# DYNAMICS OF RELATIVE MOTION OF CHARGED TEST PARTICLES IN GENERAL RELATIVITY

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The paper discusses action principles simultaneously leading to the general relativistic Lorentz equations of motion in a given background field and to the electromagnetic deviation equations of the first and second order of the Lorentzian world lines. Some consequences of the simultaneous action principles connected with their reparametrization covariance and invariance under gauges of deviations, as well as the corresponding Hamilton-Jacobi equations are also considered.

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## *Introduction*

As is known, the geodesic and geodesic deviation equations can be derived from a single, so-called simultaneous variational principle [1], which is a particular case of a general property that there is always a simultaneous action principle leading to both the Lagrange and the Jacobi equations of a given action [2]. One should thus expect the same to be true for the general relativistic Lorentz equations of motion taken together with the electromagnetic (e.m.) deviations of the first and second order, which were a subject of a recent paper of the authors [3]. The objective of the present paper is just to formulate such action principles for two types of dynamical systems. The first system consists of Lorentzian world lines in a background of a gravitational and an electromagnetic field of general relativity taken together with a first e.m. deviation vector field defined along these lines, and is a subject of Sect. 2. The next system, discussed in Sect. 3, consists of three elements, that is of Lorentzian world lines and of both a first and a second e.m. deviations considered together; all in the same background as before.

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Sect. 1 contains a discussion of several properties connected with the variational principle leading to the Lorentz equations of motion. The subject of this Section may be considered as known, but we include it here for the sake of completeness and for further reference. The action principles, discussed in Sects 2 and 3 are also justified by means of an approach introduced in [3] under the name of the $\Sigma$-approach, which is a particular case of that used in [2]. In all the sections three types of consequences of the respective action principles are studied. They are (*i*) the Hamilton stationary action principles that lead to corresponding evolution equations; (*ii*) theorems of the type of the second Noether theorem; (*iii*) procedures that allow one to derive explicit forms of the Hamilton-Jacobi equations directly from the forms of the actions, without any reference to the canonical formalism.

The results obtained in the paper can be regarded as preparatory measures to a method of explicit solving the e.m. deviation equations, which will be a subject of a forthcoming work of the authors. In the particular case of geodesic deviations such a method was recently formulated by one of the present authors [5].

## *1. The Lorentz equations*

In a four-dimensional pseudo-Riemannian space-time $(V_4, g_{\alpha\beta})$ endowed with a background electromagnetic field, let us consider a time-like Lorentzian world line $\Gamma: R \to V_4$ described in a coordinate system $\{x^\alpha\}$ by the equations $x^\alpha = \xi^\alpha(\tau)$ with $\tau \in I$ being an arbitrary scalar parameter along $\Gamma$. The four functions $\xi^\alpha(\tau) = x^\alpha \circ \Gamma(\tau)$ are a solution of the general relativistic Lorentz equations (cf. [3])

$$\mathrm{L}[\xi^\alpha]: = \frac{D}{d\tau}\left(\frac{u^\alpha}{\sqrt{u_\lambda u^\lambda}}\right) - \sigma F^\alpha{}_\beta u^\beta = 0, \tag{1.1}$$

where $\frac{D}{d\tau}(\cdot) := (\cdot)_{;\lambda}\, u^\lambda$ denotes the absolute derivative along $\Gamma$, $u^\alpha := \frac{d\xi^\alpha}{d\tau}$ is a vector tangent to $\Gamma$, $F_{\alpha\beta}$ is the electromagnetic field tensor, $\sigma = \frac{q}{\mu c^2} = \mathrm{const}$ and $u_\lambda u^\lambda > 0$ is assumed everywhere along $\Gamma$. As is known, Eqs (1.1) can be derived from a standard variational principle characterized by a $\tau$-reparametrization invariant action given by the functional

$$W[\xi^\alpha] = \int_{\tau_0}^{\tau_1} L(\xi^\alpha, u^\alpha) d\tau, \tag{1.2}$$

with the Lagrangian

$$L(\xi^\alpha, u^\alpha) := \sqrt{g_{\alpha\beta} u^\alpha u^\beta} + \sigma A_\alpha u^\alpha, \tag{1.3}$$

where the metric tensor $g_{\alpha\beta}$ as well as the electromagnetic vector potential $A_\alpha$ are evaluated along the curve $\Gamma$. For the sake of completeness and for future reference we rederive now the Lorentz equations (1.1) by a less laborious procedure based on the concept of covariant variation of geometric objects which was used e.g. in [6].

Let us take the variations $x^\alpha = \xi^\alpha(\tau, \varepsilon)$ of functions $\xi^\alpha$ along a curve $\Gamma$ and define the corresponding variation vector $\bar{\delta}\xi^\alpha$, which in what follows is for simplicity also called the variation $\bar{\delta}\xi^\alpha$, in the standard way as

$$\bar{\delta}\xi^\alpha(\tau) := \varepsilon \frac{\partial \xi^\alpha}{\partial \varepsilon}(\tau, 0), \tag{1.4}$$

where $\varepsilon$ takes arbitrary values from an open interval. These variations are components of a vector tangent to the curve $x^\alpha = \xi^\alpha(\tau)$. The complete variations $\delta\xi^\alpha$ of the functions $\xi^\alpha$, generated by the variations $\bar{\delta}\xi^\alpha$ of the world line $\Gamma$ and by the variation $\delta\tau$ of the parameter $\tau$,

$$\delta\tau(\tau) := \varepsilon \frac{\partial f}{\partial \varepsilon}(\tau, 0),$$

where $\tau \mapsto \tilde{\tau} = f(\tau, \varepsilon)$ are defined to be

$$\delta\xi^\alpha := \bar{\delta}\xi^\alpha + u^\alpha \delta\tau. \tag{1.5}$$

The covariant variation of a tensor field $t^{\alpha_1 \cdots \alpha_n}$ given along the line $\Gamma$ can be introduced as

$$\bar{\Delta} t^{\alpha_1 \cdots \alpha_n}(\tau) := \varepsilon \frac{D t^{\alpha_1 \cdots \alpha_n}}{\partial \varepsilon}(\tau, 0), \tag{1.6}$$

where

$$\frac{D t^{\alpha_1 \cdots \alpha_n}}{\partial \varepsilon} := t^{\alpha_1 \cdots \alpha_n}{}_{;\beta} \frac{\partial \xi^\beta}{\partial \varepsilon}$$

is the absolute derivative of $t^{\alpha_1 \cdots \alpha_n}$ evaluated along $\Gamma$. Then, analogously to Eqs (1.5), the complete covariant variation of a tensor $t^{\alpha_1 \cdots \alpha_n}$ takes the form

$$\Delta t^{\alpha_1 \cdots \alpha_n} := \bar{\Delta} t^{\alpha_1 \cdots \alpha_n} + \frac{D t^{\alpha_1 \cdots \alpha_n}}{d\tau} \delta\tau, \tag{1.7}$$

which, in particular, for a vector field $t^\alpha$ reads

$$\Delta t^\alpha = t^\alpha{}_{;\beta} \bar{\delta}\xi^\beta + \frac{D t^\alpha}{d\tau} \delta\tau. \tag{1.8}$$

The complete variation of the action is equal to

$$\delta W := \varepsilon \frac{dW}{d\varepsilon}(0),$$

where

$$W(\varepsilon) = \int_{\tau_0(\varepsilon)}^{\tau_1(\varepsilon)} L(\xi^\alpha(\tau, \varepsilon), u^\alpha(\tau, \varepsilon)) d\tau$$

and $\tau_i(\varepsilon) = f(\tau_i, \varepsilon)$, $i = 0, 1$, in the integration limits. Since

$$\frac{dW}{d\varepsilon} = \int\limits_{\tau_0(\varepsilon)}^{\tau_1(\varepsilon)} \left\{ \frac{d}{d\varepsilon} L(\xi^\alpha(\tau, \varepsilon), u^\alpha(\tau, \varepsilon)) + \frac{d}{d\tau}\left[ L(\xi^\gamma(\tau, \varepsilon), u^\alpha(\tau, \varepsilon)) \frac{\partial f(\tau, \varepsilon)}{\partial \varepsilon}\right]\right\} d\tau,$$

we can replace the ordinary derivative of the scalar Lagrangian (1.3) by the absolute one and take into account that $\Delta g_{\alpha\beta} = 0$. Thence we obtain

$$\delta W = \int\limits_{\tau_0}^{\tau_1} \left[ \frac{g_{\alpha\beta} u^\beta}{\sqrt{u_\lambda u^\lambda}} \bar{\Delta} u^\alpha + \sigma(u^\alpha \bar{\Delta} A_\alpha + A_\alpha \bar{\Delta} u^\alpha) + \frac{d}{d\tau}(L\delta\tau)\right] d\tau.$$

Due to the commutation of $\partial/\partial\tau$ and $\partial/\partial\varepsilon$ and the symmetry of the connection coefficients $\Gamma^\alpha{}_{\beta\gamma}$, the covariant variation of $u^\alpha$ can be expressed by means of $\bar{\delta}\xi^\alpha$ as

$$\bar{\Delta} u^\alpha = \frac{D}{d\tau} \bar{\delta}\xi^\alpha. \tag{1.9}$$

Therefore, after taking into account Eqs (1.5), (1.8), (1.9), the complete variation of the action (1.2) is given by

$$\delta W = -\int\limits_{\tau_0}^{\tau_1} (L[\xi_\alpha]\bar{\delta}\xi^\alpha) d\tau + p_\alpha \delta\xi^\alpha|_{\tau_0}^{\tau_1}, \tag{1.10}$$

where

$$p_\alpha := \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha, \tag{1.11}$$

and use was made of the definition $F_{\alpha\beta} := A_{\beta;\alpha} - A_{\alpha;\beta}$.

