathbf{1}, \quad \gamma_2^\mu = \gamma^5 \otimes i\gamma^\mu\gamma^5 \tag{5}$$

with $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$.

It is easy to see that in this case the 16×16 matrices $\Gamma^\mu = \frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu)$ satisfy the usual Dirac anticommutation relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \tag{6}$$

and so define a reducible representation of the Dirac algebra. Thus the equation

$$\left[\frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu)p_\mu - m\right]\psi = 0 \tag{7}$$

describes now a free spin-1/2 particle carrying in addition to its spin and chirality some other internal degrees of freedom giving rise to four internal states. Evidently, Eq. (7) is the Dirac equation for a free spin-1/2 particle existing in four species.

In order to describe these species more precisely it is convenient to introduce the 16×16 matrices $\Gamma^{\mu\prime} = \frac{1}{\sqrt{2}}(\gamma_1^\mu - \gamma_2^\mu)$ satisfying the anticommutation relations

$$\{\Gamma^\mu, \Gamma^{\nu\prime}\} = 0, \quad \{\Gamma^{\mu\prime}, \Gamma^{\nu\prime}\} = 2g^{\mu\nu}. \tag{8}$$

Then, changing properly the representation (5) we can write

$$\Gamma^\mu = \gamma^\mu \otimes \mathbf{1}, \quad \Gamma^{\mu\prime} = \gamma^5 \otimes i\gamma^\mu\gamma^5. \tag{9}$$

In this representation Eq. (7) assumes the familiar form

$$(\gamma^\mu p_\mu - m)\psi = 0 \tag{10}$$

with γ^μ being the usual 4×4 Dirac matrices. Here, the second 4-valued bispinor index of ψ is free [2]. It is evident that Eq. (10) transforms covariantly under two *commuting* Lorentz groups generated by the matrices

$$\begin{aligned} A_k &= \Gamma^0\Gamma^k = \alpha_k \otimes \mathbf{1}, \\ \Sigma_k &= \Gamma^5 A_k = \sigma_k \otimes \mathbf{1} \end{aligned} \tag{11}$$

and

$$\begin{aligned} A_k' &= \Gamma^{0\prime}\Gamma^{k\prime} = \mathbf{1} \otimes \alpha_k, \\ \Sigma_k' &= \Gamma^{5\prime} A_k' = \mathbf{1} \otimes \sigma_k, \end{aligned} \tag{12}$$

respectively, where

$$\Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3 = \gamma^5 \otimes \mathbf{1} \tag{13}$$

and

$$\Gamma^{5\prime} = i\Gamma^{0\prime}\Gamma^{1\prime}\Gamma^{2\prime}\Gamma^{3\prime} = \mathbf{1}\otimes\gamma^5, \tag{14}$$

while $\alpha_k = \gamma^0\gamma^k$ and $\sigma_k = \gamma^5\alpha_k$, $k = 1, 2, 3$. Only in the case of the first of these groups there exist in Eq. (10) orbital variables, x^μ and p^μ, so that the total generators of these Lorentz groups are

$$J_{\mu\nu} = L_{\mu\nu} + \frac{i}{4}[\Gamma_\mu, \Gamma_\nu] \tag{15}$$

and

$$J'_{\mu\nu} = \frac{i}{4}[\Gamma'_\mu, \Gamma'_\nu], \tag{16}$$

respectively, where $L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$. The first Lorentz group describes the behaviour of our Dirac particle in Minkowski space, while the second is responsible for the existence of four species of this particle.

The covariance under the second Lorentz group must be spontaneously or explicitly broken *if* the masses of four species are to be split. There are no a priori reasons why such a covariance has to be maintained, since the second Lorentz group does not act in the physical space-time where the theory of relativity holds. In particular, there are no a priori objections to try to interpret four species of the Dirac particle described by Eq. (10) as four fermionic generations, three of which being already observed as the leptons e, μ, τ and ν_e, ν_μ, ν_τ and the quarks d, s, b and u, c, t (?). The effective mass operator if introduced into Eq. (10) may break the second Lorentz group as follows:

$$m = m_0 - m_1\Gamma^{5\prime} - m_2\Sigma'_3 + m_3\Gamma^{5\prime}\Sigma'_3, \tag{17}$$

where m_0, m_1, m_2 and m_3 are mass-dimensional constants, while $\gamma^5 = \text{diag}\,(1, 1, -1, -1)$ and $\sigma_3 = \text{diag}\,(1, -1, 1, -1)$ are to be used in $\Gamma^{5\prime}$ and Σ'_3, respectively. In the case of charged leptons, where

$$m = \mathbf{1}\otimes\begin{pmatrix} m_e & 0 & 0 & 0 \\ 0 & m_\mu & 0 & 0 \\ 0 & 0 & m_\tau & 0 \\ 0 & 0 & 0 & m_\omega \end{pmatrix}, \tag{18}$$

