LIE-BÄCKLUND TRANSFORMATION AND GRAVITATIONAL INSTANTONS

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Lie-Bäcklund transformation relating the equation for one-sided type-D vacuum gravitational instanton with the one for self-dual vacuum gravitational instanton admitting the "rotational" Killing vector field is presented. As the examples, Euclidean Cahen-Defrise solution and Eguchi-Hanson-like metrics are examined.

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1. *Introduction*

This paper is a continuation of the previous one given by Przanowski and Baka [1] in which it has been shown that for each one-side type-D gravitational instanton, Einstein's vacuum equations can be reduced locally to a single second-order non-linear partial differential equation of one real function. The aim of the present paper is to give the Lie-Bäcklund transformation from that equation to the one defining self-dual vacuum gravitational instanton admitting the "rotational" Killing vector field.

As it has been shown by Boyer and Finley [2] and then by Gegenberg and Das [3] all self-dual vacuum gravitational instantons admitting at least one Killing vector field can be splitted into two classes (Sect. 2).

The first class consists of the instantons with "translational" Killing vector fields, to the second one belong such instantons which admit "rotational" Killing vector fields (this is Boyer and Finley terminology). The metric of any gravitational instanton from the first class is defined by a real function satisfying Laplace's equation in Euclidean R^3. The metric

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for the second class is determined by a real function fulfilling the non-linear partial differential equation $F_{,yy}+F_{,zz}+(e^F)_{,xx} = 0$.

We find (Sect. 3) that the latter equation is related by the Lie-Bäcklund transformation to the "master equation" for type-$D \otimes$ [*anything*] vacuum gravitational instanton ([1], Eq. (45)). Therefore, the class of self-dual vacuum gravitational instantons with "rotational" Killing vector fields and class of all one-sided type-D vacuum gravitational instantons are defined by real solutions of a single non-linear partial differential equation.

We hope that our result affords the tool which enables one

(*i*) to generate new vacuum gravitational instantons from the old ones,

(*ii*) to realize the twistor program for type-$D \otimes$ [*anything*] complex space-times.

In Sect. 4, we use the ansatz of Gegenberg and Das [3]. We find that this ansatz leads to the Euclidean Cahen-Defrise solution [10, 11] and to the Eguchi-Hanson-like metrics [3, 14].

Of course it would be very nice to find the general solution of our fundamental equation, $F_{,yy}+F_{,zz}+(e^F)_{,xx} = 0$, or the Lie-Bäcklund transformation leading from this equation to a linear one but, at present, we are far from understanding how to do it.

2. *Self-dual vacuum gravitational instantons admitting Killing vector fields*

In this Section we recapitulate the main results of Boyer and Finley [2], Gegenberg and Das [3].

It is well known that each self-dual vacuum gravitational instanton is locally Kählerian manifold i.e., for each point there exist local complex coordinates z^α, $\alpha = 1, 2$, such that the metric is of the form

$$ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha dz^{\bar{\beta}} \tag{1}$$

and

$$d(g_{\alpha\bar{\beta}}dz^\alpha \wedge dz^{\bar{\beta}}) = 0, \tag{2}$$

where $\bar{\beta} = \bar{1}, \bar{2}$; $z^{\bar{\beta}} := \overline{z^\beta}$; $\overline{g_{\alpha\bar{\beta}}} = g_{\beta\bar{\alpha}}$ (Refs [4–6]).

From (2) it follows that locally

$$g_{\alpha\bar{\beta}} = \hat{K}_{,z^\alpha z^{\bar{\beta}}}, \tag{3}$$

where $\hat{K}(z^\alpha, z^{\bar{\alpha}})$ is a real function and "," denotes the partial derivation. Then, Euclidean Einstein vacuum equations are reducible to a single second-order non-linear partial differential equation

$$\hat{K}_{,z^1z^{\bar{1}}}\hat{K}_{,z^2z^{\bar{2}}}-\hat{K}_{,z^1z^{\bar{2}}}\hat{K}_{,z^2z^{\bar{1}}} = 1. \tag{4}$$