The expression (1.10) for the complete variation of the action $W$ finds several applications. The most common is the Hamilton stationary action principle:

The complete variation $\delta W$ of the functional (1.2), generated by the variations $\bar{\delta}\xi^\alpha$ and $\delta\tau$ satisfying the conditions

$$\delta\tau = \bar{\delta}\xi^\alpha(\tau_0) = \bar{\delta}\xi^\alpha(\tau_1) = 0$$

and otherwise arbitrary, is equal to zero if and only if $x^\alpha = \xi^\alpha(\tau)$ fulfil the Lorentz equations (1.1) in an arbitrary parametrization.

Another applications of Eqs (1.10) are the Noether theorems. Following [7] one can say that the variations $\delta\tau(\tau)$ and $\bar{\delta}\xi^\alpha(\tau)$, considered as given functions of $\tau$, generate a dynamical symmetry iff the complete variation $\delta W$ caused by these generators is equal to

$$\delta W = \int \frac{d}{d\tau}(\delta F) d\tau, \tag{1.12}$$

where $\delta F$ is a function of the dynamical variables $\xi^\alpha$, $u^\alpha$, the parameter $\tau$, and the generators. Thus, Eq. (1.10) together with (1.12) impose a condition on the generators $\delta\tau$ and $\bar{\delta}\xi^\alpha$ which is the famous identity of E. Noether:

$$-L[\xi_\alpha]\bar{\delta}\xi^\alpha + \frac{d}{d\tau}(p_\alpha\bar{\delta}\xi^\alpha + L\delta\tau - \delta F) = 0. \tag{1.13}$$

As is known, the consequences of this identity depend on the degree of arbitrariness with which the generators $\delta\tau$ and $\bar{\delta}\xi^\alpha$ are defined. In the case considered now, the generators may be given either in terms of a finite number $N$ of parameters or in terms of an arbitrary function describing arbitrary reparametrizations of the Lorentzian world line. In the first case the Noether identity leads to $N$ conservation laws, and this will be a subject of a subsequent paper of ours. In the second case we may examine for which Lagrangians $L$ the action (1.2) transforms in accordance with (1.12) under a transformation of the form

$$\tau \mapsto \tilde{\tau} = f(\tau), \qquad V_4 \ni p \mapsto p' = p$$

or

$$\tilde{\tau} = f(\tau), \qquad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau) \tag{1.14}$$

depending on one arbitrary $C_2$ function $f\colon R \to R$, with $f' \neq 0$. Let us note that in the neighbourhood of the identity transformation Eqs (1.14) can be written in the form

$$\tilde{\tau} = \tau + \delta\tau(\tau), \qquad \tilde{\xi}^\alpha(\tau + \delta\tau(\tau)) = \tilde{\xi}^\alpha(\tau) + u^\alpha\delta\tau(\tau) = \xi^\alpha(\tau). \tag{1.15}$$

Thus the infinitesimal generators of the transformations (1.14) are defined by an arbitrary variation $\delta\tau(\tau)$ and, as it follows from (1.15), by

$$\bar{\delta}\xi^\alpha(\tau) = -u^\alpha\delta\tau(\tau). \tag{1.16}$$

Substituting (1.16) into (1.13) and taking into account that under the transformations (1.14) we have $\delta F = 0$, one obtains

$$\mathrm{L}[\xi_\alpha]u^\alpha\delta\tau + \frac{d}{d\tau}[(-p_\alpha u^\alpha + L)\delta\tau] = 0.$$

Since $\delta\tau(\tau)$ is an arbitrary function, in the expression above the coefficients at $\delta\tau$ and $\frac{d}{d\tau}\delta\tau$ must vanish separately, which gives

$$\mathrm{L}[\xi_\alpha]u^\alpha = 0, \tag{1.17}$$

$$L = p_\alpha u^\alpha, \tag{1.18}$$

and these are strong identities, satisfied independently of whether the function $\xi^\alpha$ is or is not a solution of the Lagrange equations. Thus, we have arrived at the following well-known result which will be generalized in the forthcoming sections.

PROPOSITION 1.1. An action of the form (1.2) is $\tau$-reparametrization invariant iff (1.17) and (1.18) are satisfied as strong identities.

REMARK 1.1. Let us observe that due to Euler's theorem about homogeneous functions the identity (1.18) means that $L$ is with respect to $u^\alpha$ a homogeneous function of degree one. This condition is of course satisfied by the Lagrangian (1.3), and the Lorentz equations (1.1) satisfy therefore the strong identity (1.17) (see also [3]).

A third application of Eq. (1.10) is a method of deriving from the form of the action (1.2) the corresponding Hamilton-Jacobi equations. Taking this purpose in mind, let us consider now a family of Lorentzian world lines $x^\alpha = \xi^\alpha(\tau)$, described by Eqs (1.1), all of which for $\tau = \tau_0$ pass through the same point $\xi_0^\alpha = \xi^\alpha(\tau_0)$. By means of such a family the functional (1.2) can be turned into a function $U: I \to R$, called the principal Hamilton function. The values $U(x^\alpha, \tau)$ of this function are defined as being the values of the integral (1.2) calculated along the Lorentzian curve $\Gamma$ which joins the initial point $\xi_0^\alpha$ with the point $x^\alpha$ just for the value $\tau$ of the parameter, and both these values of $x^\alpha$ and $\tau$ are the arguments of $U$. Then, due to the assumptions about $\Gamma$, the integrand in Eq. (1.10) vanishes, and the differential of the principal Hamilton function is equal to

$$dU = p_\alpha dx^\alpha.$$

Thus

$$\frac{\partial U}{\partial \tau} = 0, \tag{1.19}$$

which could be considered as the Hamilton-Jacobi equation on $U$. Besides

$$\frac{\partial U}{\partial x^\alpha} = \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha, \tag{1.20}$$

which is the well-known relation between the partial derivatives of $U$ and the generalized momentum $p_\alpha$ determined by the action (1.2) usually used in the framework of relativistic dynamics of charged test particles. Since the unit timelike vector $\frac{u^\alpha}{\sqrt{u_\lambda u^\lambda}}$ at the right hand side of (1.20) satisfies an abvious algebraic relation, a corresponding relation must also be satisfied by the left hand side. Therefore, we obtain the familiar Hamilton-Jacobi equation for a charged test particle

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha} - \sigma A_\alpha\right)\left(\frac{\partial U}{\partial x^\beta} - \sigma A_\beta\right) = 1, \tag{1.21}$$

which is a partial differential equation on a function $U$ depending, in virtue of Eq. (1.19), in general on four variables $x^\alpha$. The knowledge of the complete integral of Eq. (1.21) allows one to determine in the standard way a Lorentzian world line in a given pseudo-Riemannian manifold $V_4$ endowed with a given tensor field $F_{\alpha\beta}$ (or vector field $A_\alpha$) determining the background electromagnetic field.

## 2. *The first e.m. deviation equations*

As it has been shown by one of the present authors in [2], there always exists a unified variational principle simultaneously leading to both the Lagrange and the Jacobi (i.e. deviation) equations of a given action. For the geodesic problem the appropriate unified action which leads to the geodesic and the first geodesic deviation equations was given in [1]. We shall derive in this Section an analogous simultaneous variational principle leading to the Lorentz equations (1.1) and to the first e.m. deviation equations

$$\mathrm{L}_1[r^\alpha] := \frac{D}{d\tau}\left[\frac{1}{\sqrt{u_\lambda u^\lambda}}\left(\delta^\alpha_\beta - \frac{u^\alpha u_\beta}{u_\lambda u^\lambda}\right)\frac{Dr^\beta}{d\tau}\right] + \frac{1}{\sqrt{u_\lambda u^\lambda}} R^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta$$
$$-\sigma\left(F^\alpha{}_{\beta;\gamma}u^\beta r^\gamma + F^\alpha{}_\beta \frac{Dr^\beta}{d\tau}\right) = 0, \tag{2.1}$$

which were derived in [3] for an arbitrary parametrization.