Eq. (17) gives

$$\begin{aligned} m_\omega + m_\tau + m_\mu + m_e &= 4m_0, \\ m_\omega + m_\tau - m_\mu - m_e &= 4m_1, \\ m_\omega - m_\tau + m_\mu - m_e &= 4m_2, \\ m_\omega - m_\tau - m_\mu + m_e &= 4m_3, \end{aligned} \tag{19}$$

thus $m_0 > m_1 > m_2 > m_3$. The hypothetical mass relation [3]

$$\frac{m_\omega - m_\tau}{m_\tau - m_\mu} = \frac{m_\tau - m_\mu}{m_\mu - m_e} \simeq (3.99)^2 \tag{20}$$

holds if $m_1 - m_2 = (m_2^2 - m_3^2)^{1/2}$. Then $m_\omega \simeq 28.5$ GeV is our prediction [3] for mass of the fourth charged lepton (for analogical predictions in the case of quarks cf. the second Ref. [3]).

In conclusion, we established a formal correspondence between the Duffin-Kemmer--Petiau equation for a spin-0 or spin-1 particle and the Dirac equation for a spin-1/2 particle existing in four species (which might be identified with four fermionic generations). This Dirac particle is described by such a reducible representation of the Lorentz group in Minkowski space that corresponds formally (when Eq. (4) → Eq. (2)) to the familiar reducible representation describing a Duffin-Kemmer-Petiau particle.

The above correspondence, though formally interesting, cannot be considered as a convincing argument for the physical applicability of the Dirac equation (10) constructed along this line. In such a situation an intriguing question arises, whether our Clifford algebra defined by Eqs (6) and (8) (implying four fermionic species characteristic for Eq. (10)) could not be justified on some profound level of the elementary particle theory. In this note we would like only to hint at such a possibility.

To this end let us imagine that there is a really *close* analogy between the first (or particle) quantization level of the elementary particle theory and its second (or field) quantization level. According to this analogy we should be able to express all dynamical variables and observables appearing on the first quantization level in terms of some novel annihilation and creation operators being functions of the four-vector index $\mu = 0, 1, 2, 3$. (In an analogical way we are able to express all dynamical variables and observables appearing on the second quantization level by the familiar annihilation and creation operators being functions of the four-momentum and polarization.) So, on the first quantization level we should have to our disposal the Bose-Einstein-quantized annihilation and creation operators defined by

$$[c(\mu), c^+(\nu)] = -g^{\mu\nu}, \quad [c(\mu), c(\nu)] = 0 \tag{21}$$

as well as the Fermi-Dirac-quantized annihilation and creation operators determined by

$$\{a(\mu), a^+(\nu)\} = g^\mu_\nu, \quad \{a(\mu), a(\nu)\} = 0. \tag{22}$$

Recall that $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $g^\mu_\nu = \delta^\mu_\nu$.

In terms of $c(\mu)$ and $c^+(\mu)$ we can construct *one* canonical pair of orbital variables,

$$x^\mu \equiv \lambda \frac{c(\mu)+c^+(\mu)}{\sqrt{2}}, \quad p^\mu \equiv \frac{1}{\lambda} \frac{c(\mu)-c^+(\mu)}{i\sqrt{2}}, \tag{23}$$

where λ is a length scale. Note that all x^μ and p^μ are Hermitian [4]. From Eqs. (21) and (23) we *obtain*

$$[x^\mu, x^\nu] = 0, \quad [x^\mu, p_\nu] = -ig^\mu_\nu, \quad [p_\mu, p_\nu] = 0 \tag{24}$$

and hence $p_\mu = i\partial/\partial x^\mu$. We can see that the representation where $c(\mu)$ are diagonal is here analogous to the so called coherent representation frequently used for bosons on the second quantization level when their collective states are considered.

In terms of $a(\mu)$ and $a^+(\mu)$ we can build *two* Dirac-type variables [5],

$$\Gamma^\mu \equiv a(\mu)+g^{\mu\nu}a^+(\nu), \quad \Gamma^{\mu\prime} \equiv \frac{1}{i}\left[a(\mu)-g^{\mu\nu}a^+(\nu)\right]. \tag{25}$$

Note that Γ^0 and $\Gamma^{0\prime}$ are Hermitian, while Γ^k and $\Gamma^{k\prime}$, $k = 1, 2, 3$, are anti-Hermitian. From Eqs. (22) and (25) we *get* our anticommutation relations (6) and (8).

