ON CLASSICAL SOLUTIONS OF NONABELIAN GAUGE THEORIES*

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A comprehensive review of known solutions of Yang-Mills field equations with dynamically coupled sources is presented. A special emphasis is put on real exact and smooth solutions, except for Euclidean instantons and monopoles. A number of 'no go' theorems is presented. A systematic approach to the study of nonlinear equations is described. This article is intended to supplement the existing reviews on the classical Yang-Mills theory.

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1. *Introduction*

1a. Why the classical Yang-Mills theory?

The immediate answer is the following: because the standard Quantum Chromodynamics (QCD) does not solve the fundamental, no doubt, problem of elementary particle physics, the quark confinement. Apart from that, QCD, being renormalizable and describing some important features of high energy scattering fairly well, could still be regarded as inevitable part of any future theory of fundamental interactions. This is one reason, why, having no other serious candidates, one could study the internal structure of classical nonabelian gauge theory. The hope is that the change of vacuum in QCD would eventually remedy the existing problems while not causing new difficulties. The possible way (and promising one) to do this is to quantize the SU(3) gauge theory around a classical solution of Yang-Mills equations. This procedure changes propagators but the theory is still renormalizable. This approach is outlined, e.g., in [1]. One can expect that the modified propagators would lead to the desired "area law" [2] which according to Wilson [3] criterion signals the quark confinement. The hope that QCD should explain confinement of quarks was supported by computer simulations of nonabelian fields on lattice which show the existence of confining forces between quark sources [4].

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Another incentive to the study of the Yang-Mills theory follows from the success of bag model in hadron spectroscopy. Its basic assumption that hadrons are confined within bags is put by hand into the theory. It can be obtained within the framework of 4-dimensional σ-models [5], which, however, suffer from nonrenormalizability and therefore cannot be regarded as true theories of strong interactions. Thus there is a serious demand for solitonic solutions of the classical Yang-Mills-matter theory, which could provide the lowest order approximations to the low energy hadrons while quantum corrections are found as a series expansion. Unfortunately, as will be shown later, the only model that would allow for classical solitons is the Yang-Mills-Dirac theory, although even there a result of Magg [6] excludes the existence of spherically symmetric solutions.

The next motive is due to the complexity of QCD. Being nonlinear, the Yang-Mills theory with its surprising complexity of a solution space (including the instability, branching phenomena and the nonexistence of global differential gauges) bears a closer resemblance to the General Relativity rather than to electrodynamics. The analysis of the classical system would be of help in separating the genuine quantum effects from the classical background. However, we think that to do this one has to quantize the Yang-Mills theory in a more rigorous way than it is usually done. Perhaps the framework of Segal's ([7] and references therein) quantization procedure is the best one.

1b. Content of this article

Our main objective is to give a short account of recent progress made in solving the classical (pure) Yang-Mills theory as well as Yang-Mills fields coupled to matter and/or gravity. The present article is essentially intended to supplement Actor's excellent review [8] which presents almost all solutions known prior to the middle of 1978. Actor described complex and real solutions in Minkowski space, Euclidean instantons, monopoles and dyons in the Yang-Mills-Higgs theory. He did not discuss string-like and vortex solutions.

We limited our interest to smooth solutions with real gauge fields in order to avoid discussion of distribution-valued solutions (their status is unclear since Yang-Mills equations are nonlinear) as well as of complex gauge fields (which would yield a complex action S undesirable from the point of view of quantum theory where the functional $\int [dA] \exp(iS)$ plays an important part). To have a consistent and closed picture, we consider the Yang-Mills theory with dynamically introduced sources (the only exceptions are the two examples of Yang-Mills equations with constant external sources discussed in Sect. 6). Thus we deal with a complicated system of gauge and matter equations. Moreover, our attention is restricted to solutions in Minkowski space, with a few exceptions incorporating new results in Euclidean space. With all these limitations in mind, let us add that we aim to achieve two goals in this review. First, we intend to review all new explicit solutions of selfcontained nonabelian gauge theories. The discovery of explicit solutions in systems of equations as complicated as Yang-Mills equations is almost always accidental and therefore it should be appreciated. Secondly, we would like to present certain mathematical techniques which allow either for gaining useful information about solutions of nonlinear equations without necessity to solve them or for approximation of solutions. Twelve years after publication of papers of 't Hooft and Belyavin et al. [9, 10] which marked the beginning of the period of rapid development of the classical Yang-Mills theory, a conclusion can be drawn that one of the greatest achievements of the period is an extraordinary enrichment of mathematical methods used in theoretical physics. To supply documentary evidence (but without pretensions to completeness), we will show in more detail certain applications of functional analysis and of bifurcation and stability theory.

Before going to a more detailed presentation of this review, let us point out those topics that will not be touched.

We do not intend to give a systematic account on recent progress on instantons (which is little) and Yang-Mills-Higgs monopoles (which is impressive). There exist comprehensive reviews (e.g., by Leznov and Saveliev [11]) on instantons. The literature on monopoles is vast ([11–17] and references therein). An adequate treatment of these topics would have considerably enlarged the length of this paper. We also omit discussion of
(*i*) complex and/or singular solutions,
(*ii*) topological and geometrical setting of gauge theories (see for instance [18]),
(*iii*) symmetry considerations (see the review of Gu Chaochao and a paper of Basler [19]),
(*iv*) the Yang-Mills equations with external sources [20],
(*v*) gauge fixing problems ([8, 21, 22] and references therein).
Chakrabarti [23] considered the Yang-Mills fields in curved spaces and Magg [24] discussed selected topics in the classical Yang-Mills theory with and without external sources.

The following subsection provides a short description of foundations of nonabelian gauge theory. Section 2a is devoted to presentation of "no go" theorems for gauge fields interacting with scalar or spinor fields and gravity. The remaining parts of Sect. 2 describe: 3 families of solutions of SU(2) gauge fields interacting with scalar fields and solutions of the Yang-Mills-Dirac system.

Sect. 3 outlines progress made in Euclidean theory. Impressive results were obtained in the Yang-Mills-Higgs model (Sect. 4a). Apart from infinite number of monopoles this model is shown to possess an entirely new class of non(anti-)selfdual solutions in SU(3) (Sect. 4b) and SU(2) (Sect. 4c) theory. In that latter case the Lusternik-Šnirelman theory was helpful (Appendix B). New results on dyons are reported in Sect. 4d. Sect. 4e contains a plane-wave solution of the Yang-Mills-Higgs equations.

The next Section lists all (to the author's knowledge) known solutions to the pure Yang-Mills theory in Minkowski space and one solution in compactified Minkowski space $S^3 \times S^1$. Sect. 5b gives a short account on approximate and numerical solutions.

Section 6 describes bifurcation theory and its application in the SU(2) Yang-Mills theory with static external sources. Two families of solutions periodic in time are presented for a static abelian charge; one remains nonzero and nonabelian in the limit of a vanishing charge.

Two distinct notions of stability in field theories with constraints on initial data are discussed in Sect. 7. Some results on specific abelian solutions (Wu-Yang monopoles and Coulomb-like singular potentials) and on gauge symmetric potentials are reported.

The next Section deals with a chaotic behaviour in Yang-Mills equations.

Section 9 is devoted to miscellaneous topics in the Yang-Mills theory, including the local and global Cauchy problem, total colour screening and the status of perturbative solutions to selfcontained gauge theories.

The last part comprises a number of problems to be solved in classical nonabelian gauge theory.

List of references consists mainly of papers published between 1979–1986 which were directly related to topics discussed here. Some papers have probably been omitted; the author would like to apologize those authors, whose works have not been cited.

1c. Rudiments of the Yang-Mills classical theory. Notation

We will work mainly in 4-dimensional Minkowski space with a flat metric $\{g_{\mu\nu}\} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$. Only in Sections 2a, b and 3 we report work done in 4-D Euclidean space.

Only compact gauge groups will be exploited. It is because they possess definite the Killing-Cartan tensor, which in a group algebra plays the role of a metric tensor. For compact groups there exists a representation in which structure constants are completely antisymmetric and generators T^a satisfy the following commutation relations

$$[T^a, T^b] = f^{abc}T^c. \tag{1.1}$$

Here T^a's are assumed to be antihermitian. The components of the Killing-Cartan tensor are proportional to $H^{ab} = f^{adf}f^{bdf}$. The metric tensor could be chosen to be positive definite, which allows for positiveness of the energy of gauge fields, $E = \int d^3x[\frac{1}{2} F^a_{0i}F^b_{0i}H^{ab} + \frac{1}{4} F^a_{ik}F^b_{ik}H^{ab}]$. For noncompact gauge groups the energy is not definite.

The nonabelian gauge theory is constructed similarly as classical electrodynamics (for detailed description see [25]). There are potentials A^a_μ, where $(\mu, \nu, \ldots) = 0, 1, 2, 3$ is the space-time index while $a, b, c, \ldots = 1, \ldots, \dim G$ is a gauge (colour) label ($\dim G$ is a dimension of a gauge group). As usual, latin indices $i, j, k, \ldots = 1, 2, 3$ refer to space components of tensors. The field $A_\mu = A^a_\mu T^a$ belongs to the algebra $\mathbf{G}$ of a gauge group G; since we assume T^a to be always in fundamental representation, A_μ belongs to the fundamental representation of $\mathbf{G}$. However, it transforms under changes of basis in $\mathbf{G}$ (under a gauge transformation) as a pseudovector,

$$A'_\mu = h^{-1}A_\mu h + h^{-1}\partial_\mu h; \tag{1.2}$$

the first term transforms as a vector, i.e., according to the adjoint representation of G. For given A^a_μ one defines

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \tag{1.3}$$

which is the nonabelian analogue of a field strength tensor of classical electrodynamics. Here g is a coupling constant, often omitted. This omission is always possible because of a scale invariance of the sourceless Yang-Mills theory; in the presence of sources putting $g = 1$ requires a suitable rescaling of charges.

In some places we will use "electric" and "magnetic" fields, $E_i^a = F_{0i}^a$ and $B_i^a = \frac{1}{2}\varepsilon_{ijk}F_{jk}^a$, respectively. F^a's satisfy the Bianchi identity

$$D_\mu F_{\alpha\beta} + D_\alpha F_{\beta\mu} + D_\beta F_{\mu\alpha} = 0, \tag{1.4}$$

where $D_\mu F = \partial_\mu F + [A_\mu, F]$ is the covariant derivative. To show (1.4) one has to utilize the Jacobi identities. $F_{\mu\nu}$ transforms under a gauge group action as a vector, $F'_{\mu\nu} = h^{-1}F_{\mu\nu}h$. Generalizing the Maxwell equations one arrives at

$$D_\mu F^{\mu\nu} = j^\nu, \tag{1.5}$$

where j^ν is the nonabelian current. Since the left hand side of (1.5) transforms under the adjoint representation of G, the current j^ν has to transform like a vector,

$$j'_\mu = h^{-1}j_\mu h. \tag{1.6}$$

Since $D_\mu D_\nu F^{\mu\nu} = \frac{1}{2}[F_{\mu\nu}, F^{\mu\nu}] = 0$, it follows that

$$D_\mu j^\mu = 0. \tag{1.7}$$

Thus the current j^μ, unlike its abelian (maxwelian) analogue, is not preserved during evolution:

$$\partial_0 \int j_0 d^3x = \int (D_i j^i - f^{abc} A_0^b j_0^c) d^3x; \tag{1.8}$$

the right hand side of (1.8) is nonzero, in general. The proper definition of conserved gauge--invariant charges in nonabelian gauge theories is controversial [26]. Note, however, that for $j^i = 0$ and $A_0^a = 0$,

$$\partial_0 j^0 = 0, \tag{1.9}$$

i.e., the charge is time-independent. In Sect. 6 we construct time-dependent solutions with a static scalar current.

Equations (1.5) could be divided into two groups: the set of dynamical equations $D_\mu F^{\mu i} = j^i$ and the set of constraints $D_i F^{i0} = j^0$. The constraint equations generalize the Gauss law of classical electrodynamics. They are preserved in time, if $D_i F^{0i} - j^0 = 0$, then also $\partial_0(D_i F^{i0} - j^0) = 0$; this follows from

$$\partial_0(D_i F^{i0} - j^0) = D_0(D_i F^{i0} - j^0) = -D_k(D_\mu F^{\mu k} - j^k) = 0. \tag{1.10}$$

The YM equations (1.5) could be derived from the lagrangian $L = -\frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} - A_\mu^a j^{a\mu}$, by means of the variational principle applied to $S = \int L d^3x$.

Supposing $j^\mu = 0$ and using the Noether theorem (note that S is Poincare invariant; in fact it is even conformally invariant) one gets ten conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \tag{1.11}$$

$$\partial_\mu(x^\alpha T^{\beta\mu} - x^\beta T^{\alpha\mu}) = 0, \tag{1.12}$$

where $T_{\mu\nu} = \dfrac{-1}{4} g_{\mu\nu} F_{\alpha\beta}^a F^{a\alpha\beta} + F_{\mu\gamma}^a F^{a\cdot\gamma}_{\ \nu}$ is the energy-momentum tensor.

The current j^μ can be of dynamical origin, being the current of some matter field φ. φ's (either bosons or fermions) are usually supposed to be in the adjoint representation of a gauge group, but on several occasions they are in fundamental representation. In each case the action of covariant derivative as well as transformation of matter fields are distinct.

Assuming φ is in the fundamental representation we have

$$\varphi' = h\varphi, \quad D_\mu\varphi = \partial_\mu\varphi + A_\mu\varphi, \tag{1.13}$$

while for φ in the adjoint representation

$$\varphi' = h^{-1}\varphi h, \quad D_\mu\varphi = \partial_\mu\varphi + [A_\mu, \varphi]. \tag{1.14}$$

2. Yang-Mills fields coupled to matter fields and/or the Einstein gravity

In the first part of this Section we will present a number of 'no go' theorems. They give simple criteria to decide whether in a model of interest nontrivial solutions could appear. We focus our attention on new results which were published after 1978; the earlier ones have already been presented in [8]. Thus we describe Glassey and Strauss's [27] results concerning the Yang-Mills-Klein-Gordon system with a simple extension of their theorem. Next we discuss the YM system coupled to gravity or to the Dirac field.

In subsequent subsections we present new solutions, first in the Yang-Mills-(scalar field) theory following Vinet [28] and Actor [29] and then in the Yang-Mills-Dirac system, after Doneaux et al. [30], Meetz [31] and others.

2a. Nonexistence results

The first nonexistence result in the sourceless YM theory is due to Deser [32] who proved the absence of nonzero static finite energy solutions. Then Coleman, Weder and Pagels [33] proved nonexistence of time-dependent finite energy solutions (with localized energy). Glassey and Strauss [34] have shown even more — that all the energy of a solution of pure YM equations radiates along the light cone. In subsequent paper Glassey and Strauss [27] proved the absence of static solitons and solitary waves in the Yang-Mills-Klein-Gordon theory. Here we will follow just their second work. Let the lagrangian be

$$-L = \tfrac{1}{4} F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{1}{2} (D_\mu\varphi)^a (D^\mu\varphi)^a + V(\varphi). \tag{2.1}$$

Here the gauge group is supposed to be arbitrary, apart from being compact. The scalar field is in the adjoint representation of a gauge group.

