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THE ANALYSIS OF THE SVZ METHOD APPLIED TO THE SCHRÖDINGER EQUATION WITH THE POTENTIAL $V = \lambda \operatorname{ctg}^2 \pi x$

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The SVZ method is investigated within a non-relativistic model with the potential $V = \lambda \operatorname{ctg}^2 \pi x$. Three expansions of the non-relativistic analogon of the exponential moment are considered: the short-time, the weak-coupling and the quasiclassical (in powers of $\hbar$) expansions. For potentials which have been studied by now these expansions were indistinguishable. It turns out that the SVZ method applied to the potential $V = \lambda \operatorname{ctg}^2 \pi x$ works well for small values of the coupling strength λ only if either the short-time or the weak-coupling expansions are used. For large λ only the expansion in powers of $\hbar$ leads to satisfactory results.

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1. Introduction

In 1979 Shifman, Vainshtein and Zakharov (SVZ) introduced an interesting method of predicting the parameters of the bound states of a quark-antiquark pair (e.g. resonances J/ψ, Υ) from QCD [1]. The SVZ method relates the masses of the $q\bar{q}$ resonances to the value of the gluon condensate $\langle \Omega|GG|\Omega \rangle$, which is a nonperturbative parameter. The value of $\langle \Omega|GG|\Omega \rangle$ once determined from the data for a fixed resonance (for example J/ψ) can be applied to predict the masses of other resonances. The results of this procedure are in very good agreement with experimental data.

In order to understand the unexpected success of the SVZ method Bell and Bertlmann [2, 3] as well as Durand, Durand and Whitenton [4] have studied the SVZ method within non-relativistic potential models, where the $q\bar{q}$ system is described with use of the Schrödinger equation. In this case the analogon of the nonperturbative effects is a long range confining potential $V = \lambda v(x)$, where $v(x) \to \infty$ for $|x| \to \infty$. The coupling strength λ of the potential corresponds to the gluon condensate $\langle \Omega|GG|\Omega \rangle$ [5]. The conclusion of Refs [2–4] is that the SVZ method does not work well in the non-relativistic case (for example

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the gluon condensate is underestimated by a factor 2.5–3). In Ref. [6] Novikov, Shifman, Vainshtein, Voloshin and Zakharov suggested that the criticism of Refs [2–4] was based on misunderstanding and they gave their own orthodox non-relativistic version of the SVZ method, which we shall follow in the present work.

Let us consider the exponential moment

$$\mathcal{M}(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n}|\psi_n(0)|^2 = G_E(0, 0, \beta), \tag{1.1}$$

which is the non-relativistic analogon of the Borel-transformed vacuum polarization in the field theory [2–4]. $G_E(x, y, \beta)$ denotes the Euclidean Green function. The exponential moment $\mathcal{M}(\beta)$ can be calculated by taking the first two or three terms either of the expansion in powers of the parameter λ (the standard perturbative Born expansion [2–4]) or of the expansion in powers of the parameter β (the short-time or high temperature approximation [7–9]). Sometimes one can calculate the leading nonperturbative term [8, 9]. In the present work we consider also the Wigner-Kirkwood expansion in powers of the Planck constant $\hbar$. The relationship between the above-mentioned expansions is interesting, because it is not clear which of them is the non-relativistic analogon of the expansion of $\mathcal{M}(\beta)$ used by SVZ in Ref. [1].

The sum for $n \geqslant 1$ in (1.1) can be estimated by an integral:

$$\mathcal{M}(\beta) = e^{-\beta E_0}|\psi_0(0)|^2 + \int_{E_c}^{\infty} e^{-\beta E}\varrho(E)dE. \tag{1.2}$$

The integral on the r.h.s. is the contribution of higher states and is called the continuum contribution. The density $\varrho(E)$ is calculated in the approximation of free motion. It can be also calculated with use of the WKB approximation [8]. If there exists a domain of β where the approximations are satisfactory (the fiducial region, see the next Section), it is possible to obtain the ground state energy E_0 (which is the non-relativistic analogon of the mass of the lowest resonance) by fitting simultaneously E_0 and the effective continuum threshold E_c from Eq. (1.2). In the present work we use the partition function $Z(\beta)$ instead of the exponential moment $\mathcal{M}(\beta)$. Such a modification simplifies the calculations significantly and does not affect the final qualitative conclusions about the SVZ method.