At this point the pertinent question should be asked, what are the physical objects annihilated (created) by the first-quantization annihilation (creation) operators $c(\mu)$ and $a(\mu)$ ($c^+(\mu)$ and $a^+(\mu)$). At the moment we are able to answer this exciting question only in a formal way: they are novel Bose-Einstein and Fermi-Dirac quanta whose states $|\mu\rangle$ are fully characterized by the four-vector index $\mu = 0, 1, 2, 3$. The occupation numbers for the states $|\mu\rangle$ are eigenvalues of the operators $c^+(\mu)c(\mu)$ or $a^+(\mu)a(\mu)$ [6]. The corresponding simultaneous eigenstates span the first-quantization state space for any individual elementary particle which, therefore, can be considered as a collective state of our first-quantization bosons and fermions. We leave to the Reader's imagination the possible physical aspects of the first-quantization quanta. In the present note we wanted only to hint at the idea of such quanta.

In conclusion, under our hypothesis on the first-quantization annihilation and creation operators we derived the anticommutation relations (6) and (8) providing the existence of four species for any elementary Dirac particle because, by our hypothesis, it can be described by $c(\mu)$, $c^+(\mu)$ and $a(\mu)$, $a^+(\mu)$ treated as dynamical variables.

Finally, we would like to mention that under our hypothesis the physical space-time arises from a background dynamics, rather than being an a priori arena of physical events [7]. Such a dynamics is based on the Bose-Einstein annihilation and creation operators $c(\mu)$ and $c^+(\mu)$ which define via Eq. (23) the space-time coordinates x^μ for any elementary particle. Note that for eigenstates of $\vec{x} = (x^k)$ we get the formula

$$|\vec{x}\rangle = \sum_{\vec{n}} |\vec{n}\rangle e^{-\frac{1}{2}\vec{x}^2/\lambda^2} \prod_k (\sqrt{\pi}\, 2^{n_k} n_k!\lambda)^{-1/2} H_{n_k}(x^k/\lambda), \tag{26}$$

where $n_k = 0, 1, 2, \ldots$ are eigenvalues of $c^+(k)c(k)$ and $H_{n_k}(x^k/\lambda)$ denote Hermite polynomials, $k = 1, 2, 3$. In fact,

$$\langle x^k|n_k\rangle = (\sqrt{\pi}\, 2^{n_k} n_k!\lambda)^{-1/2} e^{-\frac{1}{2}(x^k/\lambda)^2} H_{n_k}(x^k/\lambda), \tag{27}$$

since through Eq. (23)

$$\frac{1}{\lambda}\left[c^+(k)c(k)+\tfrac{1}{2}\right] = \tfrac{1}{2}\left(\lambda p_k^2 + \frac{1}{\lambda^3} x^{k^2}\right) \tag{28}$$

and so it is the energy of a one-dimensional harmonic oscillator of mass $1/\lambda$ and zero-point frequency $1/\lambda$.

REFERENCES

[1] Cf. e.g. H. Umezawa, *Quantum Field Theory*, North-Holland, Amsterdam 1956, p. 85.

[2] Note a similarity of Eq. (10) with the Dirac-Kähler equation, E. Kähler, *Rendiconti di Matematica* **21**, 425 (1962). To this end cf. T. Banks, Y. Dothan, D. Horn, *Phys. Lett.* **B117**, 413 (1982). But there are also differences. For them cf. W. Królikowski, *Acta Phys. Pol.* **B14**, 533 (1983).

[3] W. Królikowski, *Acta Phys. Pol.* **B16**, 831 (1985); *Acta Phys. Pol.* **B17**, 557 (1986).

[4] Conservatively speaking, time $x^0 = t$ is a parameter rather than an observable, in contrast to x^k, $k = 1, 2, 3$. Cf., however, P.A.M. Dirac, *The Principles of Quantum Mechanics*, 4th ed., Oxford University Press, Oxford 1958, pp. 253 and 254.

[5] W. Królikowski, *Acta Phys. Pol.* **B13**, 783 (1982).

[6] Note that in the case of $c^+(\mu)c(\mu)$ there is for $\mu = 0$ the problem of the indefinite metric in Hilbert space. For the analogical problem in QED (on the second quantization level) cf. e.g. J. M. Jauch, F. Rohrlich, *The Theory of Photons and Electrons*, 2nd ed., Springer, New-York-Heidelberg-Berlin 1976, p. 103.

[7] For a recent interesting discussion of this long-standing alternative cf. W. Kaplunowsky, M. Weinstein, *Phys. Rev.* **D31**, 1879 (1985).