Let us take the action

$$W_1[\xi^\alpha, r^\alpha] = \int_{\tau_0}^{\tau_1} L_1\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right)d\tau, \tag{2.2}$$

being a functional depending both on the world line $\Gamma$ and on the vector field $r$ defined along it and determined at $p(\tau) \in \Gamma$ by its components $r^\alpha(\tau)$, in which the Lagrangian is

$$L_1\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right) := \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\frac{Dr^\alpha}{d\tau} + \sigma A_{\alpha;\beta}u^\alpha r^\beta, \tag{2.3}$$

where $g_{\alpha\beta}$ and $A_\alpha$ are taken at the world line $\Gamma$ described by the functions $x^\alpha = \xi^\alpha(\tau)$.

A heuristic explanation of the choice of the Lagrangian (2.3) can be obtained from the $\Sigma$-approach to families of Lorentzian world lines which was developed in a recent paper [3] of the authors. Indeed, let us observe that in the $\Sigma$-approach Eqs (1.2) and (1.3) along a fixed line $\Gamma_\varepsilon$ read correspondingly a

$$W[\xi^\alpha(\varepsilon)] = \int_{\tau_0}^{\tau_1} L(\xi^\alpha(\tau, \varepsilon), u^\alpha(\tau, \varepsilon))d\tau$$

and

$$L(\xi^\alpha(\tau, \varepsilon), u^\alpha(\tau, \varepsilon)) := \sqrt{g_{\alpha\beta}u^\alpha(\tau, \varepsilon)u^\beta(\tau, \varepsilon)} + \sigma A_\alpha(\tau, \varepsilon)u^\alpha(\tau, \varepsilon).$$

Differentiating this with respect to $\varepsilon$, taking into account the definition $r^\alpha(\tau, \varepsilon) := \frac{\partial\xi^\alpha}{\partial\varepsilon}(\tau, \varepsilon)$ and the relation $\frac{Du^\alpha}{\partial\varepsilon} = \frac{Dr^\alpha}{\partial\tau}$ (cf. [3]), one obtains

$$W_1[\xi^\alpha, r^\alpha] := \frac{\partial W[\xi^\alpha]}{\partial\varepsilon} = \int_{\tau_0}^{\tau_1} L_1\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right)d\tau, \tag{2.4}$$

where

$$L_1\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right) := \frac{\partial L(\xi^\alpha, u^\alpha)}{\partial\varepsilon} = \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\frac{Dr^\alpha}{d\tau} + \sigma A_{\alpha;\beta}u^\alpha r^\beta. \tag{2.5}$$

Such a heuristic derivation of the form of the Lagrangian does not guarantee however that (2.2) is a correct action simultaneously leading to both the Lorentz equations (1.1) and the first e.m. deviation equations (2.1). This property must be demonstrated in a direct way.

Similarly like in Sect. 1, let us compute the variation of the action (2.2) caused by independent variations $\delta\xi^\alpha$, $\delta r^\alpha$ and $\delta\tau$:

$$\delta W_1 = \int_{\tau_0}^{\tau_1}\Bigg[\frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\beta}{d\tau}\bar{\Delta}u^\alpha + \sigma\frac{Dr^\alpha}{d\tau}\bar{\Delta}A_\alpha + \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\bar{\Delta}\frac{Dr^\alpha}{d\tau}$$

$$+\sigma u^\alpha r^\beta\bar{\Delta}A_{\alpha;\beta} + \sigma A_{\alpha;\beta}(u^\alpha\bar{\Delta}r^\beta + r^\beta\bar{\Delta}u^\alpha) + \frac{d}{d\tau}(L_1\delta\tau)\Bigg]d\tau,$$

where $h_{\alpha\beta} := g_{\alpha\beta} - \dfrac{u_\alpha u_\beta}{u_\lambda u^\lambda}$. Applying the definitions (1.7) and (1.8), making use of Eqs (1.5), (1.9), and taking into account the Ricci identity

$$\Delta\frac{Dt^\alpha}{d\tau} = \frac{D}{d\tau}\Delta t^\alpha + R^\alpha{}_{\beta\gamma\delta}t^\beta\delta\xi^\gamma u^\delta, \tag{2.6}$$

we find the expression

$$\delta W_1 = -\int_{\tau_0}^{\tau_1}(L_1[r_\alpha]\delta\xi^\alpha + L[\xi_\alpha]\Delta r^\alpha)d\tau + (\pi_\alpha^{(1)}\delta\xi^\alpha + p_\alpha\delta r^\alpha)|_{\tau_0}^{\tau_1} \tag{2.7}$$

for the complete variation of the action (2.2), where $p_\alpha$ is given by Eqs (1.11),

$$\pi_\alpha^{(1)} := \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\beta}{d\tau} + \Gamma^\beta{}_{\gamma\alpha}\left(\frac{u_\beta}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\beta\right)r^\gamma + \sigma A_{\alpha;\beta}r^\beta, \tag{2.8}$$

and use was made of the relation

$$\Delta r^\alpha = \delta r^\alpha + \Gamma^\alpha{}_{\beta\gamma}r^\alpha\delta\xi^\gamma \tag{2.9}$$

which followed from (1.4), (1.8) and from the definition $\delta r^\alpha(\tau) := \varepsilon\dfrac{\partial r^\alpha}{\partial\varepsilon}(\tau, 0)$.

By analogy with Eq. (1.10), the expression (2.7) for the complete variation of $W_1$ finds also several applications, the most common of which is the Hamilton stationary action principle formulated now in the following form:

THEOREM 2.1. The complete variation $\delta W_1$ of the functional (2.2), generated by the variations $\delta\xi^\alpha$, $\delta r^\alpha$ and $\delta\tau$ satisfying the conditions

$$\delta\tau = \delta\xi^\alpha(\tau_0) = \delta\xi^\alpha(\tau_1) = \delta r^\alpha(\tau_0) = \delta r^\alpha(\tau_1) = 0$$

and being otherwise arbitrary, is equal to zero if and only if $x^\alpha = \xi^\alpha(\tau)$ fulfil the Lorentz equations (1.1) and $r^\alpha = r^\alpha(\tau)$ fulfil the first e.m. deviation equations (2.1), both in an arbitrary parametrization.

The proof follows from (2.7).

Thus, within the framework based on the unified action (2.2) the dynamical properties of the first e.m. deviation are not considered as separate from those of the motion determined by the Lorentz equations. In the terminology used in [2], from Theorem 2.1 it follows that the first e.m. deviation equations are the Jacobi equations of the action (1.2).

A second application of Eq. (2.7) is again the Noether identity. The variations $\delta\tau(\tau)$, $\delta\xi^\alpha(\tau)$ and $\delta r^\alpha(\tau)$, treated as given functions of $\tau$, will be the generators of a dynamical symmetry iff the complete variation $\delta W_1$ caused by these generators is equal to

$$\delta W_1 = \int \frac{d}{d\tau}(\delta F_1)d\tau, \tag{2.10}$$

where $\delta F_1$ is a function of the dynamical variables $\xi^\alpha$, $u^\alpha$, $r^\alpha$, $\frac{Dr^\alpha}{d\tau}$, the parameter $\tau$, and the generators. If the requirement (2.10) is satisfied, Eq. (2.7) imposes a condition on the generators of a symmetry which is the Noether identity

$$-(\mathrm{L}_1[r_\alpha]\bar{\delta}\xi^\alpha+\mathrm{L}[\xi_\alpha]\bar{\Delta}r^\alpha)+\frac{d}{d\tau}(\pi^{(1)}_\alpha\bar{\delta}\xi^\alpha+p_\alpha\bar{\delta}r^\alpha+L_1\delta\tau-\delta F_1) = 0. \tag{2.11}$$

In accordance with the general scheme, if the generators $\delta\tau(\tau)$, $\bar{\delta}\tau^\alpha(\tau)$ and $\bar{\delta}r^\alpha(\tau)$ are given in terms of a finite number $N$ of parameters, Eq. (2.11) leads to $N$ conservation laws which will be discussed in a subsequent paper of ours. Here we restrict our attention to two particular cases when $\delta\tau(\tau)$, $\bar{\delta}\xi^\alpha(\tau)$ and $\bar{\delta}r^\alpha(\tau)$ are given in terms of an arbitrary function of $\tau$.

