LETTERS TO THE EDITOR

THE COMPONENTS OF THE ELECTRIC CHARGE DISTRIBUTION AROUND A MAGNETIC MONOPOLE

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(Received July 1, 1986)

The electric charge of the vacuum around a magnetic monopole is calculated using fermionic states normalized in a spherical box of radius $R \to \infty$. The results are compared with results of other authors, who used continuum normalization and with the results for one-dimensional chiral bags. New components of the charge density, which are omitted in the continuum approach, are found.

PACS numbers: 14-80. Hv, 11.10. Qv, 12.40. Aa

1. *Introduction*

Many authors have described systems, where deformations of fermionic spectra by external fields produce fractional fermion numbers, or fractional baryon numbers, or fractional charges. The fermion number of a fermion vacuum can be calculated from the formula [1]

$$B = -\tfrac{1}{2} \lim_{t \to 0^+} \sum_E \operatorname{sign}(E) e^{-|E|t}, \tag{1}$$

where the summation extends over all the single particle fermionic energy levels. This formula is unambiguous, if the fermionic spectrum is discrete. For instance for the inside of a chiral bag sum (1) can be explicitly evaluated [1, 2] and yields the baryon number of the vacuum inside the bag. The same formula, or its analogue for the fermion number density

$$\varrho(x) = -\tfrac{1}{2} \lim_{t \to 0^+} \sum_E \operatorname{sign}(E) \Psi_E^+(x) \Psi_E(x) e^{-|E|t} \tag{2}$$

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is, however, also used for systems where the energy spectrum is continuous. Examples are: calculations for a one-dimensional soliton [3], or for magnetic monopoles [4, 5]. For a review with many references cf. [6].

A calculation of the fermion number from a continuous spectrum may be risky, as shown by the following example. Consider a system with single fermion energy levels

$$E_n = a \operatorname{sign}(n) + \frac{n+\delta}{R}, \qquad n = \pm 1, \pm 2, \dots, \tag{3}$$

where $R > 0$, $a > 0$, and $-1 < \delta < 1$ are constants. Substituting into formula (1) we find after summing geometrical progressions that

$$B = \delta \tag{4}$$

independent of R. The continuum limit corresponds to $R \to \infty$. The energy spectrum in this limit is

$$\varrho(E) = \varrho_0 \theta(E^2 - a^2), \tag{5}$$

where ϱ_0 is a constant and θ is the step function. Formula (5) being symmetric with respect to $E \to -E$ suggests (wrongly) that $B = 0$ independent of δ. Even if it were possible to define some non-zero "spectral asymmetry" (cf. [6]) for the spectrum (5), all information on δ has been lost in the continuum limit and a correct calculation of B, using only (5) as input, is impossible.

In the present paper we calculate the electric charge distribution around an Abelian magnetic monopole. We use a model proposed by Yamagishi [4], but instead of starting with a continuous spectrum, we follow another suggestion from Ref. [4] and enclose the monopole in a spherical box of radius R. The monopole is in the centre of the box. We find that the charge density has two components. One, which is neglected in the continuum approach, is

$$\varrho_1(r) = \frac{Q}{R} \frac{1}{4\pi r^2}, \tag{6}$$

where Q is the total electric charge around the monopole. For $R \to \infty$ ϱ_1 locally tends to zero, but its integral over the whole box yields Q. The other component $\varrho_2(r) = \varrho(r) - \varrho_1(r)$ yields zero when integrated over the whole box. For non-zero fermion mass, however, the limit of $\varrho_2(r)$ for r fixed and $R \to \infty$ differs from zero. This, when integrated over all space, again yields Q. In the continuum calculation only this limiting distribution (further denoted as ϱ_{21}) is calculated [4]. The obvious question is: can one in general replace the density $\varrho(r)$ by the limit $\varrho_{21}(r)$ of the component $\varrho_2(r)$. If such a rule exists, it cannot be quite general, because for zero mass fermions both ϱ_2 and ϱ_{21} vanish, while $Q \neq 0$.