Assume that $\hat{K}(z^\alpha, z^{\bar{\alpha}})$ is of the form

$$\hat{K} = K(z^1+z^{\bar{1}}, z^2, z^{\bar{2}}). \tag{5}$$

Now one can write (4) in terms of differential forms:

$$dK-xdw-qdz^2-\bar{q}dz^{\bar{2}} = 0, \tag{6}$$

$$dx \wedge d\bar{q} \wedge dz^{\bar{2}}-dw \wedge dz^2 \wedge dz^{\bar{2}} = 0, \tag{7}$$

where $w := z^1 + z^{\bar{1}}$. It means that the question of solving Eq. (4) under the assumption (5) is equivalent to the problem of finding three-dimensional integral varieties of (6) and (7) for which

$$dw \wedge dz^2 \wedge dz^{\bar{2}} \neq 0. \tag{8}$$

Those integral varieties are three-dimensional submanifolds of seven-dimensional manifold defined by the coordinates $(w, z^2, z^{\bar{2}}, K, x, q, \bar{q})$. From (6), (7), and (8) it follows that

$$dx \wedge dz^2 \wedge dz^{\bar{2}} \neq 0. \tag{9}$$

Perform the contact transformation

$$(w, z^2, z^{\bar{2}}, K, x, q, \bar{q}) \mapsto (x, z^2, z^{\bar{2}}, H, -w, q, \bar{q}) \tag{10}$$

$$H = K - wx.$$

Then (6) can be written in the form:

$$dH + wdx - qdz^2 - \bar{q}dz^{\bar{2}} = 0 \tag{11}$$

and from (7) and (11) one finds that our three-dimensional integral varieties are defined by the solutions of Laplace's equation

$$H_{,xx} + H_{,yy} + H_{,zz} = 0, \tag{12}$$

with $H_{,xx} \neq 0$; here $y := z^2 + z^{\bar{2}}$, $z := \frac{1}{i}(z^2 - z^{\bar{2}})$.

Finally (1), (3), (4), (5) and (10) yield

$$ds^2 = -\tfrac{1}{2} H_{,xx}(dx^2 + dy^2 + dz^2)$$
$$-2(H_{,xx})^{-1}(\tfrac{1}{2} H_{,xy}dz - \tfrac{1}{2} H_{,xz}dy + d\tau)^2, \tag{13}$$

with $\tau := \frac{1}{2i}(z^1 - z^{\bar{1}})$.

Substitution

$$G := -\tfrac{1}{2} H_{,x}$$

gives

$$ds^2 = G_{,x}(dx^2 + dy^2 + dz^2) + (G_{,x})^{-1}(G_{,z}dy - G_{,y}dz + d\tau)^2, \tag{14}$$

where $G = G(x, y, z)$ is any real solution of Laplace's equation (see (12))

$$G_{,xx} + G_{,yy} + G_{,zz} = 0 \tag{12a}$$

with $G_{,x} > 0$.

The Killing vector field for the metric (14) is $\frac{\partial}{\partial\tau}$ and it has been called the "translational" Killing vector field [2, 3]. One easily recognizes in (14) and (12a) the Gibbons-Hawking metric [7].

Assume now

$$\hat{K} = K(\ln|z^1|^2, z^2, z^{\bar{2}}). \tag{15}$$

Hence (4) gives

$$K_{,uu}K_{,z^2z^{\bar{2}}} - K_{,uz^2}K_{,uz^{\bar{2}}} = e^u, \tag{16}$$

where $u := \ln|z^1|^2$. Analogously to the previous case we write (16) in the language of differential forms

$$dK - pdu - qdz^2 - \bar{q}dz^{\bar{2}} = 0, \tag{17}$$

$$dp \wedge d\bar{q} \wedge dz^{\bar{2}} - e^u du \wedge dz^2 \wedge dz^{\bar{2}} = 0. \tag{18}$$

By the contact transformation

$$(u, z^2, z^{\bar{2}}, K, p, q, \bar{q}) \mapsto (p, z^2, z^{\bar{2}}, L, -u, q, \bar{q})$$

$$L = K - up \tag{19}$$

one finds

$$L_{,aa} + L_{,bb} - (e^{-L_{,p}})_{,p} = 0, \tag{20}$$

where $a := z^2 + z^{\bar{2}}$, $b := \frac{1}{i}(z^2 - z^{\bar{2}})$.