First, let us consider the case when the arbitrary function mentioned above describes an arbitrary reparametrization of the Lorentzian world line and of the first e.m. deviation vector field along it. So, we shall examine now for which Lagrangians $L_1$ the action (2.2) satisfies the condition (2.10) under transformations of the form

$$\tau \mapsto \tilde{\tau} = f(\tau), \quad V_4 \ni p \mapsto p' = p, \quad TV_4 \ni r \mapsto r' = r,$$

with $TV_4$ being the tangent bundle to the manifold $V_4$, or

$$\tilde{\tau} = f(\tau), \quad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau), \quad \tilde{r}^\alpha(\tilde{\tau}) = r^\alpha(\tau) \tag{2.12}$$

depending on one arbitrary $C_2$ function $f\colon R \to R$ (with $f' \neq 0$). In the neighbourhood of the identity transformation Eqs (2.12) can be written in the form

$$\tilde{\tau} = \tau + \delta\tau(\tau),$$

$$\tilde{\xi}^\alpha(\tau+\delta\tau(\tau)) = \tilde{\xi}^\alpha(\tau) + u^\alpha \delta\tau(\tau) = \xi^\alpha(\tau),$$

$$\tilde{r}^\alpha(\tau+\delta\tau(\tau)) = \tilde{r}^\alpha(\tau) + \frac{dr^\alpha}{d\tau}\delta\tau(\tau) = r^\alpha(\tau). \tag{2.13}$$

Thus, as it follows from (2.13), the infinitesimal generators of the transformations (2.12) are defined by an arbitrary variation $\delta\tau(\tau)$, by the variation $\delta\xi^\alpha$ given in the form (1.16), and by

$$\delta r^\alpha = -\frac{dr^\alpha}{d\tau}\delta\tau. \tag{2.14}$$

Substituting these expressions into (2.11) and taking into account that under the transformations (2.12) we have $\delta F_1 = 0$, one obtains

$$\left(\mathrm{L}_1[r_\alpha]u^\alpha + \mathrm{L}[\xi_\alpha]\frac{Dr^\alpha}{d\tau}\right)\delta\tau + \frac{d}{d\tau}\left[\left(-\pi_\alpha^{(1)}u^\alpha - p_\alpha\frac{dr^\alpha}{d\tau} + L_1\right)\delta\tau\right] = 0.$$

Due to the arbitrariness of $\delta\tau$, one obtains from here the following strong identities

$$L_1[r_\alpha]u^\alpha + \mathrm{L}[\xi_\alpha]\frac{Dr^\alpha}{d\tau} = 0, \tag{2.15}$$

$$L_1 = \pi_\alpha^{(1)}u^\alpha + p_\alpha\frac{dr^\alpha}{d\tau} \tag{2.16}$$

satisfied independently of whether the functions $\xi^\alpha$ and $r^\alpha$ are or are not solutions of the Lagrange and the Jacobi equations correspondingly. Thus, we have proved the following statement:

PROPOSITION 2.1. A unified action of the form (2.2) transforms under a $\tau$-reparametrization in accordance with (2.10) iff (2.15) and (2.16) are satisfied as strong identities.

COROLLARY 2.1. If $\Gamma$ is a Lorentzian world line defined by the Lorentz equations (1.1), then the vector field $r$ satisfies the strong identity

$$\mathrm{L}_1[r_\alpha]u^\alpha = 0 \tag{2.17}$$

independently of whether it is or is not a solution of Eqs (2.1) (see also [3]).

Secondly, let us discuss the transformations

$$\tau \mapsto \tilde{\tau} = \tau, \quad V_4 \ni p \mapsto p' = p, \quad TV_4 \ni r \mapsto r' = r + \kappa u$$

or

$$\tilde{\tau} = \tau, \quad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau), \quad \tilde{r}^\alpha(\tilde{\tau}) = r^\alpha(\tau) + \kappa(\tau) u^\alpha(\tau) \tag{2.18}$$

depending again on one arbitrary $C_2$ function $\kappa : R \to R$ (with $\kappa' \neq 0$), but now describing (in accordance with Prop. 2.2 *i* from [3]) an arbitrary gauge transformation of $r^\alpha$ under an arbitrary (but fixed) parametrization of $\Gamma$. In the neighbourhood of the identity transformation the function $\kappa(\tau)$ should be replaced by $\delta\kappa(\tau)$. This gives the infinitesimal generators of the transformations (2.18) in the following form

$$\delta\tau = 0, \quad \delta\xi^\alpha = 0, \quad \delta r^\alpha = u^\alpha \delta\kappa. \tag{2.19}$$

Inserting Eqs (2.19) into (2.2) with the Lagrangian (2.3), one obtains that

$$\delta F_1 = p_\alpha u^\alpha \delta\kappa. \tag{2.20}$$

Substituting now Eqs. (2.19) and (2.20) into the Noether identity (2.11) and making use of the arbitrariness of the function $\delta\kappa$, we obtain the strong identity

$$\mathrm{L}[\xi_\alpha] u^\alpha = 0. \tag{2.21}$$

This identity was already obtained in Sect. 1, but as a consequence of the reparametrization invariance of a different action (1.2). Now the following statement is true:

PROPOSITION 2.2. The unified action (2.2) transforms under the gauge transformations $r^\alpha \mapsto r^\alpha + \kappa(\tau) u^\alpha$ generated by an arbitrary, differentiable function $\kappa(\tau)$ in accordance with (2.10) iff the strong identity (2.21) is satisfied.

A third application of Eq. (2.7) concerns a method of deriving the Hamilton-Jacobi equations which correspond to the action (2.2). Let us consider again a family of Lorentzian world lines $x^\alpha = \xi^\alpha(\tau)$ described by Eqs (1.1), all of which for $\tau = \tau_0$ pass through the same point $\xi_0^\alpha = \xi^\alpha(\tau_0)$. Along every of these lines one can take a family of the first e.m. deviation vector fields, with the components $r^\alpha = r^\alpha(\tau)$ being a solution of Eqs (2.1), such that every of its members takes the same value $r_0^\alpha = r^\alpha(\tau_0)$ at the point $\xi_0^\alpha$ for different Lorentzian world lines passing through $\xi_0^\alpha$. By means of such a family the functional (2.2) can be turned into a function $U_1 : TV_4 \times R \to R$ analogous to the principal Hamilton function. The values $U_1(x^\alpha, r^\alpha, \tau)$ of this principal function are defined by the integral of the form (2.2) in that the upper limit is replaced by the value $\tau \in I$, and the integration is carried out along the curve $\Gamma$ which joins the initial point $(\xi_0^\alpha, r_0^\alpha)$ with the point $(x^\alpha, r^\alpha)$ entering the argument of $U_1$. Then, due to the assumptions about $\Gamma$, the integrand in Eq. (2.7) vanishes and the differential of the principal function $U_1$ is equal to

$$dU_1 = \pi_\alpha^{(1)} dx^\alpha + p_\alpha dr^\alpha.$$

Thus

$$\frac{\partial U_1}{\partial \tau} = 0 \tag{2.22}$$

and

$$\frac{\partial U_1}{\partial x^\alpha} = \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\beta}{d\tau} + \Gamma^\beta{}_{\gamma\alpha}\left(\frac{u_\beta}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\beta\right) r^\gamma + \sigma A_{\alpha;\beta} r^\beta, \tag{2.23}$$

$$\frac{\partial U_1}{\partial r^\alpha} = \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha. \tag{2.24}$$

From a formal point of view, due to the analogy with classical dynamics of charged test particles, Eq. (2.22) could be considered as the Hamilton-Jacobi equation on $U_1$ and amounts to $U_1 = U_1(x^\alpha, r^\alpha)$. Eqs (2.23) and (2.24) give relations between the partial derivatives of $U_1$ and the generalized momenta $p_\alpha$ and $\pi_\alpha^{(1)}$ determined by the action (2.2). Since the unit timelike vector $\frac{u^\alpha}{\sqrt{u_\lambda u^\lambda}}$ and the projection tensor $h_{\alpha\beta}$ at the right hand sides of (2.23) and (2.24) satisfy two obvious algebraic relations, the corresponding relations must also be satisfied by the left hand sides of (2.23) and (2.24). Therefore we obtain two partial differential equations on a single principal function $U_1$:

$$g^{\alpha\beta}\left(\frac{\partial U_1}{\partial r^\alpha} - \sigma A_\alpha\right)\left(\frac{\partial U_1}{\partial r^\beta} - \sigma A_\beta\right) = 1, \tag{2.25}$$

$$g^{\alpha\beta}\left(\frac{\partial U_1}{\partial r^\alpha} - \sigma A_\alpha\right)\left(\frac{\partial U_1}{\partial x^\beta} - \Gamma^\gamma{}_{\delta\beta} \frac{\partial U_1}{\partial r^\gamma} r^\delta - \sigma A_{\beta;\gamma} r^\gamma\right) = 0 \tag{2.26}$$

which could be called the Hamilton-Jacobi equations for the dynamical system consisting of the Lorentzian world lines and of the first e.m. deviations.