2. *The model*

The model used in the present paper for the Abelian monopole is taken from Ref. [4], where further details and references to earlier work can be found. The Dirac equation for a fermion in a spherical box with a monopole in the centre of the box is separable.

The solutions of the spin-angular equation define the partial waves. Only the lowest partial wave

$$j = |q| - 1/2 \tag{7}$$

contributes to the vacuum charge. Here $q = eg$, where e is the electric charge of the fermion and g is the magnetic charge of the monopole. From the quantization condition for magnetic monopoles $q = 1/2 \times integer$. The radial equation is

$$H_0\chi(r) \equiv (-i\hat{q}\sigma_x \frac{d}{dr} + m\sigma_z)\chi(r) = E\chi(r), \tag{8}$$

where $\hat{q} = \text{sign}\,(q)$, m is the fermion mass and E is the energy eigenvalue. The wave function (more precisely wave section cf. [7]) of the fermion (Ψ) has as usual four components, but χ has only two, because when forming Ψ each of the components of χ is multiplied by a two-component spin-orbital part. Equation (8) is formally identical with the Dirac equation for a fermion with a fixed spin projection ($\Sigma_x\, \Psi = \lambda\Psi$) in a one-dimensional bag along the x-axis (cf. e.g. [8]). In the bag case λ replaces sign (q). The normalization condition for function χ is

$$\int_0^R \chi^+(r)\chi(r)dr = 1. \tag{9}$$

Thus the contribution to the electric charge density from a filled shell corresponding to one eigenvalue E is

$$e\sum_\mu \Psi^+(r)\Psi(r) = \frac{|q|e}{2\pi r^2}\chi^+(r)\chi(r), \tag{10}$$

where μ is the angular momentum projection and (7) is assumed.

The boundary conditions for χ must be chosen so as to make the operator H_0 self--adjoint. Following [4] we assume

$$\chi(0) \sim \begin{pmatrix} i\sin(\theta_0/2+\pi/4) \\ \cos(\theta_0/2+\pi/4) \end{pmatrix}, \quad \chi(R) \sim \begin{pmatrix} i\sin(\theta_R/2+\pi/4) \\ \cos(\theta_R/2+\pi/4) \end{pmatrix} \tag{11}$$

where θ_0 and θ_R are constants. A more general boundary condition guaranteeing the self--adjointness of H_0 depends on four parameters [9] ,but the model with the boundary conditions (11) is sufficiently rich for our purpose. The corresponding condition for the one dimensional chiral bag is

$$\pm\lambda\sigma_y\chi(x) = \exp[\pm i\theta_c\sigma_x]\chi(x) \quad \text{for} \quad x = \pm R_b. \tag{12}$$

Here χ is the two-component wave function obtained from Ψ by fixing the spin projection $\lambda = \pm 1$, θ_c is the chiral angle and R_b the bag radius. Identifying $r = 0$ with $x = -R_b$

and $r = R$ with $x = +R_b$ we obtain by comparison of formulae (11) and (12)

$$\theta_0 = \theta_c + \frac{\pi}{2}(\lambda - 1); \quad \theta_R = -\theta_c - \frac{\pi}{2}(\lambda + 1). \tag{13}$$

Each of the angles θ_0, θ_R, θ_c can be changed (without changing the physics) by adding or subtracting arbitrary multiplies of 2π.

3. Total electric charge

Using the formulae (1) and (7) we find for the total electric charge

$$Q = -|q|e \lim_{t \to 0^+} \sum_E \varepsilon e^{-|E|t}, \tag{14}$$

where $\varepsilon = \mathrm{sign}\,(E)$ and the summation extends over all the eigenvalues of the equation (8) with the boundary conditions (11). These eigenvalues are

$$E = \varepsilon \sqrt{k^2 + m^2}, \tag{15}$$

where the allowed values of k are the positive roots[1] of the equation

$$\mathrm{tg}\,(kR) = \hat{q}k \frac{t_R - t_0}{E + m + (E - m)t_R t_0} \tag{16}$$

with

$$t_i = \mathrm{tg}\,(\theta_i/2 + \pi/4), \quad i = 0, R. \tag{17}$$

Since the roots k depend on ε, the energy spectrum is not symmetric with respect to $E \to -E$. In the continuum case such a symmetry, according to [6], follows from the existence of a matrix which anticommutes with the Hamiltonian. In the discrete case the matrix σ_y anticommutes with H_0, but if χ is a solution, then in general $\sigma_y \chi$ does not satisfy the boundary conditions. The symmetry of the spectrum does not, therefore, follow from this.