The following potentials have been used in order to test the non-relativistic version of the SVZ method: the harmonic oscillator potential $V = m\omega^2x^2/2$ [6, 7]; the square--well potential $V = \lim_n x^n$ [7–9]; the linear potential $V = \lambda x$ [7]; the linear plus Coulomb potential $V = \lambda x + \alpha_s/x$ [2, 4]; the cubic plus Coulomb potential $V = \lambda x^3 + \alpha_s/x$ [2]; power potentials $V = \lambda x^s$, where $\mathrm{sgn}(\lambda) = \mathrm{sgn}(s)$ [2–5]; superpositions of power potentials $V = \Sigma_s \,\mathrm{sgn}(s)\lambda(s)x^s$ or $V = \int \mathrm{sgn}(s)\lambda(s)x^s ds$, where $\lambda(s) > 0$ [2, 4]. For each of these potentials the exponential moment depends on an expression of the form $\lambda^a\beta^b\hbar^c$. Thus each of the expansions in powers of λ, β or $\hbar$ yields the same series.

In this work we consider a confining potential

$$V = \lambda \,\mathrm{ctg}^2\, \pi x. \tag{1.3}$$

The potential is chosen in such a way that the expansions with respect to λ, β and $\hbar$ are essentially different. We analyse how the SVZ method works in these three cases and come to the conclusion that the expansions in powers of λ and β lead to similar fiducial regions, but the fiducial region for the expansion in powers of the Planck constant $\hbar$ is considerably different. We find the following fact very interesting: The union of the fiducial regions for the considered expansions covers all values of the coupling constant λ (which is the non-relativistic analogon of the gluon condensate). For small λ one should use either the expansion in powers of β (the short-time approximation) or the expansion in powers of λ (the weak-coupling approximation). For large λ it is better to work with the expansion in powers of $\hbar$ (the quasiclassical approximation).

2. *The SVZ method and the partition function*

Let us remind of the formula for the exponential moment

$$\mathcal{M}(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n} |\psi_n(0)|^2 = G_E(0, 0, \beta). \tag{2.1}$$

Instead of $G_E(0, 0, \beta)$ one could put $G_E(x, x, \beta)$ for any x, because the choice of $x = 0$ is arbitrary. The dependence on x is not crucial for the SVZ method, thus we propose to replace the exponential moment by the partition function, which we get integrating $G_E(x, x, \beta)$ over the x axis

$$Z(\beta) = \int G_E(x, x, \beta) dx = \sum_{n=0}^{\infty} e^{-\beta E_n}. \tag{2.2}$$

This replacement simplifies the calculations and does not influence the final qualitative conclusions.

Let us consider the sum over the higher states

$$Z_c(\beta) = \sum_{n=1}^{\infty} e^{-\beta E_n}. \tag{2.3}$$

We shall call it the continuum contribution. Following the SVZ method we replace the sum by an integral

$$Z_c(\beta) = \int_{E_c}^{\infty} e^{-\beta E} \varrho(E) dE \tag{2.4}$$

with a density $\varrho(E)$, which we calculate using the WKB approximation [8]. From (2.3) and (2.4) one can see that

$$\varrho(E) = \frac{dn}{dE}. \tag{2.5}$$

Differentiating the Bohr-Sommerfeld quantization condition

$$\int_{x_1}^{x_2} \sqrt{2m(E-V(x))}\, dx = \pi \hbar n + a \tag{2.6}$$

with respect to E (under the assumption that a does not depend on E) we obtain

$$\varrho(E) = \frac{dn}{dE} = \frac{1}{2\pi\hbar}\int_{x_1}^{x_2} \frac{\sqrt{2m}}{\sqrt{E-V(x)}}\,dx = \frac{T_E}{2\pi\hbar}. \tag{2.7}$$

x_1 and x_2 denote the classical turning points and T_E denotes the classical period of a particle of energy E moving in a field of forces with the potential V.

From (2.2) and (2.3) we have

$$Z(\beta) = e^{-\beta E_0} + Z_c(\beta). \tag{2.8}$$

According to the SVZ procedure the continuum contribution $Z_c(\beta)$ is calculated from the approximation (2.4) with $\varrho(E)$ given by (2.7). The partition function $Z(\beta)$ is calculated by taking two or three terms of its expansion in powers of a respectively chosen parameter (β, λ or $\hbar$) and, if it is possible, also the leading nonperturbative term. In order to determine the ground state energy E_0 from Eq. (2.8) it is necessary to know the fiducial region, where the above approximations are good enough. Following the suggestions of the authors of Ref. [6] we shall use the following criteria: With respect to the leading term of the expansion of the partition function $Z(\beta)$

A) the leading power correction should not exceed 30 per cent;

B) the leading nonperturbative term (if it exists) should not exceed 30 per cent;

C) the continuum correction should not exceed 30 per cent.