It should be stressed however that if the Hamilton-Jacobi equations are presented in the form (2.25)–(2.26), it is not yet obvious whether they are connected with the ordinary differential equations (1.1) and (2.1) or even with the Hamilton-Jacobi equation (1.21). To reveal the latter connection, let us observe that Eqs (2.25) and (2.26) admit the separation of the variables $x^\alpha$ and $r^\alpha$, which can be achieved by representing $U_1$ in the form (cf. [1])

$$U_1(x^\alpha, r^\alpha) = r^\alpha \frac{\partial U}{\partial x^\alpha} + V, \tag{2.27}$$

where $U$ and $V$ are functions of $x^\alpha$ alone. Differentiating then (2.27) with respect to $r^\alpha$ and $x^\alpha$, one gets

$$\frac{\partial U_1}{\partial r^\alpha} = \frac{\partial U}{\partial x^\alpha}, \quad \frac{\partial U_1}{\partial x^\alpha} = r^\beta \left(\frac{\partial U}{\partial x^\beta}\right)_{;\alpha} + \Gamma^\beta{}_{\gamma\alpha} \frac{\partial U}{\partial x^\beta} r^\gamma + \frac{\partial V}{\partial x^\alpha},$$

and this, after substituting into (2.25) and (2.26) and taking into account the equality

$$\left(\frac{\partial U}{\partial x^\alpha}\right)_{;\beta} = \left(\frac{\partial U}{\partial x^\beta}\right)_{;\alpha},$$

shows that $U$ and $V$ must satisfy the following equations

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha}-\sigma A_\alpha\right)\left(\frac{\partial U}{\partial x^\beta}-\sigma A_\beta\right)=1, \tag{2.28}$$

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha}-\sigma A_\alpha\right)\frac{\partial V}{\partial x^\beta}=0, \tag{2.29}$$

in which the first coincides obviously with the Hamilton-Jacobi equation (1.21) for Lorentzian world lines.

## *3. The second e.m. deviation equations*

It has been shown [1] that there exists also a variational principle which leads simultaneously to the geodesic equations and to the first and the second geodesic deviation equations. In this Section, we shall derive an analogous variational principle for the system of the Lorentz equations (1.1), the first e.m. deviation equations (2.1) and the second e.m. deviation equations

$$\begin{aligned}
\mathrm{L}_2[w^\alpha]:=&\frac{D}{d\tau}\left(\frac{h^{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dw^\beta}{d\tau}\right)+\frac{1}{\sqrt{u_\lambda u^\lambda}}R^\alpha{}_{\beta\gamma\delta}u^\beta w^\gamma u^\delta-\frac{1}{\sqrt{u_\lambda u^\lambda}}\Big\{(R^\alpha{}_{\beta\gamma\delta;\varepsilon}\\
&+R^\alpha{}_{\varepsilon\gamma\delta;\beta})u^\beta u^\gamma r^\delta r^\varepsilon+4R^\alpha{}_{\beta\gamma\delta}\frac{Dr^\beta}{d\tau}u^\gamma r^\delta+2\frac{Dr^\alpha}{d\tau}\frac{d}{d\tau}\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)\\
&+u^\alpha\frac{d}{d\tau}\left[\frac{h_{\beta\gamma}}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau}+\frac{1}{u_\lambda u^\lambda}R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon-\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)^2\right]\Big\}\\
&-\sigma\Big\{F^\alpha{}_{\beta;\gamma\delta}u^\beta r^\gamma r^\delta+F^\alpha{}_{\beta;\gamma}\left[u^\beta w^\gamma+2\frac{Dr^\beta}{d\tau}r^\gamma+2u^\beta r^\gamma\left(\frac{u_\varepsilon}{u_\lambda u^\lambda}\frac{Dr^\varepsilon}{d\tau}\right)\right]\\
&+(F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon}-F^\beta{}_\varepsilon R^\alpha{}_{\gamma\delta\beta})u^\varepsilon r^\gamma r^\delta+F^\alpha{}_\beta\left[\frac{Dw^\beta}{d\tau}+2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right]\\
&+F^\alpha{}_\beta u^\beta\left(\frac{h_{\gamma\delta}}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta}{d\tau}+\frac{1}{u_\lambda u^\lambda}R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau\right)\Big\}=0,
\end{aligned} \tag{3.1}$$

which were derived in [4] for an arbitrary parametrization.

Let us take the action

$$W_2[\xi^\alpha,r^\alpha,w^\alpha]=\int_{\tau_0}^{\tau_1}L_1\left(\xi^\alpha,u^\alpha,w^\alpha,\frac{Dw^\alpha}{d\tau}\right)d\tau+\int_{\tau_0}^{\tau_1}L_2\left(\xi^\alpha,u^\alpha,r^\alpha,\frac{Dr^\alpha}{d\tau}\right)d\tau, \tag{3.2}$$

being a functional depending both on the world line $\Gamma$ and on two vector fields $r$ and $w$ defined along it; the total Lagrangian $L_3\left(\xi^\alpha,u^\alpha,r^\alpha,\frac{Dr^\alpha}{d\tau},w^\alpha,\frac{Dw^\alpha}{d\tau}\right)$ is the sum of two terms defined as follows

$$L_1\left(\xi^\alpha, u^\alpha, w^\alpha, \frac{Dw^\alpha}{d\tau}\right) := \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\frac{Dw^\alpha}{d\tau} + \sigma A_{\alpha;\beta} u^\alpha w^\beta, \tag{3.3}$$

$$L_2\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right) := \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} - R_{\alpha\beta\gamma\delta} r^\alpha\left(\frac{u^\beta}{\sqrt{u_\lambda u^\lambda}} + \sigma A^\beta\right) r^\gamma u^\delta$$
$$+ 2\sigma A_{\alpha;\beta}\frac{Dr^\alpha}{d\tau} r^\beta + \sigma A_{\alpha;\beta\gamma} u^\alpha r^\beta r^\gamma,$$

where $g_{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}$ and $A_\alpha$ are taken at the world line $\Gamma$ described by the functions $x^\alpha = \xi^\alpha(\tau)$.

Similarly like in Sect. 2, a heuristic explanation of the choice of the Lagrangian $L_3$ can be obtained from the $\Sigma$-approach. Differentiating Eqs (2.4) and (2.5) once again with respect to $\varepsilon$ and taking into account the definition $w^\alpha(\tau, \varepsilon) := \frac{\partial r^\alpha}{\partial \varepsilon}(\tau, \varepsilon)$ and the Ricci identity (2.6), one obtains

$$W_2[\xi^\alpha, r^\alpha, w^\alpha] := \frac{\partial W_1[\xi^\alpha, r^\alpha]}{\partial \varepsilon} = \int_{\tau_0}^{\tau_1} L_3\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}, w^\alpha, \frac{Dw^\alpha}{d\tau}\right) d\tau,$$

where

$$L_3\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}, w^\alpha, \frac{Dw^\alpha}{d\tau}\right) := \frac{\partial L_1\left(\xi^\alpha, u^\alpha, r^\alpha, \frac{Dr^\alpha}{d\tau}\right)}{\partial \varepsilon} = \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\frac{Dw^\alpha}{d\tau}$$
$$+ \sigma A_{\alpha;\beta} u^\alpha w^\beta + \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\alpha}{d\tau}\frac{Dr^\beta}{d\tau} - R_{\alpha\beta\gamma\delta} r^\alpha\left(\frac{u^\beta}{\sqrt{u_\lambda u^\lambda}} + \sigma A^\beta\right) r^\gamma u^\delta$$
$$+ 2\sigma A_{\alpha;\beta}\frac{Dr^\alpha}{d\tau} r^\beta + \sigma A_{\alpha;\beta\gamma} u^\alpha r^\beta r^\gamma.$$

Of course, such a derivation of the form of $L_3$ does not guarantee that (3.2) is a correct action simultaneously leading to Eqs (1.1), (2.1) and (3.1). This property must again be demonstrated in a direct way.

Let us compute the complete variation of the action (3.2) caused by independent variations $\delta\xi^\alpha$, $\delta r^\alpha$, $\delta w^\alpha$ and $\delta\tau$:

$$\delta W_2 = \int_{\tau_0}^{\tau_1}\Bigg[\frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dw^\beta}{d\tau}\bar{\Delta} u^\alpha + \sigma\frac{Dw^\alpha}{d\tau}\bar{\Delta} A_\alpha + \left(\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha\right)\bar{\Delta}\frac{Dw^\alpha}{d\tau} + \sigma u^\alpha w^\beta \bar{\Delta} A_{\alpha;\beta}$$
$$+ \sigma A_{\alpha;\beta}(u^\alpha \bar{\Delta} w^\beta + w^\beta \bar{\Delta} u^\alpha) - \left(\frac{h_{\beta\gamma}}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}\frac{u^\beta r^\gamma r^\delta u^\varepsilon}{u_\lambda u^\lambda}\right)\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}}\bar{\Delta} u^\alpha$$

$$-\frac{2h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\varepsilon u^\varepsilon}\frac{Dr^\gamma}{d\tau}\right)\bar{\Delta}u^\alpha+\frac{2h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dr^\beta}{d\tau}\bar{\Delta}\frac{Dr^\alpha}{d\tau}-\frac{1}{\sqrt{u_\lambda u^\lambda}}r^\alpha u^\beta r^\gamma u^\delta\bar{\Delta}R_{\alpha\beta\gamma\delta}$$

$$-\frac{2}{\sqrt{u_\lambda u^\lambda}}R_{\alpha\beta\gamma\delta}(u^\beta r^\gamma u^\delta\bar{\Delta}r^\alpha+r^\alpha r^\gamma u^\delta\bar{\Delta}u^\beta)-\sigma r^\alpha A^\beta r^\gamma u^\delta\bar{\Delta}R_{\alpha\beta\gamma\delta}$$

$$-\sigma R_{\alpha\beta\gamma\delta}(A^\beta r^\gamma u^\delta\bar{\Delta}r^\alpha+r^\alpha r^\gamma u^\delta\bar{\Delta}A^\beta+r^\alpha A^\beta u^\delta\bar{\Delta}r^\gamma+r^\alpha A^\beta r^\gamma\bar{\Delta}u^\delta)$$

$$+2\sigma\frac{Dr^\alpha}{d\tau}r^\beta\bar{\Delta}A_{\alpha;\beta}+\sigma u^\alpha r^\beta r^\gamma\bar{\Delta}A_{\alpha;\beta\gamma}+2\sigma A_{\alpha;\beta}\left(r^\beta\bar{\Delta}\frac{Dr^\alpha}{d\tau}+\frac{Dr^\alpha}{d\tau}\bar{\Delta}r^\beta\right)$$

$$\left.+\sigma A_{\alpha;\beta\gamma}(r^\beta r^\gamma\bar{\Delta}u^\alpha+u^\alpha r^\gamma\bar{\Delta}r^\beta+u^\alpha r^\beta\bar{\Delta}r^\gamma)+\frac{d}{d\tau}(L_3\delta\tau)\right]d\tau.$$