Define

$$(x, y, z) := (\tfrac{1}{2}p, \tfrac{1}{2}a, \tfrac{1}{2}b), \tag{21}$$

$$F := -\frac{\partial L}{\partial p}(p(x), a(y), b(z)); \tag{22}$$

then differentiating (20) with respect to p we obtain

$$F_{,yy} + F_{,zz} + (e^F)_{,xx} = 0. \tag{23}$$

Using (1), (3), (16), (19), (21) and (22) one easily finds the metric

$$ds^2 = F_{,x}[e^F(dy^2 + dz^2) + dx^2] + (F_{,x})^{-1}(F_{,z}dy - F_{,y}dz + d\tau)^2 \tag{24}$$

where $\tau := \frac{1}{2i}\ln\frac{z^1}{z^{\bar{1}}}$. To assure the positivity of the metric (24) we have to assume $F_{,x} > 0$. The Killing vector field is $\frac{\partial}{\partial\tau} = \frac{\partial}{\partial(\arg z^1)}$. The latter relation elucidates why this Killing vector field has been called the "rotational" one. As it has been shown in Refs [2] and [3] the metrics (14) with (12a) and (24) with (23), exhaust all self-dual metrics admitting at least one Killing vector field.

3. *One-sided type-D vacuum gravitational instantons*

In a paper by Przanowski and Baka [1] it has been proven that the metric of any one-side type-D vacuum gravitational instanton is of the form (1) with

$$g_{1\bar{1}} = \varepsilon[(K_{,z^1})^{-\frac{1}{2}}]_{,z^{\bar{1}}}, \quad g_{1\bar{2}} = \varepsilon[(K_{,z^1})^{-\frac{1}{2}}]_{,z^{\bar{2}}} = \overline{g_{2\bar{1}}},$$

$$g_{2\bar{2}} = (K_{,z^1})^{-\frac{1}{2}}(e^{-K} - \tfrac{1}{2} K_{,z^2 z^{\bar{2}}}), \tag{25}$$

where $\varepsilon = \pm 1$ is chosen so that $g_{1\bar{1}} > 0$; $K = K(z^1 + z^{\bar{1}}, z^2, z^{\bar{2}})$ is a real function satisfying the following equation

$$K_{,z^1 z^{\bar{1}}} K_{,z^2 z^{\bar{2}}} - K_{,z^1 z^{\bar{2}}} K_{,z^2 z^{\bar{1}}} - 2e^{-K}(K_{,z^1 z^{\bar{1}}} + 2K_{,z^1} K_{,z^{\bar{1}}}) = 0 \tag{26}$$

and $K_{,z^1} > 0$.

The question of solving Eq. (26) can be translated into the language of differential forms as the problem of finding three-dimensional integral varieties in the seven-dimensional space defined by the coordinates $(w, z^2, z^{\bar{2}}, K, u, q, \bar{q})$, where $w := z^1 + z^{\bar{1}}$, for the following system

$$dK - u dw - q dz^2 - \bar{q} dz^{\bar{2}} = 0 \tag{27}$$

$$du \wedge d\bar{q} \wedge dz^{\bar{2}} - 2e^{-K}(du \wedge dz^2 \wedge dz^{\bar{2}} + 2u^2 dw \wedge dz^2 \wedge dz^{\bar{2}}) = 0 \tag{28}$$

with the additional assumption (8).