Criteria A) and B) lead to the upper bounds β_A and β_B and criterion C) leads to the lower bound β_C of the interval of β (fiducial region)

$$\beta_C < \beta < \min(\beta_A, \beta_B), \tag{2.9}$$

where the SVZ method should work well. The ground state energy E_0 is calculated by requiring that Eq. (2.8) with $Z_c(\beta)$ given by (2.4) is satisfied in the interval (2.9) and fitting E_0 and E_c simultaneously.

3. Three expansions of $Z(\beta)$ for the potential $V = \lambda\,\mathrm{ctg}^2\,\pi x$

Consider a particle of mass m in one dimension subjected to the potential field

$$V = \lambda\,\mathrm{ctg}^2\,\pi x. \tag{3.1}$$

The energy levels of such a particle can be calculated from the formula [10]:

$$E_n = \frac{\pi^2\hbar^2}{2m}\left[(n+1)^2 + 2(n+1)a - a\right], \qquad n = 0, 1, 2, \ldots, \tag{3.2}$$

where

$$a = \tfrac{1}{2}\left[(1+8m\lambda/\pi^2\hbar^2)^{1/2} - 1\right]. \tag{3.3}$$

Let us put for convenience

$$\bar{\beta} = \beta\pi^2\hbar^2/2m, \quad \bar{\lambda} = 2\lambda m/\pi^2\hbar^2, \quad \bar{E} = 2Em/\pi^2\hbar^2. \tag{3.4}$$

We shall apply the SVZ method to the partition function

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} \exp\{-\bar{\beta}[(n+1)^2+2(n+1)a-a]\}. \tag{3.5}$$

First of all we calculate the continuum contribution from the formulas (2.4) and (2.7)

$$Z_c(\beta) = \frac{1}{2\pi\hbar}\int_{E_c}^{\infty} e^{-\beta E} T_E dE = \tfrac{1}{2}\int_{\bar{E}_c}^{\infty} e^{-\bar{\beta}\bar{E}} \frac{d\bar{E}}{\sqrt{\bar{E}+\bar{\lambda}}}. \tag{3.6}$$

The expansion of $Z(\beta)$ in powers of β can be calculated by use of the Poisson formula [7, 8]

$$\sum_{n=-\infty}^{\infty} \phi(n) = \sum_{k=-\infty}^{\infty} \hat{\phi}(k), \tag{3.7}$$

where $\wedge$ denotes the Fourier transformation. We get the expression

$$Z(\beta) = \left\{-\tfrac{1}{2} + \int_0^{\infty} \exp[-\bar{\beta}(x^2+2ax)]dx \right.$$

$$\left. +2\sum_{k=1}^{\infty}\int_0^{\infty} \exp[-\bar{\beta}(x^2+2ax)]\cos(2\pi kx)dx\right\}\sum_{n=0}^{\infty}\frac{a^n}{n!}\bar{\beta}^n \tag{3.8}$$

from which we find the short-time approximation for $Z(\beta)$

$$Z(\beta) = (-\tfrac{1}{2} + \int_0^{\infty} \exp[-\bar{\beta}(x^2+2ax)]dx)$$

$$+a\bar{\beta}(-\tfrac{1}{2} + \int_0^{\infty} \exp[-\bar{\beta}(x^2+2ax)]dx)$$

$$+2\int_0^{\infty} \exp[-\bar{\beta}(x^2+2ax)]\cos(2\pi x)dx. \tag{3.9}$$

This expression consists of the leading term as well as both the leading perturbative and the leading nonperturbative corrections.

Now consider the expansion in powers of λ. We shall apply the Born expansion

$$G_E(x, y, \beta) = \sum_{n=0}^{\infty}(-1)^n\int_0^{\beta} d\beta_n\int_0^{\beta_n} d\beta_{n-1}\cdots\int_0^{\beta_2} d\beta_1\int_0^1 dx_n\cdots\int_0^1 dx_1$$

$$G_0(x, x_n, \beta-\beta_n)V(x_n)G_0(x_n, x_{n-1}, \beta_n-\beta_{n-1})\cdots G_0(x_2, x_1, \beta_2-\beta_1)V(x_1)G_0(x_1, y, \beta_1), \tag{3.10}$$