Since L does not depend explicitly on time and space coordinates, there are ten conservation laws, which follow from the Poincare invariance via the Noether theorem. These are four momentum conservation laws

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.2}$$

and six angular momentum conservation laws

$$\partial_\mu(T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha) = 0, \tag{2.3}$$

where

$$T_{\mu\nu} = F^a_{\mu\gamma}F^{a\gamma}_{\nu.} - \tfrac{1}{4}\, g_{\mu\nu}F^a_{\alpha\beta}F^{a\alpha\beta} + (D_\mu\varphi)^a(D_\nu\varphi)^a$$

$$-\frac{g_{\mu\nu}}{2}\,[(D_\alpha\varphi)^a(D^\alpha\varphi)^a + 2V(\varphi)]. \tag{2.4}$$

In addition, there are five useful identities, which could be obtained from (2.3), (2.4) and equations of motion, by performing certain algebraic manipulations [27]. For $V(\varphi)$ being a monomial in φ of the fourth order, these identities coincide with those 5 Noether's laws which are due to conformal invariance. (If $V(\varphi/\lambda) = \lambda^{-4}V(\varphi)$, then $L(\varphi/\lambda, \lambda^{-1}\partial_\mu, A_\mu/\lambda) = \lambda^{-4}L$; thus the model happens to be dilatation invariant. As it is also Poincare invariant, it implies the full conformal invariance [35] and therefore the existence of corresponding conservation laws).

One of these identities reads

$$\partial^0[(t^2+r^2)T^{00} + 2tx_kT^{0k} + 2t(D^0\varphi)^a\varphi^a - \varphi^a\varphi^a] + 2t[(V'(\varphi))^a\varphi^a - 4V(\varphi)]$$

$$= \partial_k\{(t^2+r^2)T^{0k} + 2t[x_jT^{jk} + (D^k\varphi)^a\varphi^a]\}. \tag{2.5}$$

Integrating it over all space and discarding boundary terms, one arrives at

$$\int d^3x\{\partial_0[(t^2+r^2)T^{00} + 2tx_kT^{0k} + 2t(D^0\varphi)^a\varphi^a - \varphi^a\varphi^a]$$

$$+2t[(V'(\varphi))^a\varphi^a - 4V(\varphi)]\} = 0. \tag{2.6}$$

Let $(V'(\varphi))^a\varphi^a \geqslant 4V(\varphi)$; (2.6) yields now

$$\int d^3x[(t^2+r^2)T^{00} + 2tx_kT^{0k} + 2t(D^0\varphi)^a\varphi^a - \varphi^a\varphi^a] \leqslant \text{const}. \tag{2.7}$$

The const on the r.h.s. of (2.7) is given by

$$\text{const} = \int_{t_0=0} (r^2T^{00} - \varphi^a\varphi^a)d^3x, \tag{2.8}$$

where T^{00} is given in terms of initial data of the corresponding Cauchy problem. From that we get the required falloff of fields at spatial infinity

$$\varphi^a \sim r^{-(\varepsilon+3/2)}, \qquad \partial_i\varphi^a \sim r^{-(\varepsilon+5/2)},$$

$$T^{00} \sim r^{-(\varepsilon+5)} \text{ (which is satisfied if } A^a_\mu \sim r^{-(\varepsilon+5/4)},\ \partial_kA^a_\mu \sim r^{-(\varepsilon+5/2)}). \tag{2.9}$$

Glassey and Strauss proved that the integrand Z of the l.h.s. of (2.7) is nonnegative if $V(\varphi)$ is nonnegative, and that the following inequality holds

$$\text{const} \geqslant \int_{|\vec{x}|<R+(1-\varepsilon)t} d^3xZ \geqslant (\varepsilon t - R)^2 \int_{|\vec{x}|<R+(1-\varepsilon)t} (T^{00} + V(\varphi)/2) \geqslant 0 \tag{2.10}$$

(where $t > R\varepsilon^{-1}$); thus the energy spreads out, even if it was localized at $x_0 = 0$.

THEOREM [27]

Assume that V satisfies the inequalities $0 \leqslant 4V(s) \leqslant sV'(s)$. Let $R > 0$ and $0 < \varepsilon \leqslant 1$. Then as $t \to \infty$, $\int\limits_{|\vec{x}|<R+(1-\varepsilon)t} T^{00}d^3x = O(t^{-2})$, where T^{00} is the energy density of the YMKG system.

It could be proved, further, that there are no finite energy solutions of the form $f(x-at)$, if $a < 1$ (except the trivial one, $F^a_{\mu\nu} = \varphi^a = 0$). Thus all solutions radiate along the light cone.

Glassey and Strauss proved also absence of solitary waves for a massive scalar field, that is for $V(\varphi) = m^2\varphi^2 + c|\varphi|^{p+1}$ ($m > 0$, $c > 0$, $p > 1$), under the assumption $0 \leqslant 2V(\varphi) \leqslant V'(\varphi)^a\varphi^a$. In particular their results imply nonexistence of static finite energy solutions, supposing that either

$0 \leqslant 4V(s) \leqslant V'(s)s$ (for the massless scalar field)

or $0 \leqslant 2V(s) \leqslant V'(s)s$ (for the massive scalar field),

and assuming boundary conditions (2.9).

For static Yang-Mills-Klein-Gordon equations these conclusions hold even under weaker assumptions [36]; for $\int d^3xV'(\varphi)\varphi \geqslant 0$ and under substantially relaxed boundary conditions

$$A^a_\mu \sim r^{-(\varepsilon+3/4)}, \quad \partial_i A^a_\mu \sim r^{-(\varepsilon+3/2)}, \quad \varphi^a \sim r^{-(\varepsilon+3/2)}. \tag{2.11}$$

We do not prove this here but instead we will present similar result for YM field coupled to a Schrödinger field. Proofs are very similar.

Let the Schrödinger field ψ be in the fundamental representation of any compact group.

Supposing statics the YMS equations read

$$D_\mu F^{\mu\nu} = j^\nu, \quad \text{where} \quad j^0_a = i\psi^+ T^a\psi,$$

$$j^i_a = \frac{1}{2m}(\psi^+T^aD_i\psi - (D_i\psi)^+T_a\psi), \tag{2.12a}$$

$$-\frac{1}{2m}D_iD_i\psi - iA_0\psi + V(\varphi) = 0. \tag{2.12b}$$

Here T^a denotes antihermitian generators of the group algebra and the covariant derivative is defined in (1.13).

We will need Eqs (2.12b) and the Gauss-like YM equation

$$(D_iF^{i0})^a = i\psi^+T^a\psi; \tag{2.13}$$

multiply Eqs (2.13) and (2.12b) by A^a_0 and ψ^+, respectively, and integrate over all space. Then, integrating by parts and discarding boundary terms (which vanish because of (2.11)), one obtains

$$-\int F^a_{i0}F^a_{i0}d^3x = \int \psi^+ iA_0\psi d^3x, \tag{2.14}$$

$$\frac{1}{2m}\int (D_i\psi)^+ D_i\psi d^3x - \int \psi^+ iA_0\psi d^3x + \int \psi^+ V(\psi)d^3x = 0. \tag{2.15}$$

The first two terms in (2.15) are nonnegative (the second term is nonnegative because of (2.14)), hence if $\int \psi^+V(\psi)d^3x \geqslant 0$ then $\psi = 0$, $F^a_{i0} = 0$. The absence of nonzero magnetic part follows now from [32].

Now we prove the following

THEOREM [37]

The Einstein-Yang-Mills-Klein-Gordon system of equations does not possess static smooth solutions with nonvanishing curvatures if $\int \varphi^a \frac{\partial V(\varphi)}{\partial \varphi^a} \geqslant 0$ and conditions at spatial infinity are assumed:

$$g_{\mu\nu} = \eta_{\mu\nu} + O(1/r), \quad |\partial_l g_{ij}| = O(1/r^2),$$

$$\partial_k g_{00} = \frac{M}{r^2} + O(1/r^2) \ (M \text{ is either zero or positive}),$$

$$A^a_\mu = O(r^{-(\varepsilon+3/4)}), \quad \partial_i A^a_\mu = O(1/r^{3/2}). \tag{2.16}$$

Proof

The EYMKG equations read

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = T_{\mu\nu}(\mathrm{YM}) + T_{\mu\nu}(\mathrm{KG}), \tag{2.17}$$

$$D_i F^{i0} = j^0, \tag{2.18}$$

$$D_\mu F^{\mu k} = j^k, \tag{2.19}$$

$$D_i D^i \varphi + D_0 D^0 \varphi - V'(\varphi) = 0, \tag{2.20}$$

where $j^a_\mu = f^{abc}\varphi^b(D_\mu\varphi)^c$; $T_{\mu\nu}(\mathrm{YM}) = F^a_{\mu\gamma}F^{a\gamma}_\nu - \frac{g_{\mu\nu}}{4}F^a_{\alpha\beta}F^{a\alpha\beta}$; $T_{\mu\nu}(\mathrm{KG}) = (D_\mu\varphi)^a(D_\nu\varphi)^a - \frac{g_{\mu\nu}}{2} \times [(D_\alpha\varphi)^a(D^\alpha\varphi)^a + 2V(\varphi)]$; φ is in the adjoint representation of any compact group G. Supposing that $g_{0i} = 0$ and that all fields do not depend on time, one could show, dealing with equations (2.18), (2.20) in a way analogous to that described above in the case of the YMS theory, that $\varphi^a = 0$, $A^a_0 = 0$.

Thus we are left with the Einstein-Yang-Mills Eqs (2.18), (2.19) with vanishing sources, $j^a_\mu = 0$. The identity $T^\mu_\mu(\mathrm{YM}) = 0$ yields now $R = 0$.

Weder [34] observed that for $\frac{\partial}{\partial r} g_{00} = \frac{-M}{r^2} + o(1/r^2)$, where $M \geqslant 0$, the integral over a space oriented section W_3, $\int_{W_3} R^0_0 \sqrt{-g_{00}} d^3x = 0$. Here is the outline of the calculation. In a suitable system of coordinates the integrand $\sqrt{-g}R^0_0 = -\partial^k(g_{ik}\sqrt{-g_{00}}\partial^i\sqrt{-g_{00}})$; using the Gauss law one arrives at $\int_{W_3} R^0_0\sqrt{-g}d^3x = +M$.

Therefore $\int_{W_3} T^0_0\sqrt{-g}d^3x = \int_{W_3} R^0_0\sqrt{-g}d^3x = +M \geqslant 0$; since $T^0_0 < 0$, it implies $M = 0$ and $F^a_{ik} = 0$.

Equations (2.17) now give $R_{\mu\nu} = 0$ and $g_{00} = const = -1$ (because of boundary conditions).

This results in local flatness of W_3 [38].

The asymptotic condition on g_{00} could be relaxed to allow for positive newtonian mass $(-M) > 0$.

THEOREM [37]

Let the falloff of A^a_i, $\partial_k A^a_i$ at infinity be $A^a_i = O(1/r)$, $\partial_k A^a_i = O(1/r^2)$, respectively. Suppose $|\Gamma^\mu_{i\mu}| \in L_3(W_3)$ and $|g_{\mu\nu}-\eta_{\mu\nu}| = O(1/r)$, $|\partial_k g_{\mu\nu}| = O(1/r^2)$ at infinity. Then the EYM system does not possess solutions A^a_i small in the Sobolev $W_{1,3}$ norm provided that the L_3 norm of $|\Gamma^\mu_{i\mu}|$ is sufficiently small ($\Gamma^\mu_{i\mu}$ is the contraction of Christoffel's symbol).

The proof, which employs several integral inequalities and estimates, is sketched in Appendix A.

Using arguments as above we could see that the manifold W_3 is also locally flat.

The Einstein-Yang-Mills equations in (2+1)-dimensional space-time are considerably simpler, Deser [39] proved in that case absence of static solutions.

Only partial "no go" results were obtained for Yang-Mills-Dirac equations. Magg[1] [6] has found that for massless Dirac equations there exist no nontrivial solutions with finite energy, provided the static Yang-Mills fields coupled to stationary fermion fields are spherically symmetric. This result excludes the possibility to have spherically symmetric bag-like configuration at the classical level of nonabelian gauge theory, at least for SU(2) gauge group.

Below we sketch Magg's proof, which holds only for SU(2) gauge group.

The YMD equations are given by Eqs (2.28) (see below), with Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

Its solutions conserve the energy momentum tensor, $\partial_\nu T^{\mu\nu} = 0$, where

$$T^{\mu\nu} = \frac{-g^{\mu\nu}}{4} F^a_{\alpha\beta}F^{a\alpha\beta} - F^{a\mu\alpha}F^{a\cdot\nu}_{\alpha}$$

$$-i \operatorname{tr}[\psi^+\gamma^0\gamma^\nu(\partial^\mu + A^\mu)\psi],$$

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[1] The author thanks Dr. M. Basler from Jena for bringing his attention to this paper.

ψ is a Dirac bispinor. Let ψ be stationary, $\psi = e^{-iEt}\psi_0$. Straightforward calculation shows that

$$0 = \int \partial_i(x_k T^{ki})dV = \int T_i^i dV = \int T^{00} dV$$
$$= \int [\tfrac{1}{2}(E_i^a E_i^a + B_i^a B_i^a) + i \operatorname{tr}(\psi^+ T^a A^{a0}\psi) + E \operatorname{tr}(\psi^+\psi)]d^3x. \tag{2.21}$$

Magg proved, provided the fields are spherically symmetric, that the interaction term $i\int \operatorname{tr}(\psi^+ T^a A^{a0}\psi)d^3x$ has to vanish; then for $E \geqslant 0$ the absence of nonzero solutions is evident. The sign of E appears to be unessential and nontrivial solutions are absent also for $E < 0$ [6].

Świerczyński [40] has studied the Yang-Mills-Higgs-Dirac theory (to be specific: dyons coupled to fermions). He proved, assuming spherical symmetry and a gauge group $G = SU(2)$, that solutions have to have the following property : their Dirac current vanishes for finite energy configurations. This result generalizes a property which was characteristic for solutions presented in [41]. It implies also absence of spherically symmetric finite energy static solutions of SU(2) Yang-Mills-Dirac field equations, in full agreement with Magg [6].

In Sect. 9 we prove the absence of small solutions of Yang-Mills-Dirac equations for arbitrary compact group and for massive fermion fields. The nonzero solutions (if any) are large in a sense of certain functional norm and are singular (as functions of the gauge coupling constant "g").

The interest in soliton solutions of YM-(matter) equations stems from the success of the bag model [42]. One might wonder if the phenomenological assumptions of the bag model could be obtained within the matter-gauge classical theory. The above nonexistence results exclude such a possibility in most models.

2b. Solutions of the Yang-Mills-scalar field system

The following set of Yang-Mills equations coupled to a non-linear Klein-Gordon field,

$$(D_\mu F^{\mu\nu})^a = \varepsilon^{abc}\varphi^b(D^\nu\varphi)^c; \tag{2.22a}$$

$$(D_\mu D^\mu\varphi)^b = \lambda\varphi^b\varphi^a\varphi^a \tag{2.22b}$$

has been studied by Vinet [28].

Here ε^{abc} is the completely antisymmetric Levi-Civita tensor and $\{\varphi^a\}$ is a triplet of scalar fields in the adjoint representation of SU(2).