Perform the contact transformation

$$(w, z^2, z^{\bar{2}}, K, u, q, \bar{q}) \mapsto (v, z^2, z^{\bar{2}}, P, s, t, \bar{t}),$$

$$v = u^{\frac{1}{2}}, \quad P = w - Ku^{-1}, \quad s = 2Ku^{-\frac{3}{2}}, \quad t = -qu^{-1}, \quad \bar{t} = -\bar{q}u^{-1}. \tag{29}$$

Then (27), (28) can be written in the form

$$dP - s dv - t dz^2 - \bar{t} dz^{\bar{2}} = 0 \tag{30}$$

$$dv \wedge d\bar{t} \wedge dz^{\bar{2}} + e^{-\frac{v^3}{2}s}[(2v^{-2} + 3sv) dv \wedge dz^2 \wedge dz^{\bar{2}} + v^2 ds \wedge dz^2 \wedge dz^{\bar{2}}] = 0. \tag{31}$$

From (8), (27) and (28) one easily infers that

$$dv \wedge dz^2 \wedge dz^{\bar{2}} \neq 0. \tag{32}$$

The system (30), (31) with (32) yields the following non-linear partial differential equation

$$P_{,z^2 z^{\bar{2}}} - (2v^{-1} e^{-\frac{1}{2} v^3 P_{,v}})_{,v} = 0 \tag{33}$$

for the real function $P = P(v, z^2, z^{\bar{2}})$. Let $T = T(v, z^2, z^{\bar{2}})$ be a real function such that

$$T_{,q} = P. \tag{34}$$

Then from (33) and (34) we have

$$T_{,z^2 z^{\bar{2}}} - 2v^{-1} e^{-\frac{1}{2} v^3 T_{,vv}} = f, \tag{35}$$

where $f = f(z^2, z^{\bar{2}})$ is some real function. Without loss of generality one can put $f = 0$. Hence,

$$T_{,z^2 z^{\bar{2}}} - 2v^{-1} e^{-\frac{1}{2} v^3 T_{,vv}} = 0 \tag{36}$$

and T is now defined with the precision to the transformation

$$T \mapsto T + g(z^2) + \overline{g(z^2)}, \tag{37}$$

where $g(z^2)$ is any holomorphic function of z^2.

By the point transformation

$$(v, z^2, z^{\bar{2}}, T) \mapsto (x, z^2, z^{\bar{2}}, Y),$$

$$x = v^{-1}, \quad Y = -\tfrac{1}{2} v^{-1} T \tag{38}$$

Eq. (36) leads to

$$Y_{,z^2 z^{\bar{2}}} + e^{Y_{,xx} + 2 \ln x} = 0. \tag{39}$$

Differentiating (39) twice with respect to x and substituting

$$\hat{F} := Y_{,xx} + 2 \ln x \tag{40}$$

one finds that the real function $\hat{F} = \hat{F}(x, z^2, z^{\bar{2}})$ satisfies the equation

$$\hat{F}_{,z^2 z^{\bar{2}}} + (e^{\hat{F}})_{,xx} = 0. \tag{41}$$

Using the real coordinates $y := z^2 + z^{\bar{2}}$, $z := \frac{1}{i}(z^2 - z^{\bar{2}})$, and defining $F = F(x, y, z)$ as

$$F := \hat{F}(x, \tfrac{1}{2}(y + iz), \tfrac{1}{2}(y - iz)) \tag{42}$$

we obtain finally

$$F_{,yy} + F_{,zz} + (e^F)_{,xx} = 0 \tag{43}$$

i.e., the "master equation" (23) for self-dual gravitational instanton admitting at least one "rotational" Killing vector field. The careful analysis of the procedure leading from (26) to (43) shows that the latter equations are related by the transformation

$$x = (K_{,w})^{-\frac{1}{2}}, \quad y = z^2 + z^{\bar{2}}, \quad z = \frac{1}{i}(z^2 - z^{\bar{2}}), \quad F = -K - \ln(K_{,w}), \tag{44}$$

(with $w = z^1 + z^{\bar{1}}$) and its appropriate differential and integral consequences.

The infinite prolongation of (44) constitutes the Lie-Bäcklung transformation (see Refs [8], [9]). In that sense one calls (44) the Lie-Bäcklund transformation too.