Applying the definitions (1.7), (1.8) and $\bar{\delta}w^\alpha(\tau):=\varepsilon\frac{\partial w^\alpha}{\partial\varepsilon}(\tau,0)$, making use of Eqs (1.5), (1.9), (2.6), (2.9) and of the relation

$$\Delta w^\alpha=\delta w^\alpha+\Gamma^\alpha{}_{\beta\gamma}w^\beta\delta\xi^\gamma,$$

we get the complete variation of the action (3.2) to be equal to

$$\delta W_2=-\int_{\tau_0}^{\tau_1}[(\mathrm{L}_2[w_\alpha]-g_\alpha)\bar{\delta}\xi^\alpha+2\,\mathrm{L}_1[r_\alpha]\bar{\Delta}r^\alpha+\mathrm{L}[\xi_\alpha]\bar{\Delta}w^\alpha]d\tau$$

$$+(\pi^{(2)}_\alpha\delta\xi^\alpha+2n_\alpha\delta r^\alpha+p_\alpha\delta w^\alpha)|^{\tau_1}_{\tau_0},\tag{3.4}$$

where

$$g_\alpha:=\left[\frac{g_{\alpha\beta}}{u_\lambda u^\lambda}\left(h_{\gamma\delta}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta}{d\tau}+R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau\right)-2R_{\alpha\gamma\delta\beta}r^\gamma r^\delta\right]\mathrm{L}[\xi^\beta]$$

$$+2\left[\frac{\mathrm{L}_1[r_\alpha]}{\sqrt{u_\lambda u^\lambda}}-\mathrm{L}[\xi_\alpha]\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)\right]\left(\frac{u_\varepsilon}{u_\lambda u^\lambda}\frac{Dr^\varepsilon}{d\tau}\right),$$

$$\pi^{(2)}_\alpha:=\frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}}\frac{Dw^\beta}{d\tau}+R_{\alpha\beta\gamma\delta}r^\beta r^\gamma\left(\frac{2u^\delta}{\sqrt{u_\lambda u^\lambda}}+\sigma A^\delta\right)$$

$$-\frac{2}{\sqrt{u_\lambda u^\lambda}}\frac{Dr_\alpha}{d\tau}\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)-\frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}}\left[\frac{h_{\beta\gamma}}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau}\right.$$

$$\left.+\frac{1}{u_\lambda u^\lambda}R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon-2\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)^2\right]+\sigma A_{\alpha;\beta}w^\beta$$

$$+\sigma A_{\alpha;\beta\gamma}r^\beta r^\gamma+2\Gamma^\gamma{}_{\beta\alpha}n_\gamma r^\beta+2\Gamma^\gamma{}_{\beta\alpha}p_\gamma w^\beta,$$

$$n_\alpha:=\pi^{(1)}_\alpha-\Gamma^\beta{}_{\gamma\alpha}p_\beta r^\gamma,$$

and $p_\alpha$ and $\pi^{(1)}_\alpha$ are given by Eqs (1.11) and (2.8), correspondingly.

The most common application of the expression (3.4) is again the Hamilton stationary action principle formulated now in the following form:

THEOREM 3.1. The complete variation $\delta W_2$ of the functional (3.2), generated by the variations $\delta\xi^\alpha$, $\bar{\delta} r^\alpha$, $\bar{\delta} w^\alpha$ and $\delta\tau$ satisfying the conditions

$$\delta\tau = \delta\xi^\alpha(\tau_0) = \delta\xi^\alpha(\tau_1) = \bar{\delta} r^\alpha(\tau_0) = \bar{\delta} r^\alpha(\tau_1) = \bar{\delta} w^\alpha(\tau_0) = \bar{\delta} w^\alpha(\tau_1) = 0$$

and being otherwise arbitary, is equal to zero if and only if $x^\alpha = \xi^\alpha(\tau)$ fulfil the Lorentz equations (1.1), $r^\alpha = r^\alpha(\tau)$ the equations of the first e.m. deviation (2.1), and $w^\alpha = w^\alpha(\tau)$ the second e.m. deviation equations (3.1) in an arbitrary parametrization.

The proof follows from (3.4) and from the vanishing of $g_\alpha$ under the conditions of the theorem.

Thus, in the approach based on the unified action principle (3.2) with the Lagrangian defined by (3.3), the dynamical properties of the second e.m. deviation are not being separated from those of the first e.m. deviation and of the motion determined by the Lorentz equations.

A second applicati-on of Eq. (3.4) concerns again the Noether identity. The variations $\delta\tau(\tau)$, $\bar{\delta}\xi^\alpha(\tau)$, $\bar{\delta} r^\alpha(\tau)$ and $\bar{\delta} w^\alpha(\tau)$, treated as given functions of $\tau$, will be generators of a dynamical symmetry iff the complete variation $\delta W_2$ caused by these generators is equal to

$$\delta W_2 = \int \frac{d}{d\tau}(\delta F_2) d\tau, \tag{3.5}$$

where $\delta F_2$ is a function of the dynamical variables $\xi^\alpha$, $u^\alpha$, $r^\alpha$, $\frac{Dr^\alpha}{d\tau}$, $w^\alpha$, $\frac{Dw^\alpha}{d\tau}$, the parameter $\tau$, and the generators. If the requirement (3.5) is satisfied, Eq. (3.4) imposes a condition on the generators of a symmetry in the form of a Noether identity

$$-[(\mathrm{L}_2[w_\alpha]-g_\alpha)\bar{\delta}\xi^\alpha+2\,\mathrm{L}_1[r_\alpha]\bar{\Delta} r^\alpha+\mathrm{L}[\xi^\alpha]\bar{\Delta} w^\alpha]$$
$$+\frac{d}{d\tau}(\pi_\alpha^{(2)}\bar{\delta}\xi^\alpha+2n_\alpha\bar{\delta} r^\alpha+p_\alpha\bar{\delta} w^\alpha+L_3\delta\tau-\delta F_2) = 0. \tag{3.6}$$

Leaving the discussion of the conservation laws which follow from (3.6) for a future publication, let us confine ourselves to some particular cases when $\delta\tau(\tau)$, $\bar{\delta}\xi^\alpha(\tau)$, $\bar{\delta} r^\alpha(\tau)$ and $\bar{\delta} w^\alpha(\tau)$ are given in terms of an arbitrary function of $\tau$.

First, let us examine for which Lagrangians $L_3$ the action (3.2) satisfies the condition (3.5) under the transformations of the form

$$\tau \mapsto \tilde{\tau} = f(\tau), \qquad V_4 \ni p \mapsto p' = p, \qquad TV_4 \ni r \mapsto r' = r, \qquad TV_4 \ni w \mapsto w' = w$$

or

$$\tilde{\tau} = f(\tau), \qquad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau), \qquad \tilde{r}^\alpha(\tilde{\tau}) = r^\alpha(\tau), \qquad \tilde{w}^\alpha(\tilde{\tau}) = w^\alpha(\tau) \tag{3.7}$$

depending on one arbitrary $C_2$ function $f$ describing an arbitrary reparametrization of the Lorentzian world line and of the vector fields $r$ and $w$ along it. It may be simply checked that the infinitesimal generators of the transformations (3.7) are defined by an arbitrary variation $\delta\tau(\tau)$, by the variations $\bar{\delta}\xi^{\alpha}(\tau)$ and $\bar{\delta}r^{\alpha}(\tau)$ given in the form (1.16) and (2.14) correspondingly, and by

$$\bar{\delta}w^{\alpha} = -\frac{dw}{d\tau}\,\delta\tau.$$

Substituting these expressions into (3.6) and taking into account that under the transformations (3.7) we have $\delta F_2 = 0$, one obtains

$$\left[(\mathrm{L}_2[w_\alpha]-g_\alpha)u^{\alpha}+2\mathrm{L}_1[r_\alpha]\frac{Dr^{\alpha}}{d\tau}+\mathrm{L}\,[\xi^{\alpha}]\frac{Dw^{\alpha}}{d\tau}\right]\delta\tau$$

$$+\frac{d}{d\tau}\left[\left(-\pi_{\alpha}^{(2)}u^{\alpha}-2n_{\alpha}\frac{dr^{\alpha}}{d\tau}-p_{\alpha}\frac{dw^{\alpha}}{d\tau}+L_3\right)\delta\tau\right]=0.$$