Vinet made use of the conformal invaiiance of these equations to map them on compactified Minkowski space. This space M is diffeomorphic to $SU(2)\times U(1)$ with $SU(2, 2)$ acting globally on it. Provided that solutions are invariant under relevant subgroups of $SU(2, 2)$, the equations of motion reduce to some algebraic equations.

Vinet found a variety of solutions which are mostly complex. Those real, smooth and having finite energy are now reproduced. They are invariant under $SU(2)_L \times U(1)_R \times U(1)$ subgroup of $SU(2, 2)$. In explicit form they are:

$$\varphi^a = \frac{\delta^{a3}\varphi}{\tau}, \tag{2.23}$$

$$A_j^a = \frac{-2b_k^a}{\tau^2}[\varepsilon^{jkl}x^l - x^j x^k - \tfrac{1}{2}\delta^{jk}(1-x_\alpha x^\alpha)], \tag{2.24}$$

$$A_0^a = \frac{1}{\tau^2}(-2b_j^a x^0 x^j), \tag{2.25}$$

where

$$\{b_k^a\} = \begin{pmatrix} d & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}, \quad \tau^2 = x_0^2 + \tfrac{1}{4}(1+x_\alpha x^\alpha)^2, \tag{2.26}$$

d, b and φ are either
(*i*) $d = 3/2-(11/2)^{1/2}/2$, $b = 0.37$, $\varphi = \pm i\,2.14$;
or
(*ii*) solutions of algebraic equations:

$$(d-1)(2d-1)(d-3)+\tfrac{3}{4}(2d+1) = 0,$$

$$b^2 = \left(-d^2+3d+\frac{1}{4\lambda}-1\right)\Big/(1-2\lambda),$$

$$\varphi^2 = -(1+8b^2)/\lambda. \tag{2.27}$$

d, b and φ are real for $\lambda \in (-\infty, -6.5)$. Numerical plots show existence of two different curves of real solutions (branches II and III in Vinet's paper).

Actor [29] has studied the Yang-Mills SU(2) theory with a scalar SU(2) doublet (i.e., a scalar field is in the fundamental representation of SU(2)).

The selfinteraction of a scalar field was described as before by a term of quartic order in the lagrangian. His solutions are singular and we do not discuss them.

All solutions presented above spread out as the time x_0 tends to infinity and move inward as x_0 tends to zero. Thus they do not correspond to solitonic configurations of the fields, although their energy is finite.

Cervero and Estevez [43] have studied the Einstein-Yang-Mills theory coupled to a scalar nonlinear field. They found a few classes of solutions that are singular.

2c. Solutions of the Yang-Mills-Dirac system

The Yang-Mills fields coupled to the massless Dirac spinors form the following system of equations:

$$(\gamma^\mu)^{AB}\partial_\mu(\psi)_B^s + (-ig)(\gamma^\mu)^{AB}A_\mu^a \tfrac{1}{2}\sigma_{st}^a(\psi)_B^t = 0, \tag{2.28a}$$

$$\partial^\mu F_{\mu\nu}^a + g f_{abc}A^{\mu b}F_{\mu\nu}^c = g(\bar{\psi})_A^s(\gamma^\nu)^{AB}\tfrac{1}{2}\sigma_{st}^a(\psi)_B^t. \tag{2.28b}$$

Here ψ stands for the Dirac spinor in the fundamental representation of the SU(2) gauge group and γ's are Dirac matrices, $\{\gamma^\alpha, \gamma^\beta\} = -2\eta^{\alpha\beta}$ ($\eta^{\alpha\beta}$ is the Minkowski metric).

Using conformal transformations these equations could be projected on compactified Minkowski space $S^3 \times S^1$. To simplify equations of motion Doneux et al. [30] assumed invariance of solutions under various subgroups acting on $S^3 \times S^1$. Among nontrivial solutions, which were obtained in this way, only one type corresponds to real gauge potentials:

$$A_0^a = \frac{1}{\tau^2}\left[-2u_k^a x_0 x^k + \tfrac{1}{2} u_0^a(1 + x_0^2 + x_i x^i)\right], \tag{2.29a}$$

$$A_j^a = \frac{1}{\tau^2}\left[-2u_k^a(\varepsilon^{jkl}x^l - x^j x^k - \tfrac{1}{2}\delta^{jk}(1 - x_\mu x^\mu) - u_0^a x^0 x^j\right], \tag{2.29b}$$

$$\psi = \tau^{-3/2}\begin{pmatrix} \dfrac{1 + ix^0 - ix_j\sigma^j}{(1+2ix^0 + x_\alpha x^\alpha)^{1/2}} & 0 \\ 0 & \dfrac{1 - ix^0 + ix_j\sigma^j}{(1-2ix^0 + x_\alpha x^\alpha)^{1/2}} \end{pmatrix}, \tag{2.29c}$$

with

$$(u_i^a) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{5}{6}\varepsilon_a & 0 \\ 0 & 0 & \frac{5}{6}\varepsilon_b \end{pmatrix}, \quad (u_0^a) = \begin{pmatrix} \varepsilon_b\sqrt{61}/3 \\ 0 \\ 0 \end{pmatrix},$$

or

$$(u_i^a) = \tfrac{1}{2}\mathbf{1}, \quad (u_0^a) = 0.$$

Here $\tau = (x_0^2 + \frac{1}{4}(1 + x_\mu x^\mu)^2)^{1/2}$. The symbol ε denotes $\pm$ and subscript attached to it designates independent $\pm$; each combination gives a different solution. The Dirac spinor is complex; its current as well as contribution to energy are nonzero. As for YMKG case, these solutions are nonsolitonic. The energy spreads out at timelike infinity and concentrates near the origin at $x_0 = 0$.

Basler [44] found spherically symmetric static solutions of the SU(2) Yang-Mills equations coupled to an isospin 1/2 Dirac equations with a nonzero spinor mass. They all have infinite energy and are singular at the origin.

Meetz [31] considered the same system of equations as Doneux et al. [30]. His solution depends on the function q which solves the following nonlinear differential equation:

$$\ddot{q} + 2q(q+1)(q+2) \mp \tfrac{1}{2} = 0. \tag{2.30}$$

The Yang-Mills potential is given by:

$$A_0^a = -4q(t(x))w^2 x^0 x^a, \tag{2.31a}$$

$$A_i^a = 4q(t(x))\left[\delta_{ia}\tfrac{1}{2}(1 - x_\mu x^\mu) + \varepsilon_{aji}x^j + x^\alpha x^i\right]. \tag{2.31b}$$

Dirac fields are given by:

$$\psi^\pm = (2w)^3 \frac{1 + (x^0)^2 + x_k\sigma^k(1 \mp x^0)}{[(1 + (x^0)^2)^2 - x_k x^k(i \mp x^0)^2]^{1/2}}\chi^{(\pm)} f^\pm(t(x)), \tag{2.32}$$

where

$$\chi^+ = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \quad \chi^- = \begin{pmatrix} 0 \\ \varphi \end{pmatrix}, \quad \varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$f^{\pm} = \exp\,[\mp \tfrac{3}{2}\, i(t + \int_0^t q(t')dt')].$$

Signs $\pm$ correspond to two different solutions.

The variable t is related to Minkowski time x^0 by the formula $x_0 = \dfrac{\sin t}{\cos t + \dfrac{1 - x_k x^k}{1 + x_k x^k}}$, and the function w is given by $w = [(1 - x_\alpha x^\alpha)^2 + 4x_k x^k]^{1/2}$.

These solutions have finite energy and they are special cases of those presented in Appendix A of [30].

Akhouri and Weisberger [45] found constant Yang-Mills potentials coupled to plane-wave Dirac spinors. Their rather lengthy solution is presented in Sect. 2.3 of [45].

Quite recently the following solution of equations (2.22) [46] was discovered:

$$\psi = \begin{pmatrix} g\sigma^2 + \chi\sigma^3\sigma^2 \\ g^*\sigma^2 + \chi^*\sigma^3\sigma^2 \end{pmatrix}, \quad \arg g = \arg \chi \pm \pi \tag{2.33}$$

(there is no longer distinction between spinor and isospinor indices), g and χ are constants satisfying the identity $4\sqrt{2}(g^*\chi + g\chi^*) = -\alpha^3$, $A_3^a = A_i^3 = A_0^a = 0$; the remaining components of the vector potential are given by either formula (6.29) or (6.30) (see Section 6).

Only gauge fields of this solution contribute to the energy density, which is constant in time. For physical relevance of this, periodic in time, solution see the end of Sect. 6. Further solutions of space-independent Yang-Mills-Dirac equations are found in [47].

Note:

Such a form of ψ corresponds to the chiral representation of Dirac matrices,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}.$$

The same γ's were used by Meetz [31], while Doneux et al. [30] preferred

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i\sigma^2 \\ \sigma^2\sigma^i & 0 \end{pmatrix}.$$

In the paper of Marciano and Muzinich [48] the potential scattering of fermions in the field of a magnetic monopole has been analyzed. The fermions are described by the Dirac equation (2.28a) but, in contrast to the previous study, there appears the massive term.

Substitution of

$$\psi = e^{iEt} \begin{pmatrix} X^+ & \sigma^2 \\ X^- & \sigma^2 \end{pmatrix},$$

with

$$X^{\pm}_{im} = \delta_{im} G^{\pm}(r) + \sigma^a_{im} \frac{x^a}{r} P^{\pm}(r) \tag{2.34}$$

into the Dirac equations gives the following equations [48]

$$\left(\frac{d}{dr} - A\right) G^{\pm} = i(E \pm m) P^{\mp}, \tag{2.35a}$$

$$\left(\frac{d}{dr} + \frac{2}{r} + A\right) P^{\pm} = i(E \pm m) G^{\mp}, \tag{2.35b}$$

where

$$A \equiv \frac{1}{r}\left(\frac{Mr}{\sinh Mr} - 1\right).$$

The above equations should be solved with the initial conditions $G^{\pm} = O(1)$, $P^{\pm} = O(r)$ at $r = 0$, to ensure the regularity of ψ at the origin.

The solutions of Marciano and Muzinich are the following

$$G^{\pm} = \frac{c^{\pm}}{r}\left\{\left[\frac{2ik}{M} + \tanh\left(\frac{Mr}{2}\right)\right] e^{-ikr} - \left[\frac{2ik}{M} - \tanh\left(\frac{Mr}{2}\right)\right] e^{ikr}\right\}, \tag{2.36a}$$

$$P^{\mp} = \frac{-c^{\pm}}{r} \frac{k}{E \pm m}\left\{\left[\frac{2ik}{M} + \coth\left(\frac{Mr}{2}\right)\right] e^{-ikr} + \left[\frac{2ik}{M} - \coth\left(\frac{Mr}{2}\right)\right] e^{ikr}\right\}, \tag{2.36b}$$

where $k^2 = E^2 - m^2$.

They satisfy the required initial conditions. The vanishing of k implies $P^{\pm} = 0$; this, in turn, results in either

$$(i)\ G^{-} = 0 \quad \text{and} \quad G^{+} = \frac{-c^{+}}{r} 2 \tanh(Mr/2) \ \text{if}\ E = m, \tag{2.37a}$$

or

$$(ii)\ G^{+} = 0 \quad \text{and} \quad G^{-} = -\frac{c^{-}}{r} 2 \tanh(Mr/2) \quad \text{if} \quad E = -m. \tag{2.37b}$$

The constants $c^{\pm}$ are arbitrary.

Let us point out that Marciano and Muzinich have employed the standard representation of Dirac matrices in which the Yang-Mills current of a fermion field reads

$$j^a_0 = \tfrac{1}{2} \operatorname{tr}\{(\chi^+)^\dagger \chi^+ \sigma^a\} + \tfrac{1}{2} \operatorname{tr}\{(\chi^-)^\dagger \chi^- \sigma^a\}, \quad j^a_i = \tfrac{1}{2} \operatorname{tr}\{(\chi^+)^\dagger \sigma_i \chi^- \sigma^a + (\chi^-)^\dagger_{} \sigma_i \chi^+ \sigma^a\}.$$

The Yang-Mills current of the solutions (2.37) vanishes identically. Therefore the Dirac solution (2.34) (with P's and G's given by (2.37)) together with the Bogomolny-Prasad-Sommerfield monopole constitutes a selfconsistent solution of the Yang-Mills-Higgs-Dirac equations. To be specific, let us point out that the fermion and scalar field were supposed to noninteract directly.

For completeness we write the BPS solution.

The Yang-Mills potential reads

$$A_0^a = 0,$$

$$A_j^a = \varepsilon_{aji} \frac{x_i}{r^2} \left[\frac{Mr}{\sinh(Mr)} - 1 \right], \tag{2.38}$$

and the Higgs scalar field is given by

$$\varphi^a = \frac{x^a}{r} \left[\operatorname{ctgh}(Mr) - \frac{1}{Mr} \right]. \tag{2.39}$$

The gauge group is SU(2), so $a = 1, 2, 3$.

For the Higgs equation in the BPS limit see the Eq. (4.3) below.

The energy $\varepsilon = \int T^{00} d^3x = m + E \int d^3x \operatorname{tr}(\psi^+\psi)$ of the solution is infinite for $E \neq 0$ since $\int d^3x \operatorname{tr}(\psi^+\psi) = |c^\pm|^2 32\pi \int_0^\infty dr \tanh^2 \frac{Mr}{2} = \infty$. Here m denotes the energy of the magnetic monopole, which is finite.

Remark

By the use of a suitable gauge and for the relevant values of $c^\mp$ Marciano and Muzinich were able to exhibit the interesting physical meaning of the scattering solutions (2.36). Namely, their solutions describe processes in which only the pure charge exchange is observed. This may indicate, according to Marciano and Muzinich, that the proton in the field of the BPS monopole can decay [48], in accordance with the suggestion of Rubakov [49].

3. New results in the Euclidean gauge theory

The discovery of one-instanton solution of the Euclidean SU(2) theory [10] marked the beginning of the Yang-Mills (classical theory) boom. The main impetus had been noted just in the Euclidean gauge theory. Its results have been impressive — infinite number of exact explicit smooth and finite energy solutions was found. As they were soon proven not to be the general solutions, a procedure to generate fully general ones was invented [8]. Unfortunately, that constructive method did not produce solutions more general than those already guessed by physicists (because of complexity of algebraic constraints to be solved in the ADHM procedure — see [8]), up to the authors knowledge. In addition to references given by Actor which concern only the SU(2) gauge group, there is a number of generalizations on different compact semisimple groups [11].

Since the publication of Actor's review a relatively little progress has been made in construction of explicit solutions. Actor [50] presented new "heated" solutions, which have infinite energy, in contrast to instantons. They are self(anti)-dual, i.e.,

$$F^{a\mu\nu} = \pm\varepsilon^{\mu\nu\alpha\beta}F^{a}{}_{\alpha\beta}, \tag{3.1}$$

where $F^{a\mu\nu}$ is the strength field tensor. The gauge group is SU(2).

Assumption that a gauge potential is axially symmetric yields its representation either in the form given by Witten [51]

$$A_0^a = -\frac{x^a}{r}A_0(r, x_0),$$

$$A_i^a = \varepsilon_{aim}\frac{x_m}{r^2}(1+\varphi_2(r, x_0)) - \frac{x_a x_i}{r^2}A_1(r, x_0)$$

$$-\left(\delta_{ai} - \frac{x_a x_i}{r^2}\right)\frac{1}{r}\varphi_1(r, x_0) \tag{3.2}$$

or the one due to 't Hooft:

$$A_\mu^a = -\eta^{\pm}_{a\mu\nu}\partial_\nu \ln\varrho, \qquad \eta^{\pm}_{a\mu\nu} = \varepsilon_{0a\mu\nu} \pm \delta_{0\mu}\delta_{a\nu} \mp \delta_{0\nu}\delta_{a\mu}.$$

Here $r = |\vec{x}|$.