Now we must elucidate what we mean under the words that "equations (26) and (43) are related by (44) with its appropriate differential and integral consequences".

Let $K = (z^1+z^{\bar{1}},\ z^2,\ z^{\bar{2}})$ be a real solution of (26) with $K_{,z^1} > 0$. Substituting $w = z^1+z^{\bar{1}}$ we have $K_{,w} > 0$. Then, according to (44), defining

$$F(x, y, z) := -K(w(x, y, z), \tfrac{1}{2}(y+iz), \tfrac{1}{2}(y-iz))+2\ln x \tag{45}$$

one finds that $F(x, y, z)$ satisfies the equation (43).

Conversely, assume now that $F = F(x, y, z)$ is a real solution of (43). Define a real function $\hat{P} = \hat{P}(x, y, z)$

$$\hat{P} := -\int x^2 F_{,x} dx + x^2 F + x^2(1-2\ln x). \tag{46}$$

Using coordinates $v: = \frac{1}{x}$, $z^2: = \frac{1}{2}(y+iz)$, $z^{\bar{2}}: = \frac{1}{2}(y-iz)$ we can easily check that the function $P = P(v, z^2, z^{\bar{2}}): = \hat{P}\left(\frac{1}{v}, z^2+z^{\bar{2}}, \frac{1}{i}(z^2-z^{\bar{2}})\right)$ fulfills the following equation (compare (33))

$$P_{,z^2 z^{\bar{2}}} - (2v^{-1} e^{-\frac{1}{2} v^3 P_{,v}})_{,v} = h, \tag{47}$$

where $h = h(z^2, z^{\bar{2}})$. Then one can choose $\int x^2 F_{,x} dx$ in (46) such that $h = 0$. Henceforth we assume that $\int x^2 F_{,x} dx$ is defined in such a manner that $h = 0$. Concluding, $P(v, z^2, z^{\bar{2}})$ satisfies Eq. (33).

Assume now that (compare with (49))

$$\tfrac{1}{2} x F_{,x} - 1 \neq 0. \tag{48}$$

Performing the contact transformation inverse to the one defined by (29), we can find that (30), (31), (32) and (48) lead to (27), (28) and (8) and then, of course, to (26). Reversing considerations, one can verify that the real solution $K = K(z^1+z^{\bar{1}}, z^2, z^{\bar{2}})$ of (26), constructed as it has been just described, is related to $F = F(x, y, z)$ by (44).

Now the main problem is to express the metric defined by (1), (25) and (26) in terms of F. By simple but tedious manipulations we obtain

$$\begin{aligned} ds^2 &= \varepsilon x(\tfrac{1}{2} x F_{,x} - 1)\,[e^F(dy^2+dz^2)+dx^2] \\ &+ [\varepsilon x(\tfrac{1}{2} x F_{,x} - 1)]^{-1}[(\tfrac{1}{2}\textstyle\int x^2 F_{,x} dx)_{,z} dy \\ &-(\tfrac{1}{2}\textstyle\int x^2 F_{,x} dx)_{,y} dz + d\tau]^2, \end{aligned} \tag{49}$$

where $\varepsilon = \pm 1$.

This is the analog of self-dual metric (24). (Remember that $\int x^2 F_{,x} dx$ is defined so that $P(v, z^2, z^{\bar{2}})$ determined by (46) satisfies Eq. (47) with $h = 0$. To fix $\int x^2 F_{,x} dx$ one has to solve the Poisson equation in R^2. Any function $f = f(y, z)$ added to $\int x^2 F_{,x} dx$ and such that $\Delta f = 0$ can be absorbed by the coordinate τ).

4. Examples

In this Section we consider type-$D \otimes$ [*anything*] metrics generated by the solutions of (43) which have been presented by Gegenberg and Das [3] for the case of self-dual metrics admitting "rotational" Killing vector fields (see our Sect. 2).