In virtue of the arbitrariness of $\delta\tau$, one gets from the expression above the following strong identities

$$(\mathrm{L}_2[w_\alpha]-g_\alpha)u^{\alpha}+2\,\mathrm{L}_1[r_\alpha]\frac{Dr^{\alpha}}{d\tau}+\mathrm{L}\,[\xi_\alpha]\frac{Dw^{\alpha}}{d\tau}=0, \tag{3.8}$$

$$L_3 \doteq \pi_{\alpha}^{(2)}u^{\alpha}+2n_{\alpha}\frac{dr^{\alpha}}{d\tau}+p_{\alpha}\frac{dw^{\alpha}}{d\tau} \tag{3.9}$$

satisfied independently of whether the functions $\xi^{\alpha}$, $r^{\alpha}$, and $w^{\alpha}$ are or are not solutions of the Lagrange and the Jacobi equations correspondingly. Thus, we have proved the following statement:

PROPOSITION 3.1. A unified action of the form (3.2) transforms under a $\tau$-reparametrization in accordance with (3.5) iff (3.8) and (3.9) are satisfied as strong identities.

COROLLARY 3.1. If $\Gamma$ is a Lorentzian world line defined by the Lorentz equations (1.1), then the vector fields $r^{\alpha}$ and $w^{\alpha}$ satisfy the strong identity

$$(\mathrm{L}_2[w_\alpha]-j_\alpha)u^{\alpha}+2\,\mathrm{L}_1[r_\alpha]\frac{Dr^{\alpha}}{d\tau}=0, \tag{3.10}$$

where

$$j_\alpha=\frac{2}{\sqrt{u_\lambda u^\lambda}}\mathrm{L}_1[r_\alpha]\left(\frac{u_\beta}{u_\gamma u^\gamma}\frac{Dr^{\beta}}{d\tau}\right),$$

independently of whether or not they are solutions of Eqs (2.1) and (3.1) correspondingly.

COROLLARY 3.2. If $\Gamma$ is a Lorentzian world line defined by the Lorentz equations (1.1) and $r^{\alpha}$ is the first e.m. deviation vector defined by Eqs (2.1), then the vector field $w^{\alpha}$ satisfies the strong identity

$$\mathrm{L}_2[w_\alpha]u^\alpha = 0 \tag{3.11}$$

independently of whether it is or is not a solution of Eqs (3.1) (see also [4]).

Secondly, let us suppose that the parametrization of a world line $\Gamma$ is fixed, but the vector fields $r^\alpha$ and $w^\alpha$, in accordance with Prop. 6.1 (*ii*) from [3], undergo the gauge transformations

$$\tau \mapsto \tilde{\tau} = \tau, \quad V_4 \ni p \mapsto p' = p, \quad TV_4 \ni r \mapsto r' = r+\kappa u,$$

$$TV_4 \ni w \mapsto w' = w+2\kappa \frac{Dr}{d\tau} +\kappa^2(\sigma \sqrt{u\cdot u}F\cdot u+u \frac{d}{d\tau} \ln \sqrt{u\cdot u})$$

or

$$\tilde{\tau} = \tau, \quad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau), \quad \tilde{r}^\alpha(\tilde{\tau}) = r^\alpha(\tau)+\kappa(\tau)u^\alpha(\tau),$$

$$\tilde{w}^\alpha(\tilde{\tau}) = w^\alpha(\tau)+2\kappa(\tau)\frac{Dr^\alpha}{d\tau}(\tau)+\kappa^2(\tau)\Big[\sigma\sqrt{u_\lambda(\tau)u^\lambda(\tau)}\,F^\alpha{}_\beta(\tau)u^\alpha(\tau)$$
$$+u^\alpha(\tau)\frac{d}{d\tau}\ln\sqrt{u_\lambda(\tau)u^\lambda(\tau)}\Big] \tag{3.12}$$

depending on one arbitrary $C_2$ function $\kappa(\tau)$. Replacing the function $\kappa(\tau)$ by $\delta\kappa(\tau)$ gives the following infinitesimal generators of the transformations (3.12)

$$\delta\tau = 0, \quad \delta\xi^\alpha = 0, \quad \delta r^\alpha = u^\alpha\delta\kappa, \quad \delta w^\alpha = 2\frac{Dr^\alpha}{d\tau}\delta\kappa. \tag{3.13}$$

Inserting (3.13) into the action (3.2) with the Lagrangian determined by (3.3), up to an approximation of terms of the order $(\delta\kappa)^2$, one obtains that

$$\delta F_2 = 2\left(p_\alpha\frac{Dr^\alpha}{d\tau}+\sigma A_{\alpha;\beta}u^\alpha r^\beta\right)\delta\kappa. \tag{3.14}$$

Substituting now Eqs (3.13) and (3.14) into the Noether identity (3.6) and making use of the arbitrariness of the function $\delta\kappa$, we receive the strong identity

$$\mathrm{L}_1[r_\alpha]u^\alpha+\mathrm{L}[\xi_\alpha]\frac{Dr^\alpha}{d\tau} = 0, \tag{3.15}$$

which was already obtained in Sect. 2, but as a consequence of reparametrization invariance of a different action (2.2). Hence, the following proposition is true:

PROPOSITION 3.2. The unified action (3.2) transforms under the simultaneous gauge transformations $r^\alpha \mapsto r^\alpha+\kappa(\tau)u^\alpha$ and $w^\alpha \mapsto w^\alpha+2\kappa(\tau)\frac{Dr^\alpha}{D\tau}+\kappa^2(\tau)\Big(\sigma\sqrt{u_\lambda u^\lambda}\,F^\alpha{}_\beta u^\beta+u^\alpha$

$\frac{d}{d\tau}\ln\sqrt{u_\lambda u^\lambda}\Big)$ generated by an arbitrary, differentiable function $\kappa(\tau)$ in accordance with (3.5) iff the strong identity (3.15) is satisfied.

COROLLARY 3.3. If $\Gamma$ is a Lorentzian world line defined by the Lorentz equations (1.1), then the vector field $r^\alpha$ satisfies the strong identity

$$\mathrm{L}_1[r_\alpha]u^\alpha = 0 \tag{3.16}$$

independently of whether it is or is not a solution of Eqs (2.1).

Thirdly, let us suppose that not only the parametrization of $\Gamma$ but also of the vector field $r^\alpha$ along $\Gamma$ is fixed, and the vector field $w^\alpha$ undergoes gauge transformation, i.e.

$$\tau \mapsto \tilde{\tau} = \tau, \quad V_4 \ni p \mapsto p' = p, \quad TV_4 \ni r \mapsto r' = r,$$
$$TV_4 \ni w \mapsto w' = w + \psi u$$

or

$$\tilde{\tau} = \tau, \quad \tilde{\xi}^\alpha(\tilde{\tau}) = \xi^\alpha(\tau), \quad \tilde{r}^\alpha(\tilde{\tau}) = r^\alpha(\tau), \quad \tilde{w}^\alpha(\tilde{\tau}) = w^\alpha(\tau) + \psi(\tau)u^\alpha(\tau), \tag{3.17}$$

where $\psi(\tau)$ is an arbitrary $C_2$ function. Replacing $\psi(\tau)$ by $\delta\psi(\tau)$ results in the following form of infinitesimal generators of the transformations (3.17)

$$\delta\tau = 0, \quad \delta\xi^\alpha = 0, \quad \delta r^\alpha = 0, \quad \delta w^\alpha = u^\alpha \delta\psi, \tag{3.18}$$

and inserting (3.18) into(3.2) with the Lagrangians (3.3) gives

$$\delta F_2 = p_\alpha u^\alpha \delta\psi. \tag{3.19}$$

Substituting Eqs (3.18) and (3.19) into (3.6) provides, due to the arbitrariness of $\delta\psi$, the strong identity (cf. also (1.17), (2.21))

$$\mathrm{L}[\xi_\alpha]u^\alpha = 0 \tag{3.20}$$

independently of whether $\xi^\alpha$ is or is not a solution of Eqs (1.1).

PROPOSITION 3.3. The unified action (3.2) transforms under the gauge transformation $w^\alpha \mapsto w^\alpha + \psi(\tau)u^\alpha$ generated by an arbitrary, differentiable function $\psi$, in accordance with (3.5) iff the strong identity (3.20) is satisfied.