where $G_0(x, y, \beta)$ denotes the Euclidean Green function for the square-well potential with walls at 0 and 1;

$$G_0(x, y, \beta) = 2 \sum_{n=0}^{\infty} \sin[(n+1)\pi x] \sin[(n+1)\pi y] \exp[-\bar{\beta}(n+1)^2], \tag{3.11}$$

and the potential $V = \lambda \operatorname{ctg}^2 \pi x$ is assumed to be a small perturbation of the Hamiltonian describing a particle in a square-well potential. Setting in (3.10) $x = y$ and integrating over the x axis one can obtain the expansion

$$Z(\beta) = \sum_{n=0}^{\infty} (-1)^n \int_0^{\beta} d\beta_n \int_0^{\beta_n} d\beta_{n-1} \ldots \int_0^{\beta_2} d\beta_1 \int_0^1 dx_n \ldots \int_0^1 dx_1$$

$$G_0(x_1, x_n, \beta+\beta_1-\beta_n) V(x_n) G_0(x_n, x_{n-1}, \beta_n-\beta_{n-1}) \ldots G_0(x_2, x_1, \beta_2-\beta_1) V(x_1). \tag{3.12}$$

Since $V = \lambda \operatorname{ctg}^2 \pi x$, this expansion is in powers of λ. Taking the first two terms of this expansion and setting the expression (3.11) for the Green function $G_0(x, y, \beta)$ we get at the weak-coupling approximation

$$Z(\beta) = \sum_{n=0}^{\infty} \exp[-\bar{\beta}(n+1)^2] - \bar{\lambda}\bar{\beta} \sum_{n=0}^{\infty} (2n+1) \exp[-\bar{\beta}(n+1)^2]. \tag{3.13}$$

The quasiclassical approximation of $Z(\beta)$ can be derived from the Wigner-Kirkwood expansion of $Z(\beta)$ in powers of the Planck constant $\hbar$ [11]

$$Z(\beta) = \frac{1}{2\pi\hbar} \sum_{n=0}^{\infty} \int\int \hbar^n \chi_n(p, x) \exp\left\{-\beta\left[\frac{p^2}{2m} + V(x)\right]\right\} dp\, dx, \tag{3.14}$$

where

$$\chi_0 = 1$$

$$\chi_1 = -\frac{i\beta^2 p}{2m}\left(\frac{dV}{dx}\right)$$

$$\chi_2 = -\frac{\beta^4 p^2}{8m^2}\left(\frac{dV}{dx}\right)^2 + \frac{\beta^2 p^2}{6m^2}\left(\frac{d^2V}{dx^2}\right) + \frac{\beta^3}{6m}\left(\frac{dV}{dx}\right)^2 - \frac{\beta^2}{m}\left(\frac{d^2V}{dx^2}\right)$$

$$\vdots \tag{3.15}$$

Let us neglect in (3.14) all terms for $n \geqslant 3$. This leads to the approximation

$$Z(\beta) = \tfrac{1}{2} \int_0^{\infty} e^{-\bar{\beta}\bar{E}} \frac{d\bar{E}}{\sqrt{\bar{E}+\bar{\lambda}}} - \left(\frac{\bar{\beta}\sqrt{\bar{\lambda}}}{12} + \frac{1}{8\sqrt{\bar{\lambda}}}\right). \tag{3.16}$$