The resulting equations on A_0, A_1, φ_1, φ_2 are invariant under conformal transformations in the (r, x_0) plane. Having one explicit solution one could produce infinite number of "heated" ones, using conformal transformations (i.e., heating, in Actor's terminology) $x = x_0 + ir \to y(x)$ where the analytic function y is periodic in Euclidean time variable x_0. These new solutions are periodic in time. Some interesting properties of a few solutions are discussed in [50].

Interesting results concerning stability of solutions of the Euclidean YM theory were obtained by Bourguignon and Lawson [52] and Taubes [53].

The YM equations in Euclidean space constitute a semilinear set of elliptic equations which could be derived from the action

$$S = -\int d^4x F^a{}_{\mu\nu}F^{a\mu\nu} \tag{3.3}$$

using the variational principle.

Its solution A_μ^a is said to be stable if $S(A_\mu^a + \delta A_\mu^a) \geqslant S(A_\mu^a)$. (Note: it is evident that this inequality could not be strict, for two reasons: (*i*) gauge invariance — it is unessential since it can be removed by fixing gauge; (*ii*) zero modes [1], which are consequences of the fact that the (spatial) symmetry of a solution is smaller than that of equations of motion). Bourguignon and Lawson, Jr. [52] proved that any stable solution having finite energy has to be self (antiself)-dual. Thus nonself (antiself)-dual solutions (if they exist — they are not known up to now[2]) has to be unstable. Taubes [53] calculated the number of instability modes of such solutions, which depend only on their topological number.

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[2] The only exception is the Burzlaff's solution which is known approximately (see Sect. 4). It solves the Yang-Mills-Higgs equations in the limit of a vanishing Higgs potential and thus could be interpreted as a (Euclidean)-time independent solution of the Euclidean Yang-Mills theory — see [8] for the discussion. However its energy (which includes also integration over the time variable) is infinite.

4. *Solutions of the Yang-Mills-Higgs system*

The Yang-Mills-Higgs (YMH) action functional is

$$S(A, \varphi) = -\int [\tfrac{1}{4} F^a{}_{ik}F^{aik} + \tfrac{1}{2}(D_i\varphi)^a(D^i\varphi)^a + V(\varphi)]d^3x, \tag{4.1}$$

where φ is in the SU(2) adjoint representation and $V(\varphi)$ denotes a quartic selfinteraction term $V(\varphi) = \lambda(|\varphi|^2-1)^2$. The potential A^a_μ is supposed to vanish at spatial infinity while $|\varphi| = \sqrt{\varphi^a\varphi^a}$ should tend to 1.

The interest in classical solutions of such a model began with the pioneering work of 't Hooft [9] who obtained an approximate monopole-like solution. However up to now little is known about minimizers of the action (4.1) and none of them is known explicitly, for $\lambda \neq 0$. Some new results are reviewed in Section 4d.

The sector with $\lambda = 0$ and boundary conditions as above, which is called the Bogomolny-Prasad-Sommerfield limit has been explored much better. There is a subclass of configurations (of self- or antiself-dual fields) in which the Euler-Lagrange equations reduce to a system of first order equations and form a completely integrable system. Apart from them, there are also nonself(anti)-dual extrema of (4.1), which do not satisfy the above mentioned set of first order equations.

4a. New solutions of the Bogomolny-Prasad-Sommerfield equations

The Bogomolny equations read

$$B_i = \pm D_i\varphi. \tag{4.2}$$

Let us remark that any solution of (4.2) satisfies the YMH equations. Indeed, differentiation of (4.2) with respect to D^* or D yields

$$0 = D_iB^i = D_iD^i\varphi,$$

or

$$-D_k\varepsilon^{kij}B_j = D_jF^{ji} = \mp\varepsilon^{ijk}D_kD_i\varphi = \mp\tfrac{1}{2}\varepsilon^{ijk}[F_{jk}, \varphi] = \mp[B_i, \varphi] = [\varphi, D^i\varphi], \tag{4.3}$$

respectively, in accordance with Eqs (2.22) (note that here $\lambda = 0$ and $A^a_0 = 0$).

After the discovery of the Bogomolny-Prasad-Sommerfield solution in 1975–6 [8], no one had succeeded in finding new solutions of (4.2) up to 1981.

As late as in 1981 the equivalence of the axially and mirror symmetric Bogomolny equations and the Ernst equation has been proved [13]. It allows for systematic generating of SU(2) monopoles of arbitrary charge, using the Bäcklund transformation [13] or the Belinski-Zakharov inverse scattering method. Since that time many papers appeared which describe a few methods of generating solutions. Nahm [54] adapted the ADHM procedure [8] to the Bogomolny equations. Recently Corrigan and Goddard [14] demonstrated the completeness of ADHM construction for SU(2) monopoles.

All solutions to the Bogomolny equations are stable [53, 55] for all compact gauge groups provided their energy is finite. The BPS model is actually a rapidly developing branch of YM theory and these remarks do not pretend to be complete. For details and further references see [13, 14, 54]. An interesting application of the Bogomolny-Prasad--Sommerfield monopole is discussed in [48] (see also the discussion at the end of Sect. 2c).

The review of Giacomelli [17] contains an extensive bibliography of more than 400 papers (mainly experimental) on monopoles including also the references to the complete bibliography.

4b. Nonminimal solutions in a SU(3) model

Burzlaff [56] looked for solutions of the Yang-Mills-Higgs SU(3) theory supposing that

$$A_0^a = 0, \quad A_i = (1-B(r))\,[E, \partial_i E], \quad \varphi = A(r)K, \tag{4.4}$$

where $E = [\lambda_7, -\lambda_5, \lambda_2]$ is the multiplet of antihermitian generators of the maximal SU(2) subalgebra of SU(3), K_{ij} is the complementary 5-plet of SU(3) algebra and $E = \frac{x_i}{r} E_i$, $K = \frac{x_i x_j}{r^2} K_{ij}$.

Since the Higgs potential $V(\varphi)$ is supposed to vanish, the equations of motion reduce to

$$(r^2 A')' = 6AB^2, \tag{4.5a}$$

$$r^2 B'' = B(B^2 + r^2 A^2 - 1). \tag{4.5b}$$

Burzlaff did not find exact solutions of Eqs (4.5), but proved, using a function theoretic argument, the existence of a smooth nonzero finite-energy solution. He also found some approximate solutions at infinity and near the origin. Unfortunately, it is not clear whether these asymptotics could be smoothly matched.

A solution of (4.5) cannot satisfy the Bogomolny equations (4.2) unless the solution is zero.

Indeed, the vector magnetic part reads

$$B_i = \frac{1}{r^2}(B^2-1)x_i E + B'(r)\partial_i K, \tag{4.6a}$$

while the covariant derivative of φ is

$$D_i \varphi = A'(r) x_i K - AB\partial_i E. \tag{4.6b}$$

Thus $B_i = \pm D_i \varphi$ if and only if

$$B_i = D_i \varphi = 0, \tag{4.7}$$

i.e., $B^2 = 1$ and $A = 0$.

This is the first known example of a solution to the YMH equations with the vanishing Higgs potential which does not satisfy the Bogomolny equations. It has to be unstable since all solutions which attain local minima of the action in each topological section corresponding to nontrivial monopole numbers [53] satisfy the Bogomolny equations.

4c. The nonminimal SU(2) solution of Taubes

Taubes' proof [57] that SU(2) YMH theory without Higgs potential has a nonselfdual solution, is extremely technical and it is based on several advanced results of functional analysis. Taubes uses extensively the Lusternik-Šnirelman theory (Appendix B; for other applications in field theory, see [58, 59]) and various concepts from homotopy theory. Thus we give only a short description of his results as well as a brief account of basic ideas of the work.

The variational equations following from (4.1) read

$$(D_i F^{ik})^a = \varepsilon^{abc} \varphi^b (D^k \varphi)^c, \quad D_i D^i \varphi = 0 \tag{4.8}$$

(the Higgs potential is zero).

They are supplemented by the boundary conditions

$$|\varphi(\infty)| = 1, \quad A_i(\infty) = 0. \tag{4.9}$$

The structure of the configuration space

$$\mathscr{H} = \{A_i^a : A_i^a(\infty) = 0\} \times \{\varphi^a : |\varphi(\infty)| = 1\}$$

(with configurations, which are gauge equivalent under gauge transformations that tend to 1 at spatial infinity, being identified) is topologically nontrivial. Taubes proved the existence of a monomorphism between homotopy classes:

$$\Pi_k((S^2, e) \to (S^2, e)) = \Pi_k(\mathscr{H}).$$

Also the structure of the k-th homotopy sector of the configuration space $\tilde{\mathscr{H}} = \mathscr{H}/\{$*modulo constant gauge transformations*$\}$ is nontrivial.

Taubes defined a function set Λ of noncontractible maps of spheres into $\mathscr{H}$. Some properties of Λ made it possible to construct a flow which lowers the action and, moreover, to prove those properties of the action and of its gradient (that is, of equations (4.8)) which allow for application of the Minimax Theorem (Appendix B). The Minimax Theorem immediately yields the existence of a trivial solution $F^a{}_{ik} = 0$, $\varphi^a = \delta^{a3}$ which minimizes the action in the $k = 0$ sector of $\mathscr{H}$ (then $S = 0$). The main problem is to show that there still exists a solution at which the action is nonzero; that this solution is a saddle point of S, follows from the fact that stable minimizers of S should satisfy the Bogomolny equations and in the $k = 0$ sector there is only one such a minimizer, the trivial one.

Taubes succeeded in proving the existence of nontrivial critical point of the action; this is an unstable monopole with trivial topological number. It is proved to have the energy (which is here equal to the action) smaller than the doubled energy of the Bogomolny--Prasad-Sommerfield monopole. Unfortunately, the Lusternik-Šnirelman theory does not give a constructive method for searching for predicted solutions.

One could easily check that it is not possible to reach this new solution perturbatively, starting either from $F^a{}_{ik} = 0$, $\varphi^a = \delta^{a3}$ or from the already known one-monopole solution of BPS [8].

The last fact is implied by the following. The perturbed BPS monopole could be (alternatively) integrated into a new (unstable) solution, but its topological number cannot be altered by small perturbations; thus it cannot be identified with Taubes monopole.

For the remaining claim, let us consider Eqs (4.8) linearized at the trivial solution. They are

$$\Delta\delta A_i^3 = 0, \tag{4.10}$$

$$\Delta\delta A_i^a = \delta A_i^a + \varepsilon^{a3c}\partial_i\delta\varphi^c, \quad (a = 1, 2) \tag{4.11}$$

$$\Delta\delta\varphi^a = 0, \tag{4.12}$$

with homogeneous boundary conditions $\delta A_i^a = \delta\varphi^a = 0$ at spatial infinity.

Requiring finiteness of the energy, one arrives at

$$\delta\varphi^a = 0, \quad \delta A_i^a = 0, \tag{4.13}$$

as the only solution of linearized equations. This implies the absence of those nontrivial solutions of (4.8) that could be obtained perturbatively from the trivial one. The rigorous proof needs specification of suitable function spaces and uses the Implicit Function Theorem.

4d. Dyons in the Yang-Mills-Higgs theory

The only solution having both electric and magnetic charges which is known explicitly is that of Julia and Zee [8]. It corresponds to the Bogomolny limit, $\lambda = 0$.

For the nonvanishing Higgs potential $V(\varphi)$ some simplifying ansatzes were made which reduce Yang-Mills-Higgs equations to a system of ordinary differential equations. For G = SU(2), supposing Julia-Zee ansatz

$$A_i^a = \varepsilon_{aij}x_j\frac{1-K(r)}{r^2}, \quad A_0^a = x^a\frac{H(r)}{r}, \quad \varphi^a = \frac{x^a J(r)}{r},$$

the equations read

$$\begin{aligned}\ddot{K} &= \frac{K}{r^2}(K^2-1)+(-H^2+J^2)K,\\ \ddot{J} &= -\frac{2}{r}\dot{J}+\frac{2JK^2}{r^2}+\frac{\partial}{\partial J}V(J),\\ \ddot{H}+\frac{2}{r}\dot{H} &= \frac{2HK^2}{r^2},\end{aligned} \tag{4.14}$$

where dot denotes the derivative with respect to r.

It is clear that because of required smoothness of A_μ^a and φ the radial functions K, J, H have to satisfy the following conditions at $r = 0$:

$$K(0) = 1, \quad \dot{K}(0) = 0, \quad J(0) = 0, \quad H(0) = 0. \tag{4.15}$$

These conditions could be regarded as a part of Cauchy data for the equations of motion (4.14). The derivatives of $J(0)$ and $K(0)$ with respect to r are the only free initial data. Horvath and Palla [60] generalized the Julia-Zee ansatz to all unitary groups SU(N). The resulting equations for radial functions are rather lengthy. We do not reproduce them. Schechter and Weder [61] proved the existence of smooth solutions of the above equations (including Horvath and Palla's generalization), for all SU(N) groups.

4e. Plane-wave solutions of the Yang-Mills-Higgs theory

Brihaye [62] used the following ansatz

$$A_\mu = p_\mu \alpha(k_1, k_2)\sigma_1 + q_\mu \beta(k_1, k_2)\sigma_2,$$
$$\varphi = \varphi(k_1, k_2)\sigma_3, \tag{4.16}$$

where $k_t = s_t^\mu x_\mu$ and p, q, s_1, s_2 are constant mutually orthogonal 4-vectors (except s's whose relative orientation is arbitrary) and $p_\mu p^\mu = \varepsilon_1$, $q_\mu q^\mu = \varepsilon_2$. The Yang-Mills-Higgs equations of motion reduce then to

$$\Box\alpha + g^2\alpha(\varepsilon_2\beta^2 + \varphi^2) = 0,$$
$$\Box\beta + g^2\beta(\varepsilon_1\alpha^2 + \varphi^2) = 0,$$
$$\Box\varphi + g^2\varphi(-\varepsilon_1\alpha^2 - \varepsilon_2\beta^2 + g^{-2}\lambda(\varphi^2 - 1)) = 0. \tag{4.17}$$

Brihaye found two types of solutions, existing for $s_1 = s_2 = s$, $\varepsilon_1 = \varepsilon_2 = -1$, $s_\mu s^\mu = -1$, $k_1 = k_2 = k$. They are

$$\text{(i)} \qquad \alpha = \pm\beta = \frac{\sqrt{\lambda}}{g}\cos(\sqrt{\lambda}\,k), \qquad \varphi = \frac{\sqrt{\lambda}}{g}\sin(\sqrt{\lambda}\,k), \qquad \lambda = 2g^2. \tag{4.18}$$

This solution is manifestly nonsingular and periodic.
(*ii*) for the potential $V(\varphi) = \lambda|\varphi|^4$ (the Higgs potential is replaced by one with nondegenerate vacuum), the only smooth solution is

$$\varphi = \frac{\gamma}{\sqrt{2g^2 - \lambda}}\,\mathrm{cn}\,(\gamma k, \tfrac{1}{2}), \qquad \alpha = \beta = \sqrt{1 - \lambda/g^2}\,\varphi. \tag{4.19}$$

Here cn denotes one of Jacobi elliptic functions and γ is an arbitrary parameter determining the period and amplitude of the solution.