First consider the trivial solution of (43)

$$F = \text{const.} \tag{50}$$

Then one can put $\int x^2 F_{,x} dx = 0$. Without loss of generality we assume that $F = 0$. Hence

$$ds^2 = x(dx^2+dy^2+dz^2)+x^{-1}d\tau^2. \tag{51}$$

This is the Gibbons-Hawking metric (14) with $G = \frac{1}{2}x^2 - \frac{1}{4}y^2 + z^2 + const.$ Therefore the metric (51) is of the type $D \otimes [-]$.

Now let us examine more interesting case

$$F = f(y, z)+g(x). \tag{52}$$

Substituting (52) into (43) we obtain

$$(e^g)_{,xx} = 2a, \tag{53a}$$

$$f_{,yy}+f_{,zz} = -2ae^f, \tag{53b}$$

where $a = const.$

Integration of (53a) gives

$$g = \ln(ax^2+bx+c); \quad b, c = \text{const.} \tag{54}$$

If $a \neq 0$ then (53b) is the elliptic Liouville equation with the general solution

$$f = \ln\left[-\frac{1}{a}\frac{\varphi_{,z^2}\psi_{,\bar{z}^{\bar{2}}}}{(\varphi+\psi)^2}\right], \tag{55}$$

where $\varphi = \varphi(z^2)$ — holomorphic function and $\psi = \psi(\bar{z}^{\bar{2}})$ — anti-holomorphic function of the complex variable $z^2 = \frac{1}{2}(y+iz)$. Functions φ and ψ are so defined that $-\dfrac{1}{a}\dfrac{\varphi_{,z^2}\psi_{,\bar{z}^{\bar{2}}}}{(\varphi+\psi)^2}$ is a real positive function.

If $a = 0$ then (53b) is the Laplace equation in R^2 and its general solution is

$$f = \varphi(z^2)+\overline{\varphi(z^2)}, \tag{56}$$

where $\varphi = \varphi(z^2)$ is any holomorphic function of z^2. Consider now that latter case, i.e., $a = 0$. Then by (52), (54) and (56)

$$F = \ln(bx+c)+\varphi(z^2)+\overline{\varphi(z^2)}. \tag{57}$$

It is easy to show that, if $b \neq 0$

$$\int x^2 F_{,x} dx = \tfrac{1}{2} x^2 - \frac{c}{b} x + \left(\frac{c}{b}\right)^2 \ln (bx+c) - 2c|\omega|^2, \tag{58}$$

where $\omega = \omega(z^2)$ is a holomorphic function such that $\omega_{,z^2} = e^{\varphi}$. Substituting (57) and (58) into (49) and defining

$$\zeta := |b|^{\frac{1}{2}}\omega, \quad \xi := x + cb^{-1}, \quad l := c|b|^{-1} \tag{59}$$

one finds

$$ds^2 = \varepsilon \left\{ \frac{\xi^2 - l^2}{2\xi} d\xi^2 + (\operatorname{sgn} \xi)(\xi^2 - l^2) 2 d\zeta d\bar{\zeta} + \frac{2\xi}{\xi^2 - l^2} [d\tau + il(\zeta d\bar{\zeta} - \bar{\zeta} d\zeta)]^2 \right\}. \tag{60}$$

The metric (60) is the Euclidean analog of the Cahen-Defrise solution with $k = 0$, $\Lambda = 0$, $e = 0$ (see [10] and [11] Eqs (11.11), (11.42)). The Petrov-Penrose-Plebański type of (60) is $D \otimes D$. (If $b = 0$ then (57) yields the metric of the form (51).)