For mR sufficiently large equation (16) has no solution for $kR < \pi/2$. Let us first consider the case $t_0 t_R > 0$. Then in each interval $(n - 1/2)\pi < kR < (n + 1/2)\pi$, $n = 1, 2, \ldots$ there is one positive energy solution and one negative energy solution. Therefore, the contribution to sum (14) from the energy levels corresponding to $kR < (N - 1/2)\pi$, where $N \gg mR$ is fixed arbitrarily, vanishes. For $kR > (N - 1/2)\pi$ the fermion mass can be neglected and equation (16) reduces to

$$\mathrm{tg}\,(kR) = \varepsilon \hat{q}\, \mathrm{tg}\,((\theta_R - \theta_0)/2) \tag{18}$$

and

$$|E| = k = n\pi/R + \varepsilon \hat{q} \Delta / R, \quad n = N, \quad N + 1, \ldots . \tag{19}$$

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[1] There may be also energy levels corresponding to purely imaginary k [4, 8]. We will not discuss them in the present paper.

Here

$$\Delta = (\theta_R - \theta_0)/2 + \nu\pi, \tag{20}$$

where the integer ν is chosen so that $|\Delta| \leqslant \pi/2$. Substituting (19) into (14), summing the geometrical progressions and taking the limit $t \to 0$ one finds

$$Q = 2qe\Delta/\pi, \quad \text{for} \quad t_0 t_R > 0. \tag{21}$$

For $t_0 t_R < 0$ the interval where $E+m+(E-m)t_0 t_R$ changes sign must be discussed separately. One finds that in this interval one root is added or lost and the corrected formula for the charge is

$$Q = 2qe\Delta/\pi - qe \operatorname{sign}[\operatorname{tg}((\theta_R - \theta_0)/2)], \quad \text{for} \quad t_0 t_R < 0, \quad t_0 t_R \neq -1. \tag{22}$$

As a special case we consider the one-dimensional chiral bag model with $|\theta_c| < \pi/2$. From formula (13) one finds

$$\Delta = -\theta_c + \frac{\pi}{2} \operatorname{sign} \theta_c; \quad \operatorname{sign} \operatorname{tg}((\theta_R - \theta_0)/2) = \operatorname{sign} \theta_c \tag{23}$$

and $t_0 t_R < 0$. Thus, replacing $\hat{q}$ by λ and removing the factor $2j+1 = 2|q|$, one rederives the well-known formula (cf. e.g. [8])

$$Q_{\text{Bag}} = -\lambda\theta_c/\pi. \tag{24}$$

Formulae (21) and (22) were derived for $m \neq 0$. The case $m = 0$ is much simpler. Then the denominator in (18) never vanishes and for $t_R t_0 \neq -1$ formula (18) is exact for all the energy levels. Summing the geometrical series and including (for ε sign $(q \operatorname{tg} \Delta) > 0$) the additional energy level corresponding to $kR = |\Delta| < \pi/2$ one finds formula (22) for all $t_R t_0 \neq -1$. Thus there is nothing exceptional about the $m = 0$ case, unlike in the continuum approach.

In the continuum approach θ_R is undefined and one finds [4]

$$Q = -qe\bar{\theta}_0/\pi, \tag{25}$$

where $\bar{\theta}_0 = \theta_0 - 2\pi\nu_0$ and ν_0 is an integer chosen so that $|\bar{\theta}_0| < \pi$. This result can be also derived from general symmetry considerations [10]. Let us note that the result (25) can be obtained from (21) or (22) by averaging over θ_R. Indeed, defining θ_R for the continuum case by formula (11) one finds that all the values of θ_R are possible there.