These solutions describe an infinitely long plane waves propagating with phase velocity (being defined as velocity of the spacelike hyperplanes of extremal energy) greater than 1 ($c = 1$). The velocity of transport of energy, $V^i = T^{0i}/T^{00}$ is smaller than the velocity of light.

4f. Nonminimal solutions in the Weinberg-Salam model

The work of Taubes [53] inspired the investigation of field theories in the search for other saddle point solutions. Forgács and Horvath [59] have reviewed a number of field theories, where explicitly known saddle point solutions are related to the topology of the field configuration. Manton [63] has shown that the field configuration space of the classical Weinberg-Salam model has noncontractible loops passing through the vacuum configuration. In analogy with Taubes [53], one expects the existence of nonminimal solutions (sphalerons, in Manton's terminology). The authors of [63, 64] were not able to prove the existence of sphaleron rigorously, but instead they presented variational and numerical arguments. Here we shortly review the content of [64].

Fields are supposed to be static and both the fermion fields and the time component of the gauge fields are set to zero. The field equations are

$$(D_jF_{ij})^a = -\tfrac{1}{2}\, ig[\varphi^+\sigma^a D_i\varphi-(D_i\varphi)^+\sigma^a\varphi],$$

$$\partial_j f_{ij} = -\tfrac{1}{2}\, ig'[\varphi^+ D_i-(D_i\varphi)^+\varphi],$$

$$D_iD_i\varphi = 2\lambda(\varphi^+\varphi-\tfrac{1}{2}v^2)\varphi. \tag{4.20}$$

Here the F's, D's and φ's are defined as in Sect. 1c; $f_{ij} = \partial_i f_j-\partial_j f_i$, where a_j are components of U(1) potential; scalar field φ is in the fundamental representation of SU(2); g, g' are gauge coupling constants. The gauge group is here $SU(2)\times U(1)$.

In the case of $g' = 0$ the numerical analysis of resulting equations [65, 66] suggests the existence of a nontrivial, finite energy solution. Its existence has been proved by Burzlaff [67]. The main body of the article [64] consists of variational approximation of the solution for $g' = 0$ and then is considering the effect of nonzero U(1) field f_{ij} (that is, $g' \neq 0$). Some characteristics of the solution are known. It is axially symmetric, classically unstable and it is argued to possess fractional baryonic and leptonic charges. A recent work on this topic is [68].

5. *Solutions in Minkowski space*

5a. Exact solutions

The number of known solutions of the Yang-Mills equations in Minkowski space is very limited. Four of them, De Alfaro, Furlan and Fubini's solution, Coleman's nonabelian plane wave and two elliptic solutions of Cervero, Jacobs and Nohl and Schechter and Lüscher, are reviewed in [8]. The next solution was found by several authors [69]. Our presentation follows that of Baseyan et al. [70]. Assume $A_0^a = 0$, $A_i^a = \pm f(t)\delta_i^a$.

The Yang-Mills equations reduce then to

$$\ddot{f}+2f^3 = 0. \tag{5.1}$$

(Here G = SU(2)).

The Eq. (5.1) is equivalent to a nonlinear equation

$$\dot{f}+f^4 = 2\varepsilon, \tag{5.2}$$

where ε is a density of energy.

Given ε, (5.2) possesses a unique real and smooth solution,

$$f = (4\varepsilon)^{1/4}\,\mathrm{cn}\,(2\varepsilon^{1/4}(t+t_0), \sqrt{2}/2), \tag{5.3}$$

where t_0 is an integration constant.

As a special case of this solution, assuming $A^a_3 = 0$, one gets

$$A^1_1 = \pm A^2_2 = (8\varepsilon)^{1/4}\,\mathrm{cn}\,((8\varepsilon)^{1/4}(t+t_0), \sqrt{2}/2), \tag{5.4}$$

the remaining components of a potential being zero.

Both solutions are periodic with a real period $4(2\varepsilon)^{1/4}K(\sqrt{2}/2)$ or $4(8\varepsilon)^{1/4}K(\sqrt{2}/2)$, respectively, where $K(k)$ is the full elliptic integral

$$K(k) = \int\limits_0^{\pi} \frac{d\theta}{(1-k^2\sin^2\theta)^{1/2}}\,. \tag{5.5}$$

Batsula and Gusynin [71] investigated a sector of the Yang-Mills theory which generalizes the so-called Yang-Mills mechanics ([70], see also Sect. 6c). Their ansatz for Yang--Mills fields

$$A^a_0 = p^a J(px)$$

$$A^a_i = \frac{p^k K(px)}{p^2}\,\varepsilon_{aik} + \left(\delta^a_i - \frac{p^a p^i}{p^2}\right) H(px) + \frac{p^a p^i}{p^2}\,F(px), \tag{5.6}$$

where $\{p^\mu\}$ is a constant 4-vector and $px = p_0x_0 - p_ix_i$, resembles Witten's ansatz for spherically symmetric potentials. For $p^2 > 0$ the resulting equations of motion are equivalent to those of Baseyan et al. (Eq. (5.1) above). The equivalence can be shown by the use of a suitable Lorentz transformation. For $p^2 = 0$ the only nonzero solutions are the abelian ones; the nonlinear part of a strength field tensor vanishes and equations of motion reduce to the Maxwell electrodynamics. And finally for $p^2 < 0$ and in such a Lorentz frame that $p^\mu = \delta^{\mu 3}\ p$, one arrives at the following equations [71]

$$-p^2J'' + 2J(K^2+H^2) = 0 \tag{5.7a}$$

$$-p^2K'' - KJ^2 + H(K^2+H^2) = 0 \tag{5.7b}$$

$$-p^2H'' - HJ^2 + H(K^2+H^2) = 0, \tag{5.7c}$$

$$KH' - HK' = 0. \tag{5.7d}$$

The function F in the formula (5.6) vanishes since the Lorentz gauge condition is imposed. Above prime denotes differentiation with respect to (px_3).

Eq. (5.7d) implies $H = cK$, where c is a constant. The remaining equations reduce then to

$$-p^2J'' + 2JK^2(1+c^2) = 0, \qquad -p^2K'' - KJ^2 + K^3(1+c^2) = 0. \tag{5.8}$$

There are singular solutions which can be expressed via Jacobi elliptic functions [71]. They describe a motion tending to infinity as x_3 approaches singular points. The last property can be shown to hold for a large class of solutions of Eqs (5.8).

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For given initial data $J(z_0)$, $K(z_0)$, $J'(z_0)$, $K'(z_0)$ a solution of (5.8) does exist locally and is unique. In addition, for $(JJ'+KK')(z_0)$ positive, the solution grows as z grows till the nearest singularity (possibly to infinity if the solution branch continues to $z = \infty$), that is J^2+K^2 increases; a motion described by Eqs (5.8) is in fact unbounded. We omit the proof, which is almost standard.

The above property of the Yang-Mills mechanics corresponding to $p^2 \leqslant 0$ is in a sharp contrast to the sector $p^2 > 0$, in which solutions are essentially bounded. Similar results were obtained by Arodź and Serda [72]. Their ansatz $A^a_\mu = \delta^a_1 g_{\mu 0} u(kx) + \delta^a_2 g_{\mu 1} v(kx)$ leads to the following equations

$$k^2 v'' = -vu^2, \quad k^2 u'' = uv^2. \tag{5.9}$$

Qualitative analysis for $k^2 < 0$ shows absence of periodic motions and existence of unbounded trajectories there. An interesting observation is the possible lack of chaos in that sector of the Yang-Mills mechanics [72], in contradiction with the Frøyland-Savvidi sector $p^2 > 0$ (see Sect. 8 below).

Barbier et al. [73] obtained a solution of the Yang-Mills equations on the conformal compactification of Minkowski space $S^3 \times S^1$. Assuming that a potential A^a_μ is invariant under the action of SO(4), they arrived at a nonlinear ordinary differential equation, whose solution was given implicitly by an elliptic integral.

Antoine and Jacques [74] have investigated classical SU(2) Yang-Mills fields in Minkowski space on the assumption that solutions are invariant under certain noncompact maximal subgroups of the conformal group C(3, 1). They have shown the absence of nonabelian solutions; the only nonzero invariant solutions are abelian.

A detailed discussion of nonabelian plane wave solutions is given in [75–77]. Most of the solutions are singular, some coincide with solutions already being known.

5b. Approximate and numerical solutions

As is well known [78–81] the local Cauchy problem is well posed for Yang-Mills equations. It follows that given the magnetic potential A^a_i and the electric field E^a_i at a time $t = t_0$, such that the constraint equations $D_i E^i = 0$ are satisfied, there exists a unique solution of full YM equations. This is true for any compact gauge group and for any finite time t; the last statement has been proven quite recently and it follows as a byproduct of the work of Eardley and Moncrief [80]. (See also Sect. 9a.) Thus using numerical or approximation procedures one can study evolution of YM systems in finite intervals of time. The existence of scattering states ($t \to \pm\infty$) is still unclear and only partial results were obtained [82, 83].

Time-dependent generalization of the Coulomb solution was discussed in [84]. For an interesting discussion on approximate time-dependent potentials related to the Wu-Yang monopole see [85].

In the static sector of the YM SU(3) theory some numerical investigations were made by Hädicke and Pohle [86]. They found families of spherically symmetric solutions that are localized but apparently are singular.

6. Application of bifurcation theory to Yang-Mills equations

6a. Preliminaries

Bifurcation theory was founded in the late nineties of XIX century. Among its founders we can find H. Poincare and Lyapunov. The object of bifurcation theory is to study the existence of solutions of nonlinear equations in the vicinity of a solution which is explicitly known — this is the qualitative bifurcation theory — and, moreover, to find approximate solutions — this is the analytical bifurcation theory.

Let us present a precise definition [87]: "Bifurcation theory is a study of the branching of solutions of nonlinear equation $F(x, s) = 0$, where F is a nonlinear operator and s is a parameter referred to as the bifurcation parameter. It is of particular interest in bifurcation theory to study how the solutions $x(s)$ and their multiplicities change as s varies. A bifurcation point of a solution branch $x(s)$ is a point $(x_0, x(s_0))$ from which another solution $x_1(s)$ branches. That is, $x(s_0) = x_1(s_0)$ and $x(s) \neq x_1(s)$ for all s in an interval about s_0."

From the "engineer's" point of view the search for new solutions consists of 3 steps:
(*i*) linearization at a given solution;
(*ii*) investigation of a nonlinear part of a mapping;
(*iii*) approximation of bifurcation solutions (if any exist). Linearization of F at a known branch (s_0, x_0) yields

$$F'(s_0, x_0)\delta x = 0. \tag{6.1}$$

The existence of nonzero solutions δx of (6.1) satisfying homogeneous boundary conditions is a hint for bifurcation; the case when their number is odd is particularly promising. Then it suffices that the part of F which is nonlinear as a function of $(x-x_0)$ be continuously differentiable and that it vanishes together with its first derivatives at (s_0, x_0). There are also some additional conditions which we omit here. For a more detailed discussion of the above claim see Appendix C.

To summarize: an odd number of solutions of linearized equations (6.1) and a "sufficient" nonlinearity ensure the existence of new solutions which bifurcate from the old solution $x_0(s)$ at s_0.

Having established the existence of bifurcation we can look for the approximate form of a solution. From now on we will follow Vainberg and Trenogin [88]. Suppose that $x = (x_1, ..., x_n) \in X$, where X is a suitable Banach space and $\dim \ker F'(x_0, s_0) = 1$ (that is, there is only one solution of (6.1)). Let $F'^*(x_0, s_0)$ be an operator which is L_2 adjoint to $F'(x_0, s_0)$; since in examples of physical interest the operators are usually hermitian, it follows that $\ker F' = \ker F'^*$, that is δx solves the equation

$$F'^*\delta x = 0 \tag{6.2}$$

as well as (6.1).

Vainberg and Trenogin [88] proposed the following expansion for the bifurcating solutions

$$x = x_0 + \begin{bmatrix} \sum\limits_{i/1}^{\infty} w^i X_{1io} + \sum\limits_{i/0}^{\infty} w^i \sum\limits_{j/1}^{\infty} (\delta s)^j X_{1ij} \\ \vdots \\ \sum\limits_{i/1}^{\infty} w^i X_{nio} + \sum\limits_{i/0}^{\infty} w^i \sum\limits_{j/1}^{\infty} (\delta s)^j X_{nij} \end{bmatrix}, \tag{6.3}$$

where X_{aij}, $(a = 1, 2, \ldots, n)$ are unknown functions, $\delta s = s - s_0$, w is a parameter. One can obtain X_{aij} after inserting (6.3) into the original equation $F(s, x) = 0$ and equating terms with the same degree of δs, w.

The dependence of a w on the bifurcation parameter is not arbitrary, as will be shown now.

The equation $F(s, x) = 0$ could be rewritten as

$$F'(s, x_0)(x - x_0) + T(s, x - x_0) = 0, \tag{6.4}$$

where we subtracted from F the linear part.

Multiply (6.4) by the transpose of δx, δx^+ and integrate over the support V of the functions $x \in X$ (our original equation is supposed, for definiteness, to be a differential equation).

Thence

$$\int\limits_V \delta x^+(F'(s, x_0))(x - x_0) dV + \int\limits_V \delta x^+ T(s, x - x_0) dV$$

$$= \int\limits_V \delta x^+(F'(s, x_0) - F'(s_0, x_0))(x - x_0) dV + \int\limits_V dx^+ T(s, x - x_0) dV = 0, \tag{6.5}$$

where we used $\delta x^+ F'(s_0, x_0) = (F'(s_0, x_0)\delta x)^* = 0$.

So the equation (6.5) gives a restriction upon the space of solutions of (6.4). Inserting the expansion (6.3) into (6.5) yields the so-called Lyapunov-Schmidt equation (or bifurcation equation):

$$\sum_{i/2}^{\infty} L_{io} w^i + \sum_{i/0}^{\infty} w^i \sum_{j/1}^{\infty} L_{ij} (\delta s)^j = 0, \tag{6.6}$$

where

$$L_{ij} = \int\limits_V dV \delta x^+ X_{ij}. \tag{6.7}$$

The equation (6.6) allows, at least in principle, one to express w as a function of the bifurcation parameter $s - s_0$. Since it is assumed that bifurcation solutions x tend to x_0 as s tends to s_0, the parameter w should vanish at $s = s_0$ (that is, it should be small, in the sense that $\lim_{s \to s_0} w(s) = 0$). It implies the one-to-one correspondence of small solutions to the bifurcation equation (6.6) and to the full nonlinear equations (6.4) [88].

Let us emphasize that the analytical approach gives simultaneous information about existence of bifurcation solutions and their approximate form. Its disadvantage is the complexity of calculations, particularly in the case of multiplicities of kernels of linearized operators greater than it was assumed above.

Remark

The above procedure reduces the initial infinite-dimensional problem to an algebraic one (which is finite-dimensional), that is to finding solutions of bifurcation equations. It allows us to use topological methods to get qualitative criteria for bifurcation (see Appendix C and for more details [89–91]).

