A METHOD FOR NUMERICAL CALCULATION OF LARGE-n PROBABILITIES FROM AN ANALYTICALLY GIVEN GENERATING FUNCTION

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A method for numerical calculation of large-n probabilities from an analytically given form of a generating function is proposed. The method, which uses the Cauchy integral formula, is found to be relatively quickly convergent and gives good control of errors of calculations.

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In phenomenological models of particle production one can often relatively easily derive the closed analytic formula for the generating function $\phi(z)$ of the multiplicity distribution. It is then easy to calculate the multiplicity moments of low order (related to the derivatives of $\phi(z)$ at $z = 1$) and probability distribution for small n (related to the derivatives of $\phi(z)$ at $z = 0$).

However, large-n probabilities, that is high-order derivatives of the generating function, though in principle calculable even analytically, are not calculable practically due to the increasing complication of formulae with increasing n.

The aim of this paper is to propose a simple method for numerical calculations of large-n probabilities based on Cauchy formula. The proposed method is very efficient even when implemented on personal computers. Furthermore, it provides simple and reliable way for estimating errors of the calculation. It was checked practically in Ref. [1] and found rather efficient.

To begin with, we remind some classical results in statistical analysis [4]. The generating function of a probability distribution, defined as follows:

$$\phi(z) = \sum_{n=0}^{\infty} P_n z^n, \qquad z \in C, \tag{1}$$

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is always at least analytic in the open circle $|r| < 1$ and continuous in its closure due to the positiveness of P_n and normalization condition: $\sum_{n=0}^{\infty} P_n = 1$.

This fact allows one to use the Cauchy integral formula [2] to calculate derivatives of ϕ at $z = 0$ (which are related with P_n's):

$$\phi^{(n)}(0) = n!P_n = \frac{n!}{2\pi i} \int\limits_{C_r} \frac{\phi(\zeta)}{\zeta^{n+1}}\, d\zeta \tag{2}$$

where C_r is the circle centred at 0 with the radius $0 < r < 1$. Usual substitution $\zeta = re^{i\varphi}$ gives:

$$P_n = \frac{1}{2\pi r^n} \int\limits_0^{2\pi} \phi(re^{i\varphi})e^{-in\varphi}\, d\varphi. \tag{3}$$

Discretization of this integral for numerical integration by dividing the circle into $N > n$ equal parts yields:

$$P_n \simeq \frac{1}{Nr^n} \sum_{k=1}^{N-1} \phi(re^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ink}{N}}. \tag{4}$$

This is of course only approximation of the Riemann integral of (3).

To determine what we in fact calculate in the approximation sum (4) we substitute (1) in the right-hand side of the last formula and get:

$$\frac{1}{r^n N} \sum_{k=0}^{N-1} \phi(re^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ink}{N}} = \sum_{k=0}^{\infty} P_{n+kN} r^{kN}$$

$$= P_n + r^N P_{n+N} + r^{2N} P_{n+2N} + \dots \tag{5}$$

due to the fact that:

$$\sum_{k=0}^{N-1} e^{\frac{2\pi i}{N}(l-n)k} = \begin{cases} N \text{ when } l-n \text{ is divisible by } N, \\ 0 \text{ in the other case.} \end{cases}$$

Even if ϕ is not analytic in the closure of the unit circle, the formula (2) still holds and gets a simpler form:

$$P_n + P_{n+N} + P_{n+2N} + \dots = \frac{1}{N} \sum_{k=0}^{N-1} \phi(e^{\frac{2\pi ik}{N}})e^{-\frac{2\pi ink}{N}}. \tag{6}$$

One can see now that if N is large enough, the right-hand side of (6) gives the estimation for P_n with the deviation equal to $P_{n+N}+P_{n+2N}+ \dots$. In practice, the P_n's drop very fast with increasing n, so to calculate P_n with deviation beneath 1% it is enough to take $N \approx 3n$. In [2] we have calculated the derivatives up to the 200$^{\text{th}}$ one of moderately complicated generating function taking $N = 500$.

The importance of N is due to the fact that it counts the number of points in which we have to evaluate ϕ. Computer-time consuming problems require N not to be taken too large; especially when a single evaluation of ϕ takes too much time. On the other hand, in order to improve the exactness of P_n's one has to enlarge N.

Another source of errors of the results for P_n's is the finiteness of floating-point arithmetic carried out on computers. It influences the calculations in such a way that it decreases the reliability of all the obtained P_n's due to the accumulation of errors during the addition of large number of terms. This kind of errors, however, may be easily estimated looking at the imaginary part of the result of (6) which has to be equal to zero in exact calculations.

The method has straightforward generalizations for calculation of the probability distribution of pairs (and more) of natural numbers from the generating function of two and more variables. It may be of interest when calculating the correlation of pairs, triples etc. of produced particles.

The proposed method was applied in Ref. [2] for calculating multiplicity distribution of particles and related quantities in the Giovannini-Van Hove model (Ref. [1]) with identical clusters. Although this model gives rather complicated generating function, it was possible to calculate, with very good accuracy, the first two hundred derivatives. Also, the backward-forward correlations in the above-mentioned model were easily calculated using, as previously, a personal computer (Ref. [6]).

In conclusion, we would like to stress the fact that the proposed method enables one to calculate numerically, with sufficient accuracy, probability distribution from a given analytical form of the generating function up to reasonably large n. The values of systematic errors may be reliably estimated.

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