An additional application of Eq. (3.4) concerns again a method of deriving the Hamilton-Jacobi equations which correspond to the action (3.2). Let us consider again a family of Lorentzian world lines $x^\alpha = \xi^\alpha(\tau)$ all of which for $\tau = \tau_0$ pass through the same point $\xi_0^\alpha = \xi^\alpha(\tau_0)$. Along every of these lines one can take the families of the first and the second e.m. deviation vector fields, with the components $r^\alpha = r^\alpha(\tau)$ and $w^\alpha = w^\alpha(\tau)$ correspondingly, such that every of their members takes the same values $r_0^\alpha = r^\alpha(\tau_0)$ and $w_0^\alpha = w^\alpha(\tau_0)$ at the point $\xi_0^\alpha$ for different Lorentzian world lines passing through $\xi_0^\alpha$. By means of such a family the functional (3.2) can be turned into a function $U_2: TV_4 \times TV_4 \times R \to R$ analogous to the principal Hamilton function. The values $U_2\,(x^\alpha, r^\alpha, w^\alpha, \tau)$ of this principal function are defined by the integral of the form (3.2) in that the upper limit is replaced by the value $\tau \in I$, and the integration is carried out along the curve $\Gamma$ which joins the initial point $(\xi_0^\alpha, r_0^\alpha, w_0^\alpha)$ with the point $(x^\alpha, r^\alpha, w^\alpha)$ entering the argument of $U_2$. Then, due to the assumptions about $\Gamma$, the integrand in Eq. (3.4) vanishes, and the differential of the principal function $U_2$ is equal to

$$dU_2 = \pi_\alpha^{(2)} dx^\alpha + 2n_\alpha dr^\alpha + p_\alpha dw^\alpha.$$

Thus

$$\frac{\partial U_2}{\partial \tau} = 0 \tag{3.21}$$

and

$$\frac{\partial U_2}{\partial x^\alpha} = \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dw^\beta}{d\tau} + R_{\alpha\beta\gamma\delta} r^\beta r^\gamma \left( \frac{2u^\delta}{\sqrt{u_\lambda u^\lambda}} + \sigma A^\delta \right) - \frac{2}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\alpha}{d\tau}$$
$$\times \left( \frac{u_\beta}{u_\lambda u^\lambda} \frac{Dr^\beta}{d\tau} \right) - \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} \left[ \frac{h_{\beta\gamma}}{u_\lambda u^\lambda} \frac{Dr^\beta}{d\tau} \frac{Dr^\gamma}{d\tau} + \frac{1}{u_\lambda u^\lambda} R_{\beta\gamma\delta\varepsilon} u^\beta r^\gamma r^\delta u^\varepsilon \right.$$
$$\left. - \left( \frac{u_\beta}{u_\lambda u^\lambda} \frac{Dr^\beta}{d\tau} \right)^2 \right] + \sigma (A_{\alpha;\beta} w^\beta + A_{\alpha;\beta\gamma} r^\beta r^\gamma) + 2\Gamma^\beta{}_{\gamma\alpha} \left( \frac{h_{\beta\delta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\delta}{d\tau} \right.$$
$$\left. + \sigma A_{\beta;\delta} r^\delta \right) r^\gamma + \Gamma^\beta{}_{\gamma\alpha} \left( \frac{u_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\beta \right) w^\gamma, \tag{3.22}$$

$$\frac{\partial U_2}{\partial r^\alpha} = 2 \left( \frac{h_{\alpha\beta}}{\sqrt{u_\lambda u^\lambda}} \frac{Dr^\beta}{d\tau} + \sigma A_{\alpha;\beta} r^\beta \right), \tag{3.23}$$

$$\frac{\partial U_2}{\partial w^\alpha} = \frac{u_\alpha}{\sqrt{u_\lambda u^\lambda}} + \sigma A_\alpha. \tag{3.24}$$

From a formal point of view, Eq. (3.21) could be considered again as the Hamilton-Jacobi equation on $U_2$. Eqs (3.22)–(3.24) give relations between the partial derivatives of $U_2$ and the generalized momenta $p_\alpha$, $2n_\alpha$ and $\pi_\alpha^{(2)}$ determined by the action (3.2). The algebraic relations fulfilled by $\frac{u^\alpha}{\sqrt{u_\lambda u^\lambda}}$ and $h_{\alpha\beta}$ result now in three partial differential equations on a single function $U_2$:

$$g^{\alpha\beta} \left( \frac{\partial U_2}{\partial w^\alpha} - \sigma A_\alpha \right) \left( \frac{\partial U_2}{\partial w^\beta} - \sigma A_\beta \right) = 1, \tag{3.25}$$

$$g^{\alpha\beta} \left( \frac{\partial U_2}{\partial r^\alpha} - 2\sigma A_{\alpha;\gamma} r^\gamma \right) \left( \frac{\partial U_2}{\partial w^\beta} - \sigma A_\beta \right) = 0, \tag{3.26}$$

$$g^{\alpha\beta} \left( \frac{\partial U_2}{\partial w^\alpha} - \sigma A_\alpha \right) \left( \frac{\partial U_2}{\partial x^\beta} - \Gamma^\gamma_{\delta\beta} \frac{\partial U_2}{\partial r^\gamma} r^\delta - \Gamma^\gamma_{\delta\beta} \frac{\partial U_2}{\partial w^\gamma} w^\delta - \sigma A_{\beta;\gamma} w^\gamma - \sigma A_{\beta;\gamma\delta} r^\gamma r^\delta \right)$$

$$= -\tfrac{1}{4} g^{\alpha\beta}\left(\frac{\partial U_2}{\partial r^\alpha} - 2\sigma A_{\alpha;\gamma} r^\gamma\right)\left(\frac{\partial U_2}{\partial r^\beta} - 2\sigma A_{\beta;\delta} r^\delta\right) - R^\alpha{}_\beta{}^\gamma{}_\delta$$

$$\times\left(\frac{\partial U_2}{\partial w^\alpha} - \sigma A_\alpha\right) r^\beta \frac{\partial U_2}{\partial w^\gamma} r^\delta, \tag{3.27}$$

which form a system of equations for the function $U_2$ depending, because of Eq. (3.21), in general on twelve variables $x^\alpha$, $r^\alpha$ and $w^\alpha$.

To reveal the connection of Eqs (3.25)–(3.27) with the Hamilton-Jacobi equations from the previous sections, let us observe that this system of equations admits the separation of variables $x^\alpha$, $r^\alpha$ and $w^\alpha$, which can be achieved by representing $U_2$ in the following form (cf. also [1])

$$U_2(x^\alpha, r^\alpha, w^\alpha) = w^\alpha \frac{\partial U}{\partial x^\alpha} + r^\alpha r^\beta \left(\frac{\partial U}{\partial x^\alpha}\right)_{;\beta} + 2r^\alpha \frac{\partial V}{\partial x^\alpha} + W, \tag{3.28}$$

where $U$, $V$ and $W$ are functions of $x^\alpha$ alone. Differentiating then (3.28) with respect to $w$, $r$ and $x^\alpha$, one gets

$$\frac{\partial U_2}{\partial w^\alpha} = \frac{\partial U}{\partial x^\alpha}, \quad \frac{\partial U_2}{\partial r^\alpha} = 2\left[\left(\frac{\partial U}{\partial x^\alpha}\right)_{;\beta} r^\beta + \frac{\partial V}{\partial x^\alpha}\right],$$

$$\frac{\partial U_2}{\partial x^\alpha} = w^\beta \left(\frac{\partial U}{\partial x^\beta}\right)_{;\alpha} + \Gamma^\beta_{\gamma\alpha} \frac{\partial U}{\partial x^\beta} w^\gamma + \left(\frac{\partial U}{\partial x^\gamma}\right)_{;\beta\alpha} r^\beta r^\gamma$$

$$+\Gamma^\beta_{\gamma\alpha}\left(\frac{\partial U}{\partial x^\delta}\right)_{;\beta} r^\gamma r^\delta + \Gamma^\beta_{\gamma\alpha}\left(\frac{\partial U}{\partial x^\beta}\right)_{;\delta} r^\gamma r^\delta + 2\left(\frac{\partial V}{\partial x^\beta}\right)_{;\alpha} r^\beta$$

$$+2\Gamma^\beta_{\gamma\alpha} \frac{\partial V}{\partial x^\beta} r^\gamma + \frac{\partial W}{\partial x^\alpha},$$

and this after substituting into (3.25)–(3.27) and taking into account the Ricci identity, shows that $U$, $V$ and $W$ must satisfy the following equations

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha} - \sigma A_\alpha\right)\left(\frac{\partial U}{\partial x^\beta} - \sigma A_\beta\right) = 1, \tag{3.29}$$

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha} - \sigma A_\alpha\right)\frac{\partial V}{\partial x^\beta} = 0, \tag{3.30}$$

$$g^{\alpha\beta}\left(\frac{\partial U}{\partial x^\alpha} - \sigma A_\alpha\right)\frac{\partial W}{\partial x^\beta} = -g^{\alpha\beta}\frac{\partial V}{\partial x^\alpha}\frac{\partial V}{\partial x^\beta}, \tag{3.31}$$

in which the first two coincide obviously with Eqs (2.28), (2.29).

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