6b. Bifurcation solutions in the SU(2) statics

Jackiw, Jacobs and Rebbi [92] were the first who pointed out the applicability of bifurcation theory for analysis of nonabelian gauge theories. They found numerically two branches of bifurcation solutions which appeared at sufficiently strong sources. In the subsequent paper Jackiw and Rossi [84] outlined an analytical approach to the study of bifurcation theory which is in essence similar to the one already described. Their conclusions overlap in part with those in [93, 94] which we now present in greater detail.

Let us assume statics and let the only nonzero components of SU(2) potential be

$$A_0^3 = \phi(\varrho, x_3),$$

and

$$A_i^1 = \varepsilon_{i3j}\frac{x_j}{\varrho}A(\varrho, x_3), \qquad \varrho = \sqrt{x_1^2+x_2^2} \tag{6.8}$$

(this is Sikivie and Weiss's ansatz of cylindrical symmetry [95]). Assume that vector components of a current vanish and that the only nonzero component of charge density is $j^{0,3}$. The Yang-Mills equations reduce then to

$$-\Delta\phi+g^2A^2\phi = b(\varrho, x_3), \tag{6.9a}$$

$$\left(\Delta-\frac{1}{\varrho^2}\right)A+g^2A\phi^2 = 0, \tag{6.9b}$$

where $b(\vec{x})$ is a cylindrically symmetric charge density of the Hölder class $C^{0+\mu}$ at least and square of the coupling constant, g^2, plays role of a bifurcation parameter. Suppose that our boundary conditions are

$$\begin{pmatrix}\phi\\ A\end{pmatrix}_{\partial V} = \begin{pmatrix}h(\partial V)\\ 0\end{pmatrix}. \tag{6.10}$$

Then equations (6.9) admit the Coulomb solution (i.e., with Abelian holonomy group)

$$A_0 = 0, \qquad \phi_0 = \int_V G*b+\int_{\partial V} G_s*h, \tag{6.11}$$

where G, G_s are relevant Green functions.

Insertion of (6.11) into (6.9) yields

$$-\Delta B+g^2A^2\phi_0+g^2A^2B = 0, \tag{6.12a}$$

$$\left(\Delta-\frac{1}{\varrho^2}+g^2\phi_0^2\right)A+2g^2\phi_0AB+g^2B^2A = 0, \tag{6.12b}$$

where B and A denote perturbations from ϕ_0 and A_0, respectively, which vanish on the boundary ∂V.

The linearized equations are

$$\begin{pmatrix} -\Delta & 0 \\ 0 & \Delta-\dfrac{1}{\varrho^2}+g^2\phi_0^2 \end{pmatrix}\begin{pmatrix}\delta B\\ \delta A\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{6.13}$$

and it is clear that there can exist a nonzero eigenfunction $\delta x = (0, \delta A)$ such that $\delta A(\partial V) = 0$ for certain isolated positive values of g^2.

Assume that we deal with a simple eigenvalue g_L^2 with an eigenfunction $\delta x = (0, \delta A)$. Applying the analytical approach of Vainberg and Trenogin [88] one arrives at the following result:

there appear two new solutions bifurcating from the old Coulomb potential at $g^2 = g_L^2$. Bifurcation solutions are approximated to the first order of the zero mode solution:

$$\begin{pmatrix}\phi\\ A\end{pmatrix} = \begin{pmatrix}\varphi_0\\ 0\end{pmatrix} \pm \left(\frac{-L_{11}w}{L_{30}}\right)^{1/2}\begin{pmatrix}0\\ \delta A\end{pmatrix} + o(w^{1/2}) \tag{6.14}$$

where $w = g^2-g_L^2$ and L_{11}, L_{30} are constants (their explicit expressions are given in [94]). This bifurcation is overcritical; real bifurcating solutions exist above g_L^2 and they are stable [94]. These new solutions are nonabelian and they possess a nonzero magnetic part.

Let us present an example.

Suppose the charge vanishes, the boundary ∂V is a sphere of radius R and the boundary conditions are $\phi = \phi_0 = \text{const.}$, $A = 0$. Then the abelian solution is

$$A_0^3 = \phi_0, \quad A_i^1 = 0 \tag{6.15}$$

and the bifurcating cylindrically symmetric solutions are given by

$$A_0^3 = \phi_0+o(w^{1/2}),$$

$$A_i^1 = \pm\varepsilon_{i3j}x_j\sqrt{\frac{-L_{11}w}{L_{30}}}\,J_k(g_L\phi_0 r)r^{-1/2}+o(w^{1/2}), \tag{6.16}$$

all remaining components of a potential being zero. Here $k = L+\frac{1}{2}$, J_k are Bessel functions, $g_L > 0$. Let us emphasize that bifurcation solutions appear for sufficiently large values of the coupling constant.

Paul and Khare [96] have treated the problem of bifurcation from another viewpoint. They have guessed two couples of functions (ϕ, A) (see (6.8)) which after insertion into Eqs (6.9) produce certain charge density $b(\varrho, x_3)$ (the same for all solutions) as well as a vector current $j_i^a = -\varepsilon_{i3j}\frac{x_j}{\varrho} m\left(\varrho, \frac{x}{3}\right)$ (different for each pair of functions). The two solutions (one of which is abelian while the second is nonabelian) depend on a parameter; for a particular value of the parameter both solutions coincide and then $j_i^a = 0$. (In our opinion these solutions do not exemplify bifurcation, since they solve two different equations, with different vector currents.) This line of research has been continued in [97].

Oh et al. [98] have studied the cylindrically symmetric YM equations by the use of the Lyapunov-Schmidt procedure. Their results overlap in part with those obtained earlier in [93, 94]. Caution: the reader of [98] would get the impression that the Lyapunov--Schmidt procedure works on open R^3 spaces. It is not proven; the Lyapunov-Schmidt iteration procedure gives a series (6.3) which is expected to be divergent for generic sources.

For numerical investigation of bifurcation solutions in the SU(3) theory with external sources see [99].

The physical relevance of static bifurcation is far from being proved. The best we can do is to repeat Mandula's [100] expectation that such phenomena as loss of stability (which is often related to bifurcation) can somehow be reflected as, e.g., decay of elementary particles. More cautious prediction is that the knowledge of bifurcation structure could be useful for quantum theory, since the new branching vacuum sometimes happens to have smaller action than the original one. The quantum implications are not studied, as yet. Future investigation should base on more strict formulations of quantum theory than the standard QCD.

6c. Time-dependent periodic solutions

Assuming $A_0^a = 0$, $A_i^3 = A_a^3 = 0$, defining

$$A_1^1 = s, \quad A_1^2 = t, \quad A_2^1 = u, \quad A_2^2 = w \tag{6.17}$$

and supposing that gauge potentials depend only on time we are lead to the following Yang-Mills SU(2) equations.

$$\begin{aligned} \ddot{s} &= -w(sw-uz), \\ \ddot{u} &= z(sw-uz), \\ \ddot{z} &= u(sw-uz), \\ \ddot{w} &= -s(sw-uz), \end{aligned} \tag{6.18a}$$

$$\varepsilon^{abc} A_i^b \dot{A}_i^c = -\delta^{a3}\,\text{const.} \tag{6.18b}$$

The equations (6.18a) describe the so-called Yang-Mills mechanics [101, 102]. The last one defines the charge. (Notice that in [102] the constraint equations are missing; contrary to the author's claim, he considers YM equations with an abelian source.)

Further we will study only dynamic equations (6.18a).

Introducing new variables θ_1, θ_2, r_1, r_2 such that

$$(s+w)/\sqrt{2} = r_1 \cos\theta_1,$$

$$(s-w)/\sqrt{2} = r_2 \cos\theta_2,$$

$$(u+z)/\sqrt{2} = r_2 \sin\theta_2,$$

$$(u-z)/\sqrt{2} = r_1 \sin\theta_1, \tag{6.19}$$

and inserting them into (6.18a) yields [102]

$$r_1^2\dot{\theta}_1 = L_1, \quad r_2^2\dot{\theta}_2 = L_2, \quad (L\text{'s are constants}) \tag{6.20}$$

$$\ddot{r}_1 - \frac{L_1^2}{r_1^3} + r_1(r_1^2 - r_2^2)/2 = 0,$$

$$\ddot{r}_2 - \frac{L_2^2}{r_2^3} + r_2(r_2^2 - r_1^2)/2 = 0. \tag{6.21}$$

It is easy task to show that static solutions exist either for one of L's being zero (say $L_1 = 0$, $L_2 \neq 0$) or both L's being zero.

In the first case we get

$$\theta_2 = \frac{r_2}{(2)^{1/2}}t, \quad r_2 = \left(\frac{L_2}{\sqrt{2}}\right)^{1/3} \equiv \alpha. \tag{6.22}$$

The bifurcating solutions $r_1 = x$, $r_2 = \alpha + y$ should satisfy the following equations

$$\ddot{x} + x[x^2 - (y+\alpha)^2]/2 = 0, \tag{6.23a}$$

$$\ddot{y} - \frac{\alpha^6}{2(y+\alpha)^3} - (y+\alpha)[x^2 - (y+\alpha)^2]/2 = 0. \tag{6.23b}$$

The qualitative results of bifurcation theory together with the investigation of corresponding Lyapunov-Schmidt equations enable one to prove [103] the following

THEOREM 6.1

The system (6.23) possesses one and only one nontrivial solution which bifurcates from ($x = y = 0$) at values $\alpha_n = \dfrac{n}{\sqrt{3}T}$ for each fixed positive value of T and satisfies periodic conditions $x(0) = y(0) = x(\omega(\alpha)T = \pi) = y(\omega(\alpha)T = \pi) = 0$. For α sufficiently close to $\alpha_n = \dfrac{n\pi}{\sqrt{3}T}$ the bifurcating solution is given by the following formula

$$x(t, \alpha) = 0,$$

$$y(t, \alpha) = \mu \sin(\sqrt{3}\,\alpha_n t) + O(\mu^2), \tag{6.24a}$$

where μ is a new parameter related to α by

$$\alpha^2(\mu) = \frac{n^2\pi^2}{3T^2} + \frac{2n\pi\mu}{\sqrt{3}\,T} + O(\mu^2). \tag{6.24b}$$

Remark

The charge in the Eq. (6.18b) is equal to $-L_2 = \frac{-\alpha^3}{\sqrt{2}}$, for the original solution as well as for that bifurcating one (which could be also found explicitly—see discussion below).

In the case when both L's are zero the equations (6.18) read

$$\ddot{s}+sw^2 = 0, \quad \ddot{w}+ws^2 = 0; \tag{6.25}$$

now $u = z = 0$.

There exists a static solution $s = 0$, $w = \alpha$ and one is able to prove [103]:

THEOREM 6.2

The system (6.26) possesses a solution $(s(t, \mu), w(t, \mu))$ which bifurcates from $(0, \alpha_n)$ at $\alpha_n = \frac{n\pi}{T}$ and satisfies periodic boundary conditions.

It is given by

$$\begin{pmatrix} s \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + \mu \begin{pmatrix} \sin \alpha_n t \\ 0 \end{pmatrix} + O(\mu^2), \tag{6.26}$$

where μ is related to α by $\alpha^2 = \frac{n^2\pi^2}{T^2} + O(\mu)$.

The solution described in Theorem 6.1 can be found explicitly. Since $x \equiv 0$, Eqs (6.23) reduce to a single equation whose solution is Weierstrass elliptic function [104],

$$y = \sqrt{-\wp(t'+C, 32\varepsilon, 8\alpha6)} - \alpha. \tag{6.27}$$

Here $t' = \frac{t}{2}$, ε is the density of energy and the imaginary part of C must be equal to

$$\omega_2 = K\left(\sqrt{\frac{e_1-e_2}{e_1-e_3}}\right)\Big/\sqrt{(e_1-e_2)}, \tag{6.28}$$

in order to ensure that $\wp$ is real, negative and has no singularities for $t \in (-\infty, \infty)$. e's are the roots of the equation $4z^3 - 32\varepsilon z - 8\alpha^6 = 0$, $e_1 > e_2 > e_3$; they are real in our case. K is the complete elliptic integral.

The real part of C is the usual integration constant. The solution (6.27) is periodic with period $4K\left(\sqrt{\frac{e_2-e_3}{e_1-e_3}}\right)\Big/(e_1-e_2)^{1/2}$. It could be checked that this solution coincides with that described in Theorem 6.1 for energy ε being close to its minimal value $3\alpha^4/8$.

More surprising is the fact that it remains nonzero in the limit of $\alpha \to 0$, and it is defined for all α. (In bifurcation theory it is always very difficult to define the allowed range of bifurcation parameters and usually it is bounded.)

At the end we will present the obtained solutions in the original denotation.

For both solutions, the static one given by (6.22) and the Weierstrass function solution, most of components of the potential vanish, $A_0^a = 0$, $A_i^3 = 0$, $A_3^a = 0$. The remaining components are

(*i*) for the solution (6.22)

$$A_1^1 = -A_2^2 = \alpha' \cos \alpha' t,$$

$$A_2^1 = A_1^2 = \alpha' \sin \alpha' t, \qquad \alpha' = \frac{\alpha}{\sqrt{2}}; \tag{6.29}$$

(*ii*) for the solution (6.27)

$$A_1^1 = -A_2^2 = \frac{1}{\sqrt{2}}\left(-\mathscr{P}\left(\frac{t}{2} + C, 32\varepsilon, 8\alpha^6\right)\right)^{1/2} \cos \theta(t)$$

$$A_2^1 = A_1^2 = \frac{1}{\sqrt{2}}\left(-\mathscr{P}\left(\frac{t}{2} + C, 32\varepsilon, 8\alpha^6\right)\right)^{1/2} \sin \theta(t), \tag{6.30}$$

where

$$\theta(t) = -\sqrt{2}\,\alpha^3 \int_0^t dt' \frac{1}{\mathscr{P}\left(\frac{t}{2} + C, 32\varepsilon, 8\alpha^6\right)}.$$

Note that the second solution persists in the limit of a vanishing charge $\varrho^a = -L_2\delta^{a3}$ (that is in the limit $\alpha \to 0$). It coincides then with the elliptic cosine solution of Baseyan et al. (see Sect. 5).

Note a remarkable fact that not only the charge but also the density of energy is constant, although fields are periodic in time.

These solutions together with a massless constant spinor field constitute selfconsistent solutions of the Yang-Mills-Dirac equations (see (2.33) in Sect. 2c).

Such solutions, periodic in time, could be physically relevant if the suggestion of Palumbo [10] that the imposition of time-dependent boundary conditions will remedy QCD is true. The periodic condition (at spatial infinity) cannot be imposed directly, since then Yang-Mills equations (which are hyperbolic) might have no solutions at all. Periodic conditions on the boundary are known to be inconsistent with hyperbolic equations. The natural way to implement them is to quantize QCD around solutions which are periodic in time. There are only 3 candidates[3], that of Baseyan et al. [70] and those two solutions previously described.

[3] Further space-independent solutions are found in [47].

7. *Two notions of stability in Minkowski space*

At the beginning let us point out the existence of two distinct notions of stability in field theory. The first, which is due to mechanical concepts of stability deals with evolution of a perturbation of a given solution of fields equations.

The solution is said to be stable, if its perturbation is controllable, that is

$$\|\delta\psi(t)\| \leqslant C\|\delta\psi(t_0)\|, \tag{7.1}$$

where C is a constant independent of time and $\|\delta\psi(t_0)\|$, $\delta\psi(t_0)$ is an initial perturbation; the norms $\|\cdot\|$ are those of a suitable function space.