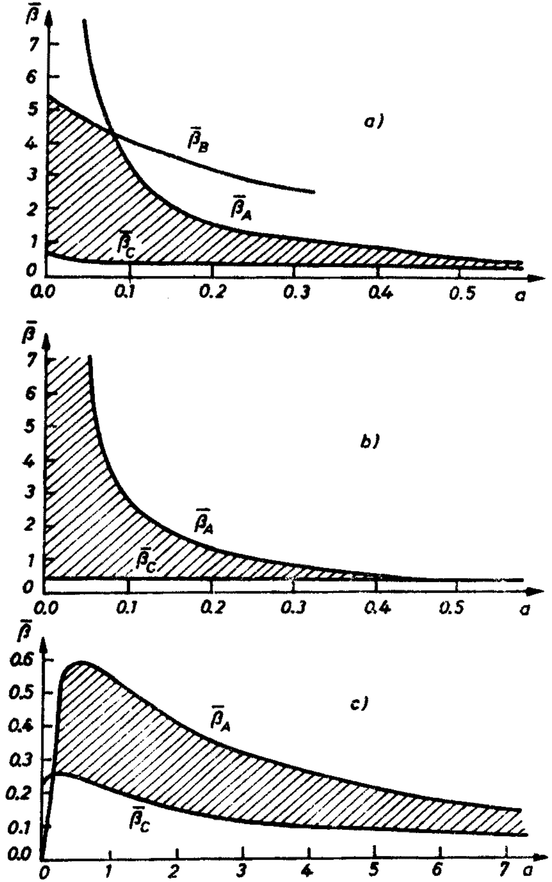

Fig. 1

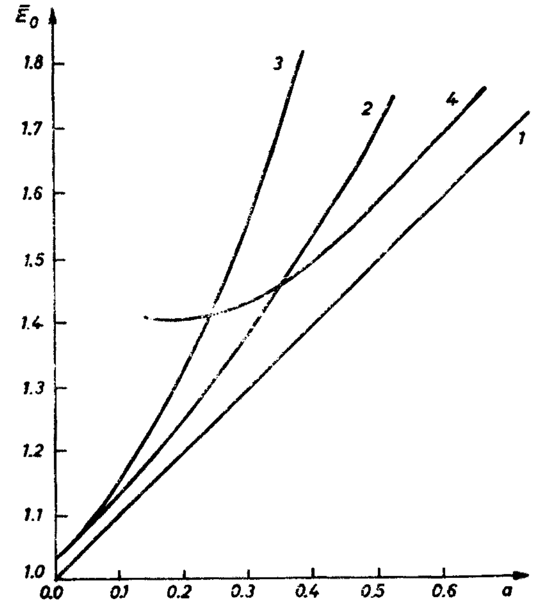

Fig. 2

Fig. 1. Fiducial regions on the plane $(a, \bar\beta)$ — regions where the SVZ method can be applied if a) the short-time; b) the weak-coupling; c) the quasiclassical approximation for $Z(\beta)$ is used. The parameter a depends on $\bar\lambda$: $a = [(1+4\bar\lambda)^{1/2} - 1]/2$

Fig. 2. Ground state energy $\bar E_0$ as a function of $a = [(1+4\bar\lambda)^{1/2} - 1]/2$: exact (1); calculated by use of the SVZ method from the short-time approximation (2), weak-coupling approximation (3), quasiclassical approximation (4)

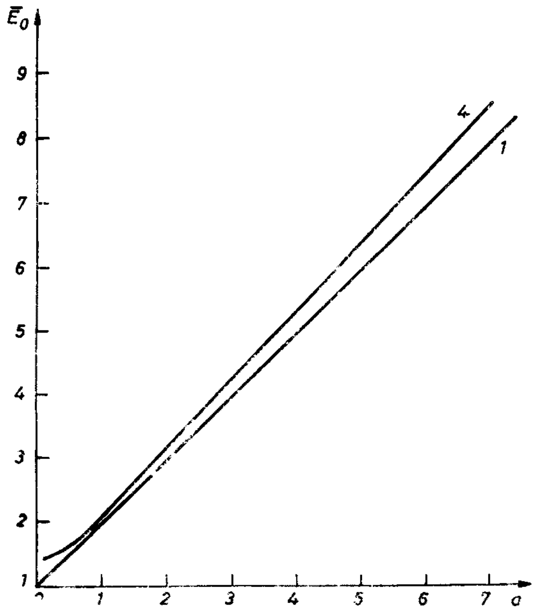

Fig. 3. Ground state energy $\bar{E}_0$ as a function of $a = [(1+4\bar{\lambda})^{1/2} - 1]/2$: exact (1); calculated by use of the SVZ method from the quasiclassical approximation (4)

Now we can apply the SVZ method using the short-time, the weak-coupling and the quasiclassical approximations of $Z(\beta)$. The fiducial regions calculated from criteria A), B) and C) are shown in Fig. 1a, b, c. The ground state energy E_0 calculated for these three approximations is compared with the exact values of E_0 in Fig. 2 and 3.

4. *Conclusions*

It follows from Fig. 1a, b, c that if either the short-time or the weak-coupling approximation is used, the SVZ method should work well for small values of λ. For large λ the SVZ method is applicable, provided that the expansion in powers of the Planck constant $\hbar$ is used. Fig. 2 and 3, showing the values of the ground state energy E_0 calculated by use of the SVZ method, confirm this observation. Moreover the three above-mentioned fiducial regions taken together cover the whole domain $[0, \infty)$ of values of λ. However the last property may be specific for the potential $V = \lambda \operatorname{ctg}^2 \pi x$.

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