4. *Charge density distribution*

According to the formulae (2) and (10) the charge density distribution is

$$\varrho(\boldsymbol{x}) = -\frac{|q|e}{4\pi r^2} \lim_{t \to 0^+} \sum_E \varepsilon \chi_E^+(r) \chi_E(r) e^{-|E|t}. \tag{26}$$

The solution $\chi_E(r)$ of equation (8), which satisfies the boundary conditions (11) can be written in the form

$$\chi_E(r) = \mathcal{N}_0 \left[\cos\left(\frac{\theta_0}{2}+\frac{\pi}{4}\right)\begin{pmatrix} \frac{ik\hat{q}}{E-m} \sin kr \\ \cos kr \end{pmatrix} + i \sin\left(\frac{\theta_0}{2}+\frac{\pi}{4}\right)\begin{pmatrix} \cos kr \\ \frac{ik\hat{q}}{E+m} \sin kr \end{pmatrix}\right], \tag{27a}$$

or equivalently

$$\chi_E(r) = \mathcal{N}_R \left[\cos\left(\frac{\theta_R}{2}+\frac{\pi}{4}\right)\begin{pmatrix} -\frac{ik\hat{q}}{E-m} \sin k\varrho \\ \cos k\varrho \end{pmatrix} + i \sin\left(\frac{\theta_R}{2}+\frac{\pi}{4}\right)\begin{pmatrix} \cos k\varrho \\ -\frac{ik\hat{q}}{E+m} \sin k\varrho \end{pmatrix}\right]. \tag{27b}$$

Here $\mathcal{N}_i$ are normalization constants, which for $R \to \infty$ behave as $1/\sqrt{R}$, and $\varrho = R-r$. Substituting formula (27a) into (26) we obtain

$$\varrho(x) = -\frac{|q|e}{4\pi r^2} \lim_{t\to 0^+} \sum_E \varepsilon \mathcal{N}^2 \left[1 - \mathrm{Re}\, \frac{m(m-E\sin\theta_0 + ik\hat{q}\cos\theta_0)}{E(E-m\sin\theta_0)} e^{2ikr}\right], \tag{28}$$

where $\mathcal{N}$ is another normalizing constant behaving as $1\sqrt{R}$ for $R \to \infty$ and Re means real part of.

We have not been able to perform the summation (28) analytically and finally the sum was evaluated numerically. Some contributions to the sum, however, can be evaluated analytically and are interesting. We discuss them in turn.

a) Divergent part

The evaluation of the sum including only the leading term $1/R$ of $\mathcal{N}^2$ and the first term (unity) from the square bracket in (28) is a repetition of the calculation performed in the preceding section. It yields (cf. (6))

$$\varrho_1(x) = \frac{Q}{R}\frac{1}{4\pi r^2}, \tag{29}$$

where Q should be taken from formulae (21) or (22). For $R \to \infty$ this contribution can be neglected in any fixed finite region of space. It is neglected in the continuum calculation [4]. Its integral over the whole box, however, yields the whole charge Q. Nevertheless, inside the box, the potential due to chargedist ribution (29) at a distance r from the centre is

$$V(r) = \frac{Q}{R} = \text{const} \to 0. \tag{30}$$

Thus for $r \to \infty$ it is possible to develop inside the box a scattering theory with noninteracting |in⟩ and |out⟩ states. This supports the calculations performed using formal scattering theory, e.g. in Ref. [4].

Sum (28), after the divergent part (29) has been extracted, is convergent for $t = 0$ and the limit $t \to 0$ can be performed term by term. One finds for the remaining charge density

$$\varrho_2(x) = \frac{|q|e}{4\pi r^2} \operatorname{Re} \sum_E \varepsilon \mathcal{N}^2 \frac{m(m - E \sin\theta_0 + ik\hat{q}\cos\theta_0)}{E(E - m\sin\theta_0)} e^{2ikr}, \tag{31}$$

where corrections $O(R^{-2})$ to (29) due to the fact that $\mathcal{N}^2 = R^{-1} + O(R^{-2})$ have been dropped. In the $R \to \infty$ limit they do not contribute.