Note that lowering the initial perturbation $\|d\psi(t_0)\|$ yields $\|\delta\psi(t)\|$ tending to zero It could be shown (e.g., [106]) that if the evolution system is

$$\dot{x} = f(x), \tag{7.2}$$

with x, f belonging to a Banach space, f being nonlinear in x and satisfying certain (smoothness) conditions, then a solution x_0 of (7.2) is stable if the eigenvalues λ of the equation

$$f'(x_0)\delta x = \lambda\delta x$$

have a negative real part. Here $f'(x_0)$ denotes a linearization of f at x_0. And conversely, for real part of λ being positive the condition (7.1) is violated and a solution appears to be unstable. That spectral condition on eigenvalues of $f'(x)$ is known as condition of "linearized stability" (or, sometimes, as "stability in the sense of Lyapunov").

The name "linearization stability" refers to an entirely distinct notion of stability. It has been extensively used in investigations of nonlinear field theories with nonlinear constraint equations on initial data. To explain this concept, let us consider an evolution equation

$$\dot{x} = f(x), \tag{7.3}$$

with nonlinear differential constraints on initial data

$$K(x) = 0. \tag{7.4}$$

Suppose that these constraints are compatible with the evolution of the system, that is if the initial data x_0 satisfy $K(x_0) = 0$ then also $x(t)$ satisfy $K(x) = 0$; thus the constraint equations are supposed to be preserved in time.

Let x_0 be the given initial data and let δx be a perturbation satisfying the constraint equations linearized at x_0

$$K'(x_0)\delta x = 0. \tag{7.5}$$

If every perturbation δx satisfying (7.5) could be integrated to a solution of full constraint equations (7.4) (i.e., δx is tangent to a curve of exact solution) then x_0 is said to be linearization stable. That this property is not trivial, it is clear from Sect. 6; we already know that the linear perturbation sometimes cannot be integrated to a full solution of a nonlinear equation.

7a. The Lyapunov stability

The Yang-Mills equations in the temporal gauge read:

$$\dot{A}^i = E^i, \quad (E^i = F^{0i}), \tag{7.6a}$$

$$\dot{E}^i = D_j F^{ji}. \tag{7.6b}$$

They are supplemented by the constraint equations

$$D_i E^i = 0 \tag{7.7}$$

which are the generalization of the well known Gauss equation in the Maxwell electrodynamics.

Inserting $A_i^a = \boldsymbol{A}_i^a + \delta A_i^a$ and $E_i^a = \boldsymbol{E}_i^a + \delta E_i^a$ (where $\boldsymbol{A}_i^a$, $\boldsymbol{E}_i^a$ are the solutions of (7.7), (7.6)) into the above equations, yields the following linearized equations

$$\delta \dot{A}^i = \delta E^i, \tag{7.8a}$$

$$\delta \dot{E}^{ai} = (\partial_j \delta^{ac} + f^{abc} \boldsymbol{A}_j^b) \delta F^{cji} + f^{abc} \delta A_j^b \boldsymbol{F}^{cji}, \tag{7.8b}$$

$$(\partial^i \delta^{ac} + f^{abc} \boldsymbol{A}_i^b) \delta E^{ci} + f^{abc} \delta A_i^b \boldsymbol{E}^{ci} = 0. \tag{7.8c}$$

The standard procedure consists now in separating the time dependence of solutions via substitution

$$\delta E^i(\boldsymbol{x}, t) = e^{i\lambda t} \delta E^i(\boldsymbol{x}), \tag{7.9a}$$

$$\delta A^i(\boldsymbol{x}, t) = e^{i\lambda t} \delta A^i(\boldsymbol{x}), \tag{7.9b}$$

and in estimating the sign of λ^2. If λ^2 is positive it means that the linear perturbations do not grow in time and the original solution is stable (see, for instance, [106]).

The resulting eigenvalue equations are usually difficult to study. This is why the stability of a very few and very simple solutions was investigated. Mandula [100] studied stability of a Coulomb solution $A_\mu^a = \delta_{\mu 0} \delta^{a3} \dfrac{q}{r}$ in SU(2) gauge theory. That solution has been shown to be unstable for sufficiently large values of gq (where g is the gauge coupling constant and q is a charge of point source). Brandt and Neri [107] shown the instability of the Dirac monopole solution embedded in SU(2) theory under spherically symmetric perturbations, for the monopole charge greater than 1/2 (in their paper the gauge coupling constant $g = 1$). Their result was affirmed in a more rigorous way by Lohiya [108], who also considered stability of corresponding solutions in curved space-time.

The same phenomenon, the instability of field configurations for sufficiently large values of a coupling constant (which is equivalent, in classical theory, to strong gauge fields), has been observed by other authors. See, for instance, [109], for the analysis of certain constant fields.

Stability of solutions bifurcating from an abelian solution has been discussed in [94].

7b. The linearization stability

Now the constraint equations

$$D_i E^i = 0 \tag{7.10}$$

are to be analyzed.

Our aim is to study the structure of their solution space; if certain second order condition is satisfied then linear perturbations which solve linearized equations are tangent to a full solution of the constraint equations. The structure of the solution space of constraint equations determines the structure of the solution space of the whole Yang-Mills system (7.6), (7.7).

Let A_i^a, E_i^a be a set of initial data.

The insertion of $A_i^a = A_i^a + \delta A_i^a$ and $E_i^a = E_i^a + \delta E_i^a$ into (7.10) yields

$$(\partial_i \delta^{ac} + f^{abc} A_i^b) \delta E_i^c + f^{abc} \delta A_i^b E_i^c + f^{abc} \delta A_i^b \delta E_i^c = 0. \tag{7.11}$$

Thus the linearized equations are

$$K(A, E)\begin{pmatrix} \delta A \\ \delta E \end{pmatrix} \equiv D_i \delta E^i + [\delta A_i, E_i] = 0. \tag{7.12}$$

We already noted in Sect. 6 the crucial role of the L_2-adjoint to linearized equations. The triviality of a set of their solutions $\left(\text{(i.e., of the kernel of } K^* = \begin{pmatrix} D_i^* \\ [E_i, \cdot\,] \end{pmatrix} \right)$ means that the solution space is smooth and the linear perturbations can be integrated to a full solution. In our case, supposing compact Cauchy data, the kernel of K^* could be nontrivial only for nongeneric data A, E, that is for initial data whose holonomy group is smaller than the whole gauge group (in other words, the solution has a gauge symmetry). Projection of (7.11) on the kernel of K^* gives certain second order conditions (some integral constraints) on perturbations, which exclude the possibility of integration of some of them.

What was said above reveals the main strategy. For some important technical questions and resolutions of difficulties that could arise here (for instance when dim Ker $K^* > 1$) see [78, 110, 111]. The main result of those investigations can be stated as follows [110].

The solution set for the source-less Yang-Mills equations on a space-time with compact Cauchy surface is linearization stable except at solutions that are gauge symmetric.

These results were also extended to the Einstein-Yang-Mills equations [78]. For noncompact Cauchy surfaces the Yang-Mills equations seem to be always stable, at least if the asymptotic falloff of initial data is not too strong.

As we have already pointed out, this notion of stability is completely independent of the Lyapunov stability.

The stability in the sense discussed above could have some impact on quantum theory. Moncrief [112] studied quantum gravity built on a symmetric (and therefore linearization unstable) vacuum. The instability of the vacuum results in some constraints on quantum states.

8. *Chaos in the Yang-Mills theory*

There is no strict definition of chaos in physical systems. The meaning of chaos, which emerges after study of various related papers, could be expressed as follows. A physical system shows chaotic behaviour if perturbing its initial state one cannot predict its further evolution. Thus chaos is related to instability; a system which is unstable could be chaotic, however there are known unstable systems which are not chaotic[4] (that is, in their evolution appear some regularities).

For a discussion of chaotic phenomena in various branches of physics see the comprehensive review (with a selection of relevant papers) by Cvitanovič [113].

Some numerical investigation of solutions of Yang-Mills mechanics (see Sect. 6c for its description) was made by Matinyan et al. [101]. They lead to the conclusion that this system could be stochastic, since numerical trajectories of solutions appeared to be unstable with respect to small changes in initial conditions.

Frøyland [102] performed numerical studies which suggest existence of multifurcations in the Yang-Mills mechanics. Periodic orbits seem to bifurcate at certain values of energy. It seems likely that the passage from regular (periodic) motion to chaos is unusual in that there appear not only period doubling but also more untypical phenomena. Matinyan et al. [114] considered Higgs fields coupled to the Yang-Mills mechanics. Chaotic behaviour appeared there for sufficiently large values of the coupling constant g. Introduction of Higgs fields eliminates stochasticity when the value of Higgs field exceeds certain critical value. Analytical studies [115] confirm this result and suggest strict connection between instability and stochasticity in this model. Similar topics are also treated by Chang and Chirikov [116, 117]. For further references see [115, 72].

As we have said above, the investigation of chaos has been made in a simplified version of the YM theory (the so-called YM mechanics), which constitute a (hamiltonian) system of ordinary differential equations. Therefore results obtained here say nothing about the possible chaotic structure of the YM theory in more than 1-dimensional space. The truly interesting case — of the full YM theory in Minkowski space — has not been touched, as yet.

9. *Miscellaneous results*

There is a tendency in the physical literature to look for those properties of Classical Field Theory (CFT) which correspond to certain phenomena that are expected to be inherent in Quantum Field Theory (QFT). Let us review several important directions of the investigation.

QFT is assumed to possess scattering (asymptotic) states, but the problem of the existence of scattering states (S-matrix) cannot be solved in the framework of present QFT (in Minkowski space). This sad situation seems to inspire people to do what can be done — to study CFT, where the problem could be formulated and (sometimes) solved. We report results of the corresponding investigation concerning the Yang-Mills theory in Sect. 9a.

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[4] The author is grateful to Prof. Eryk Infeld for pointing out this fact.

The next important problem is the validity of the perturbation approach in QFT; it cannot be solved within the present (realistic) QFT. The systematic study of the validity of perturbation approach in CFT has not been accomplished, but the first results [118] suggest the negative answer. For greater detail see Sect. 9b.

The most important problem in high energy physics is, no doubt, the confinement of quarks. Still unsolved in quantum chromodynamics, it inspires the study of classical gauge theories in search for a related phenomenon — so-called classical colour screening [119–121]. It has been found in the Yang-Mills theory with external sources, which (in our opinion) is not satisfactory; it is difficult to justify the occurrence of external sources in the theory of fundamental interactions. In Sect. 9c we consider the problem of the existence of classical colour screening in selfcontained gauge-matter systems.

9a. The Cauchy problem in the Yang-Mills theory

Let the evolution equation read

$$\dot{X} = f(X), \tag{9.1}$$

where $X \in B_1, f: B_1 \to B_2$ is a (nonlinear) operator, B_1, B_2 are Banach spaces. The Cauchy problem consists in the following: does a solution $X(t)$ of (9.1) exist, such that $X(0) = X_0$, where X_0 is a prescribed initial value?

If the answer is affirmative for sufficiently small time t, then it is said that the local Cauchy problem is well posed. The problem of the continuation of the local solution for arbitrary (but finite) time is the global Cauchy problem. Leray [122] pointed out that one solves only the local Cauchy problem; no one succeeded to prove the global existence of solutions to nonlinear hyperbolic equations at the time when the book [122] of Leray appeared.

In the Yang-Mills theory (including both gauge and matter fields) the situation looks even worse, since there appear constraint equations in addition to the evolution equations. The constraint equations

$$D_i F^{i0} = j^0$$

(see Sect. 1c for adnotation) are nonlinear, so it is a nontrivial task even to find an admissible set of initial data. This is a serious obstacle in the way to prove that the local Cauchy problem is well posed for Yang-Mills equations. Difficulties that arise there resemble very much the situation in general relativity theory.

The first to solve the local Cauchy problem in the Yang-Mills theory was Segal [79]. His results were improved (in the part concerning a choice of suitable Banach spaces to which solutions are supposed to belong) by Eardley and Moncrief [80] and Ginibre and Velo [81]. They extended Segal's results on Yang-Mills fields coupled to nonlinear scalar fields. Eardley and Moncrief have proven also the existence of global solutions by the use of a priori estimates. This is a significant achievement but it requires too strong integrability conditions on scalar and gauge fields; they are demanded to be L_2-integrable, which is incompatible with magnetic monopoles and nontrivial topologies. Burzlaff and O'Murchadha [123] extended results of [80] to include also configurations of the form

$$A_\mu(t, x) = \mathring{A}_\mu(x) + a_\mu(t, x), \quad \mathring{A}_0 = 0,$$

$$\phi(t, x) = \mathring{\phi}(x) + \varphi(t, x),$$

where a_μ, φ belong to suitable Banach spaces while $\mathring{A}_\mu$, $\mathring{\phi}$ are two static background fields which behave like static monopole solutions (at least asymptotically). This work (which combines techniques of [80] and [81]) proves the global existence of solutions in the important sector of Yang-Mills-Higgs theory. It is worth to notice that the affirmative solution of the global Cauchy problem is not obvious — there are known counterexamples in nonlinear Klein-Gordon and Schrödinger theories [124].

Christodoulou and Choquet-Bruhat [125] have proved the global Cauchy problem for Yang-Mills fields coupled to massless spinor fields, provided that a suitable Banach norm of initial data is small enough. Their approach bases on the conformal invariance of YM-Dirac massless equations and therefore cannot be generalized to include the case of massive spinor fields. That case is still unsolved, which is not surprising since even in the Maxwell-Dirac theory (with massive spinors) the existence of global solutions is not solved.

9b. On the absence of perturbative solutions of classical chromodynamics

We will say that a scalar field is non-Higgs if its potential in the lagrangian (2.1) is positive, $V(s) \geqslant 0$. For dynamical (i.e., selfcontained) gauge theories we have:

THEOREM 9.1

Suppose that matter and YM fields are static. Then for non-Higgs scalar fields and for linear massive Dirac fields perturbative solutions are absent, for arbitrary compact gauge group.

For non-Higgs scalar fields the above statement follows from the "no go" theorems of Sect. 2. The remaining part has been proven in [118] and it follows from a more general result which we now formulate.

THEOREM 9.2

Suppose that the Dirac and Yang-Mills fields are static and $A^a_\mu \in L_3$, $\psi \in W_{1,2} \cap L_6$ [5]. Define $\eta = g||A||_{L_3}$. Then, the nonzero solutions of Yang-Mills-Dirac equations are absent for sufficiently small η.

Outline of the proof [118].

It consists of two parts. First, the Dirac equation is shown to have no nonzero finite energy solutions for sufficiently small η. It is done by algebraic manipulations over the Dirac equation which yield an integral identity of the form:

$$\int (|\nabla\psi|^2 + m^2|\psi|^2) d^3x = \int N(A^a, \psi, \bar{\psi}) d^3x \tag{9.2}$$

[5] $W_{1,2}$ is a Sobolev space of functions f such that $f \in L_2$, $|\nabla f| \in L_2$.

where N is trilinear in fields; then the use of Hölder, Minkowski and Sobolev inequalities gives the following estimation

$$\int d^3x(|\nabla\psi|^2+m^2|\psi|^2) \leqslant Cg^2\|A\|^2_{L_3}\int d^3x(|\nabla\psi|^2+m^2|\psi|^2). \tag{9.3}$$

In (9.2), C is a constant independent of fields and the coupling constant g. For sufficiently small η the only possibility to satisfy (9.3) is to put $\psi = 0$.