Assume now that $a \neq 0$. Hence

$$F = \ln (ax^2 + bx + c) + \ln \left[-\frac{1}{a} \frac{\varphi_{,z^2} \psi_{,\bar{z}^2}}{(\varphi + \psi)^2} \right]. \tag{61}$$

Then one finds

$$\int x^2 F_{,x} dx = h + \frac{c}{a} \ln \left[-\frac{1}{a} \frac{\varphi_{,z^2} \psi_{,\bar{z}^2}}{(\varphi + \psi)^2} \right], \tag{62}$$

where $h = h(x)$ is any real function of x such that $h_{,x} = x^2 F_{,x}$. Substitution of (61) and (62) into (49) and the analysis of the obtained result show that without loss of generality we can put

$$\frac{\varphi_{,z^2} \psi_{,\bar{z}^2}}{(\varphi + \psi)^2} = \begin{cases} -\tfrac{1}{2}(1 + \tfrac{1}{2} z^2 \bar{z}^2)^{-2} & \text{if} \quad a > 0 \quad (63a) \\ \tfrac{1}{2}(1 - \tfrac{1}{2} z^2 \bar{z}^2)^{-2} & \text{if} \quad a < 0. \quad (63b) \end{cases}$$

Then one has

(*i*) if $b \neq 0$.

$$ds^2 = \varepsilon \left\{ \frac{\xi^2 - l^2}{k(\xi^2 + l^2) - 2m\xi} d\xi^2 + (\xi^2 - l^2) \frac{2 d\zeta d\bar{\zeta}}{\left(1 + \frac{k}{2} \zeta \bar{\zeta}\right)^2} + \frac{k(\xi^2 + l^2) - 2m\xi}{\xi^2 - l^2} \left(dt + il \frac{\zeta d\bar{\zeta} - \bar{\zeta} d\zeta}{1 + \frac{k}{2} \zeta \bar{\zeta}} \right)^2 \right\}, \tag{64}$$

where

$$l: = c|2ab|^{-\frac{1}{4}}, \quad k: = \operatorname{sgn} a,$$

$$m: = -\frac{1}{2ab}\left(\frac{1}{2}\left|\frac{b}{a}\right|\right)^{\frac{1}{4}}(b^2-2ac),$$

$$\xi := \left(\frac{1}{2}\left|\frac{b}{a}\right|\right)^{\frac{1}{4}}(x+cb^{-1}), \quad \zeta := z^2, \quad t := \left(2\left|\frac{a}{b}\right|\right)^{\frac{1}{4}}\tau.$$

The metric (64) is the Euclidean equivalent of the Cahen-Defrise solution with $\Lambda = 0$, $e = 0, k = \pm 1$ (see [10] and [11] Eqs (11.11), (11.42)). For $k = +1$ and $\varepsilon = +1$ the metric (64) is the well known Taub-NUT metric [12, 13], written in the special coordinates.
(*ii*) if $b = 0$

$$ds^2 = \left(k+\varepsilon\frac{\alpha^4}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4}\frac{2d\zeta d\bar{\zeta}}{\left(1+\frac{k}{2}\zeta\bar{\zeta}\right)^2}$$

$$+\frac{r^2}{4}\left(k+\varepsilon\frac{\alpha^4}{r^4}\right)\left(dt+\frac{i}{2}\frac{\zeta d\bar{\zeta}-\bar{\zeta}d\zeta}{1+\frac{k}{2}\zeta\bar{\zeta}}\right)^2, \tag{65}$$

where

$$k := \operatorname{sgn} a, \quad \varepsilon := \operatorname{sgn} c, \quad \alpha := 2\left|\frac{c}{a}\right|^{\frac{1}{4}},$$

$$r := 2\left(\left|\frac{c}{a}\right|x\right)^{\frac{1}{4}}, \quad \zeta := z^2, \quad t := \frac{|a|}{c}\tau.$$

(Of course $k+\varepsilon \neq -2$.)

Metrics determined by (65) are precisely those obtained by Gegenberg and Das [3] for self-dual vacuum gravitational instantons admitting "rotational" Killing vector fields and defined by solutions of Eq. (23) ($\equiv$ 43)) of the form (52). Therefore, we conclude that the metrics (65) are of type $D \otimes [-]$.

For $k = +1$, $\varepsilon = +1$, one has the Eguchi-Hanson metric I; for $k = +1$, $\varepsilon = -1$, (65) is the Eguchi-Hanson metric II [14].

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