b) Leading term for r fixed and $R \to \infty$

For r fixed and $R \to \infty$ it is legitimate to replace the sum (31) by an integral over k from zero to infinity. Extending this integration region from $-\infty$ to $+\infty$ (the integrand is even in k), deforming the integration path in the complex k plane and dropping a possible contribution from a pole[2] one finds [4] the contribution to the charge density

$$\varrho_{20}(r) = -\frac{qem\sin\theta_0}{4\pi^2 r^2} \int_m^\infty \frac{dk}{\sqrt{k^2 - m^2}} \frac{k}{k + m\cos\theta_0} e^{-2kr}. \tag{32}$$

This is the only term calculated in the continuum approach. When integrated over the whole box it gives in the limit $R \to \infty$ (cf. (25))

$$Q_0 = -qe\frac{\theta_0}{\pi}. \tag{33}$$

This is the standard result (cf. e.g. [4]). Contribution (32) is peaked at $r = 0$ and is negligible for $r \gg m$.

c) Leading term for ϱ fixed and $R \to \infty$

For ϱ fixed and $R \to \infty$ it would be possible to repeat the procedure described in the preceding point starting from expression (27b) instead of (27a). It is simpler, however, to notice that formula (27b) may be obtained from (27a) after the following operations:
(*i*) Replace r by ϱ.
(*ii*) Take the complex conjugation.
(*iii*) Replace θ_0 by $-\theta_R - \pi$.
Since the complex conjugation does not change the product $\chi^+\chi$, the analogue of formula (33) is

$$Q_R = qe\frac{\overline{\theta_R + \pi}}{\pi}, \tag{34}$$

[2] Including this contribution while omitting solution with imaginary k is inconsistent cf. [4].

where the bar denotes (as before) that $\theta_R + \pi$ is shifted by a multiple of 2π so that $|\theta_R+\pi| < \pi$. The corresponding charge density distribution is peaked at $\varrho = 0$ i.e. $r = R$ and is negligible for $R-r \gg m$. Contribution (34) to the integrated charge is a priori not less important than (33). One could argue that since $R \to \infty$, we are not interested in what happens at $r \approx R$,.but the spectral asymmetry is correlated with the full charge in the box and if it is used to define the charge around the monopole, it is irrelevant how far from the origin a contribution to the charge density has its support.

d) Numerical summation of sum (31)

Series (31) converges very slowly, because the n-th term is of order $1/n$. It is easy, however, to extract from each term in (31) the part of order $1/n$ and to sum the extracted sum analytically[3]. The result is

$$\varrho_{\log}(x) = \frac{qem}{2\pi^2 r^2} \sin\left[\theta_0 + \frac{r}{R}(\theta_R - \theta_0)\right] \ln\left|2 \sin\frac{\pi r}{R}\right| . \tag{35}$$

These logarithmic singularities at $r = 0$ and $r = R$ can be also obtained from integral (32) and its analogue for the case (6). The remaining sum can be easily performed, even on a pocket calculator. We find that for some values of the parameters θ_0 and θ_R the sum in the region $r = \alpha R$, where α is fixed for $R \to \infty$, tends to cancel the contribution ϱ_1 (29). We have not made, however, a complete analysis in the full range of the angles θ_0 and θ_R.

5: *Conclusions*

The electric charge distribution around an Abelian magnetic monopole can be calculated using the finite box normalization and then performing the limiting transition $R \to \infty$, where R is the box radius.

Many details of the resulting picture differ from those obtained from continuum calculation: The divergent term, which is dropped in the continuum calculation, here gives all the electric charge. The peak at $r \approx 0$ gives a contribution which cancels with that of another peak at $r \approx R$ and that from the charge distribution at intermediate distances. The angle θ_R, which does not appear in the continuum calculation, strongly affects the results.

Nevertheless, the charge distributions obtained by the two methods for any finite region around the monopole coincide. The use of scattering theory as e.g. in Ref. [4] can be justified. An averaging over θ_R can remove the extra charge given by the box calculation.

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[3] cf. Ref. [8] for a similar calculation in the case of a one-dimensional chiral bag.

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