Secondly, for $\psi = 0$ we are left with the pure (sourceless) Yang-Mills theory and theorems of Sect. 2a imply the absence of nonzero potentials.

A similar result (although proven in a different way) holds for the Yang-Mills-Higgs (YMH) system.

THEOREM 9.3 [118]

Assume that the energy of the YMH system is finite, the gauge group is compact and semisimple. Then topologically nontrivial smooth solutions are absent if

$$\lim_{g\to 0} gA^a_\mu(x, g) = 0. \tag{9.4}$$

Thus nonzero solutions of YM-Dirac (YMH) equations must be sufficiently singular in g at $g = 0$. This is in agreement with all explicitly known solutions. This result puts into question attempts to use perturbative techniques to the study of weak coupling region of gauge theories (e.g. [92]). At least these methods cannot be applied in systems with dynamical sources.

The above theorems hold for any compact gauge group, in 3 (and higher) dimensions. All that has been said previously can be restated as follows.

THEOREM 9.4

There is no classical "asymptotic" freedom in static Yang-Mills-matter theories.

Proof

In case of asymptotic freedom we would have $\lim_{g\to 0}\eta = 0$. Thus solutions of Yang-Mills-Dirac equations are absent. For non-Higgs scalar fields the result follows from "no go" theorems of Sect. 2 while for YMH fields it is implied by Theorem 9.3.

9c. On the existence of colour screening in selfcontained Yang-Mills-matter systems

The total charge of the YM theory is defined to be

$$I^a = \int d^3x\partial_i F^a_{i0}(\vec{x}), \tag{9.5}$$

that is, taking into account the Gauss-like equation $D_i F^{i0} = j^0$:

$$I^a = \int d^3x(j^{a0} - gf^{abc}A^b_i F^{ci0}). \tag{9.6}$$

Remark

There are some subtleties that are due to the obvious fact that the above definition is not gauge invariant. It is easy to overcome this problem by a suitable redefinition; we do not go into details, since it is not relevant for our purposes. The controversy related to this topic was eventually explained in [121].

Sikivie and Weiss [119] have presented solutions of YM equations with external sources which screen the external charge; the charge carried by gauge fields compensates the external one so that $I^a = 0$. This phenomenon is called "total colour screening".

The question arises whether or not this phenomenon could take place in selfcontained systems. It surely occurs in YMH systems since there are known explicit solutions with vanishing total charge. The results of Sect. 2 show that — apart from the YMH theory — total colour screening can occur only in Yang-Mills-Dirac equations. Theorem 9.2, in turn, states the absence of total colour screening in YMD theory as well, provided that $\eta = |g| \, ||A||_{L_3}$ is small enough. The condition that η is small can be interpreted as the demand that selfinteraction of YM fields is negligible (notice that nonlinearities in YM theory are of the form $gA\partial A$ or g^2A^3), yielding in the limit $\eta = 0$ linear equations for YM potentials. This indicates the absence of total colour screening in asymptotically free sector of the Yang-Mills-Dirac theory.

10. Prospects

We end this article by pointing out some problems which are still unsolved in classical gauge field theory.

The main problem, outstanding since appearance of 't Hooft's [9] work, is the absence of explicit solutions of Yang-Mills-Higgs (for nonzero Higgs potential). Up to now only one dyon solution of Julia and Zee is known (see [8]), which possesses both magnetic and electric charges. The question of existence of other dyons, with higher charges, is still open.

Taubes [57] and Manton [58] proved the existence of nonminimal solutions in the SU(2) YMH model and in the Weinberg-Salam model, respectively. Unfortunately, their approach does not give even approximate solutions.

An open question is the existence of soliton-like solutions in the Yang-Mills-Dirac SU(3) theory. Solitons in the YMD theory would provide justification for existing bag models in hadron physics.

Almost nothing was done in study of the Lyapunov stability of explicitly known solutions (apart from instantons and the BPS monopoles). Also the linearization stability is not, as yet, conclusive for gauge theories with large gauge groups as well as for solutions which are highly symmetric.

Chaos theory has revealed several interesting features of evolution of gauge fields, but the model considered there is oversimplified (the class of solutions is too narrow) since only time-dependence of potentials is allowed. The much more interesting problem is chaotic behaviour in the full Yang-Mills theory. The mathematical setting of chaos theory probably does not allow for realization of such an ambitious programme as to investigate stochastic behaviour of systems described by nonlinear partial differential equations.

The local Cauchy problem for the Yang-Mills theory is known to be well posed. Much has been done in the way to prove the existence of global solutions. However, even for source-less Yang-Mills equations little is known about scattering states.

Górski [126] began investigation of solutions which are regular in the limit $g \to \infty$. Yang-Mills equations are singular in this limit since the coefficients at highest derivatives vanish. There is a number of mathematical problems which should be solved before conclusive results could be obtained.

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APPENDIX A

We will prove the following theorem:

Let the falloff of A_i^a, $\partial_k A_i^a$ at infinity be $A_i^a = o(r^{-1})$, $\partial_k A_i^a = O(r^{-2})$, respectively. Suppose that $\Gamma^{\mu}_{i\mu} \in L_3$ ($\Gamma^{\mu}_{i\mu}$ is the Christoffel symbol) and $|g_{\mu\nu}-\eta_{\mu\nu}| = O(r^{-1})$ at spatial infinity. Then the system of Einstein-Yang-Mills equations does not possess solutions A_i^a small in $W_{1,3}$ norm provided that the L_3 norm of $\Gamma^{\mu}_{i\mu}$ is sufficiently small. Thus the space-like section $x_0 = \text{const}$ is locally flat (as in Sect. 2 we assume statics).

The Einstein-Yang-Mills equations read

$$\frac{1}{\sqrt{-g}}\,\partial_i(\sqrt{-g}\,g^{il}\partial_c)A^{ak}-\partial^k\partial_i A^{ai}$$
$$= -f^{abc}A^{bi}\partial_i A^{ck}-\Gamma^{\mu}_{i\mu}(F^{aik}-\partial^i A^{ak})-f^{abc}A^b_i F^{cik},$$
$$R^{\mu}_{\nu} = 0. \tag{A1}$$

Here we omitted equations for A_0^a and discarded terms with A_0^a, since A_0^a can always be assumed to be zero (see Sect. 2). In the proof we will use several integral inequalities and estimates of Hölder and Minkowski. Especially important is the following one

$$\int \varphi^6 dV \leqslant C_1[\int (\partial\varphi)^2 dV]^3. \tag{A2}$$

We make also the important conjecture that in our case the covariant Sobolev $W_{p,k}$ and L_p norms are equivalent to the usual $W_{p,k}$ and L_p norms. For definitions of these norms see [91].

Using (A2) and Minkowski and Hölder inequalities one can obtain the following estimates:

$$\|F\|_{L_2} \leqslant C_2\|\partial A\|_{L_2}(1+\|A\|_{L_3}), \tag{A3}$$

$$\|A\|_{L_4} \leqslant C_3\|A\|_{L_3}\|\partial A\|_{L_2}. \tag{A4}$$

Let us multiply (A1) by A_k^a. Integration by parts and omission of boundary terms yields

$$\|\partial A\|_{L_2}^2 = -\int f^{abc} A_k^a A_i^b \partial^i A^{ck} dV - \int \Gamma_{i\mu}^{\mu} A_k^a (F^{aik} - A^{ak}) dV - \int f^{abc} A_k^a A_i^b F^{cik} dV. \quad \text{(A5)}$$

Minkowski, Hölder and (A3), (A4) inequalities yield a crude estimation

$$\|\partial A\|_{L_2}^2 \leqslant C_4 \|\partial A\|_{L_2}^2 (\|A\|_{W_{1,3}} (2 + \|A\|_{W_{1,3}})). \quad \text{(A6)}$$

Here we used the evident inequalities

$$\|A\|_{L_3} \leqslant \|A\|_{W_{1,3}}, \qquad \|\partial A\|_{L_3} \leqslant \|A\|_{W_{1,3}}.$$

Supposing that $\|A\|_{W_{1,3}}$, $\|\Gamma\|_{L_3}$ are very small we arrive at contradiction

$$\|\partial A\|_{L_2}^2 \leqslant a \|\partial A\|_{L_2}^2, \qquad a < 1. \quad \text{(A7)}$$

This implies the absence of sufficiently small in $W_{1,3}$ norm solutions of the Einstein-Yang-Mills equations. Now they reduce to the Einstein equations

$$R^{\mu\nu} = 0. \quad \text{(A8)}$$

From $R^{00} = 0$ it follows $g^{00} = -1$ and therefore the vanishing of the Ricci tensor $R^{\mu\nu}$ is equivalent to the vanishing of the Ricci tensor R_3^{ik} of the space-like section given by $x_0 = \text{const}$. This implies its local flatness.

APPENDIX B

Here we will illustrate basic facts of the Lusternik-Šnirelman theory, using a simple compact manifold, a torus.

Let T^2 be a two torus standing on end and let f denote a real function from T^2 to R^1.

Let us define ϕ_t, $t \in [0, 1]$ to be a deformation of a torus, such that it moves all points down (to be precise — almost all points) and $\phi_0(x) = x$ be the identity transformation. Such a deformation could be defined by the equation

$$\frac{d}{dt}\phi = -\nabla f(\phi)$$

with the initial condition

$$\phi_0(x) = x$$

where f is any function decreasing as the height h decreases.

To fix attention, assume $f = h$, the height of a torus. Let us define

$$h^c = \{x \in T^2 : h(x) \leqslant c\},$$

$$h^{-1}(c) = \{x \in T^2 : h(x) = c\},$$

$$w = \{x \in T^2 : dh(x) = 0\},$$

$$w^c = w \cap h^{-1}(c).$$

The height h has 4 critical points. Note the following facts

(*i*) if $x \notin w$ then there exists $\varepsilon > 0$ and an open neighbourhood U of x such that $\phi_1(U) \subseteq h^{c-\varepsilon}$,

(*ii*) for all c there exists $\varepsilon > 0$ such that

$$\phi_1\left(h^{c+\varepsilon} - \begin{matrix}\text{open neighbourhood}\\ \text{of critical points}\\ \text{of } x\end{matrix}\right) \subseteq h^{c-\varepsilon}.$$

(*iii*) Let α_1, α_2, α_3 be a family of sets, each contractible to a point, such that they cover all surface of a torus. The minimal number of such sets (which is called "the category of a torus") denoted cat (T^2) equals 3. The Lusternik-Šnirelman category can be defined for each compact manifold.

Of great importance is the following theorem.

Minimax principle

Let $\mathbf{F}$ be a nonempty family of subsets of X satisfying the property (P): if $F \subset \mathbf{F}$, then $\phi_1(F) \subset \mathbf{F}$. Then $c = \inf_{F \subset \mathbf{F}} \sup \{f(p) : p \in F\}$ is a critical value of f.

Proof (for $X = T^2$, following [127])

Let $c = \inf \{b \in R^1 : \text{there exists } F \subset \mathbf{F} \text{ with } F \subseteq h^c\}$; if c is not a critical value of f then $\phi_1(h^{c+\varepsilon}) \subseteq h^{c-\varepsilon}$. Thus $\phi_1(F) \subset h^{c-\varepsilon}$, and c is not a minimum.

Now we shall construct the family $\mathbf{F}$ which possess the desired property (P). For a torus we could build 3 families $\mathbf{F}_k$, each consisting of sets of category at least k. These are

$$\mathbf{F}_3 = \{T^2\}; \quad \mathbf{F}_2 = \{T^2, V_2, W_2 \ldots\},$$

where V_2 and W_2 have the category 2;

$$\mathbf{F}_1 = \{p : p \in T^2\}.$$

It is evident that $\mathbf{F}$'s satisfy the property (P).

We infer directly from definitions of $\mathbf{F}$'s that there are 3 different $c_k = \inf_{F \subset \mathbf{F}} \sup \{h(p) : p \in F\}$. The minimax principle tells us that the number of critical points of any scalar function on a compact manifold M is greater or equal to the Lusternik-Šnirelman category of M. This is in accord with what we know about our sample function f, which is the height h and has 4 critical points.

This result can be generalized as to include the real-valued mappings defined on infinite dimensional manifolds, however it requires much more subtle definition of a deformation (which, in turn, imposes certain conditions on the mapping). This is of interest for field theory. The action S plays there the role of a real valued functional, whose critical points (that is, solutions of Euler-Lagrange equations) are searched for. For more details see [127, 128].

APPENDIX C

Let F be a mapping of a Banach space X into a Banach space Y which vanishes at $x_0 \in X$ (say, $x_0 = 0$) and depends on a parameter s. Suppose that $F(0, s) = 0$ for all $s \in \boldsymbol{R}^1$ and $\ker F'(0) = \ker \overset{*}{F'(0)}$, where $\overset{*}{F'(0)}$ is L_2-adjoint to $F'(0)$.

Moreover, let the kernel of $F'(0)$ be n-dimensional, $n < \infty$. Then, using the Lyapunov--Schmidt method (which is called also the alternative method), the existence problem becomes n-dimensional, as is shown in Section 6a.

Hence it is sufficient to consider mappings in finite dimensional spaces.

Let F be given by

$$F(s, x) = sx + T(s, x), \qquad x \in \boldsymbol{R}^n, \tag{C1}$$

where $T(s, x) = O(x^2)$ uniformly in a neighbourhood of $s = 0$.

The following result is due to Krasnoselsky [89], but here we follow Chow and Hale [127].

THEOREM

If n is odd, then $(s, x) = (0, 0)$ is a bifurcation point of $F(s, x)$, i.e., there are solutions (s, x) with $x \neq 0$ of the equation

$$F(s, x) = 0 \tag{C2}$$

in every neighbourhood of $(s, x) = (0, 0)$.

Proof consists in showing that there exists a function $s(x)$ such that $F(s(x), x)$ is a vector field on the $(n-1)$ dimensional sphere $||x|| = \varepsilon$ for any ε small. Since this sphere is even--dimensional, it follows that $F(s(x), x)$ has to have a zero vector. This is a well known fact in differential topology. Thus there are nontrivial solutions of $F(s, x) = 0$ for (s, x) near $(0, 0)$.

Therefore we have to prove only the existence of a relevant $s(x)$.

Consider the scalar-valued function

$$f(s, x) = \frac{1}{x^2} \langle x, sx + T(s, x) \rangle.$$

Note that $f(0, 0) = 0$ and $\frac{d}{ds} f(0, 0) = 1$. By the implicit function theorem there exists a unique function $s(x)$ for $||x||$ small such that $f(s(x), x) = 0$.

Remark

This proof reveals the decisive role of the odd- or even-dimensionality of kernel $F'(x_0)$. For instance in the case of even-dimensional kernels the sphere $||x|| = \varepsilon$ is odd dimensional and the preceding argument concerning zeros of a tangent vector field does not work.

Note added in proof. Recently D. Horvath (*Phys. Rev.* **D34** 1197 (1986)) discussed the analytical bifurcation in YM theory with external sources and D. Sivers (*Phys. Rev.* **D35**, 707 (1987)) studied spherically symmetric Yang-Mills-Dirac equations.

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