ON FIRST INTEGRALS OF THE ELECTROMAGNETIC DEVIATION EQUATIONS*

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The starting point of the paper are Lorentz equations of motion in a given background field in General Relativity as well as the corresponding to them differential equations of the first and the second electromagnetic deviations derived recently by the authors (*Acta Phys. Pol.* **B18**, 601 (1987)). Certain first integrals to all these equations are now derived under the assumption that the space-time admits symmetries. A characteristic feature of the method used here for deriving these integrals is its certain universality which manifests itself in being in principle the same procedure leading to the integrals in the case of every one of the equations considered. It is also shown that a Killing vector field or a symmetric Killing tensor field generate along a Lorentzian world line a field of a natural first electromagnetic deviation.

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Introduction

The connection between symmetries of space-time and first integrals of several types of equations of motion is well established. However, in the case of the geodesic deviation equations in general relativity, the form of an integral generated by a Killing vector field was recently found by Fuchs [1]. The first integrals discussed in the past (cf. e.g. [2–5]) were usually of a different type, just securing the compatibility of constraint relations which one must impose on the initial data for the geodesic deviation equations on one side with the accepted differential equations of geodesic deviation on the other.

The objective of this paper is to derive a number of first integrals to the first and the second electromagnetic (e.m.) deviation equations, which are a generalization of the geodesic deviation equations to the case of both electromagnetic and gravitational tidal forces (see [6] for details). The existence of these integrals is a consequence of symmetries of space-

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-time in a broader sense, i.e. of those which are generated not only by Killing vector fields, but in a sense by the second order symmetric Killing tensor field as well.

In Sect. 1 we formulate some technical lemmas which enable a uniform approach to deriving the first integrals.

In Sect. 2 it is shown that under certain very natural conditions on the electromagnetic field the Killing vector field ξ^α, when restricted to a Lorentzian world line, satisfies the natural first e.m. deviation equations (in terminology introduced in [6]). The same property enjoys also the vector $\zeta^\alpha := \frac{K^\alpha{}_\beta u^\beta}{\sqrt{|u_\lambda u^\lambda|}}$, where u^α is tangent to a Lorentzian world line and $K_{\alpha\beta}$ is a symmetric Killing tensor admitted by the space-time. An application of the lemmas formulated in Sect. 1 and of the properties of ξ^α and ζ^α mentioned above is the derivation of the well-known integrals of the Lorentz equations of motion.

In Sects 3 and 4 the same lemmas and properties are used to deriving first integrals of the first and second e.m. deviation equations. The specialization of these integrals for different types of the first and second deviations is given, and a simple connection between first integrals of the Lorentz equations and those of the first and the second e.m. deviation is obtained. In the limiting case of geodesic deviation equations, one of the integrals passes over into that found by Fuchs [1]. The remaining were however not known so far even in the limiting case of geodesics world lines.

1. Some technical lemmas

The procedure of deriving several first integrals of the Lorentz as well as of the first and the second e.m. deviation equations in an arbitrary parametrization is simplified when one makes the following observations.

PROPOSITION 1.1. If a vector a^α defined along an arbitrary curve Γ in V_n fulfils the equations of transport of the form

$$\frac{Da^\alpha}{d\tau} := a^\alpha{}_{;\beta}u^\beta = b^\alpha, \tag{1.1}$$

where b^α is another vector field along Γ, and if there exists a vector field ι^α such that

$$b_\alpha \iota^\alpha + a_\alpha \frac{D\iota^\alpha}{d\tau} = \frac{d\Phi_1}{d\tau}, \tag{1.2}$$

where Φ_1 is a scalar function of the parameter τ, then

$$a_\alpha \iota^\alpha - \Phi_1(\tau) = \text{const} \tag{1.3}$$

along Γ.

The proof is obvious and will be omitted.

If, in particular, a^α is the vector $u^\alpha := \frac{d\xi^\alpha}{d\tau}$ tangent to an arbitrary (i.e. not necessarily geodesic) non-null world line Γ parametrized by a scalar parameter τ, then the differential equations (1.1) may be always represented in the form

$$\frac{Du^\alpha}{d\tau} = b^\alpha := \lambda u^\alpha + g^\alpha, \tag{1.4}$$

where $\lambda := \frac{u_\alpha b^\alpha}{u_\lambda u^\lambda} = \frac{d}{d\tau}\ln\sqrt{|u_\lambda u^\lambda|}$, $g^\alpha := h^\alpha{}_\beta b^\beta$, $h^\alpha{}_\beta := \delta^\alpha{}_\beta - \frac{u^\alpha u_\beta}{u_\lambda u^\lambda}$, and where the assumption $u_\lambda u^\lambda \neq 0$ is accepted anywhere along Γ.

PROPOSITION 1.2. If along a world line Γ defined in V_n by Eqs (1.4) there exists a vector field ι^α such that

$$\text{a)}\ u_\alpha \frac{D\iota^\alpha}{d\tau} = 0, \qquad \text{b)}\ \frac{g_\alpha \iota^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = \frac{d\Phi_2}{d\tau}, \tag{1.5}$$

where Φ_2 is a scalar function of τ, then

$$\frac{u_\alpha \iota^\alpha}{\sqrt{|u_\lambda u^\lambda|}} - \Phi_2(\tau) = \text{const} \tag{1.6}$$

along Γ.

Proof. Contracting (1.4) with ι_α and adding the result to Eq. (1.5a) yield

$$\frac{D}{d\tau}(u_\alpha \iota^\alpha) = g_\alpha \iota^\alpha + \lambda(u_\alpha \iota^\alpha).$$

Dividing this expression by $\sqrt{|u_\lambda u^\lambda|}$ and taking into account (1.5b), after recalling the definition of λ, complete the proof.

PROPOSITION 1.3. If along the world line Γ defined in V_n by Eqs (1.4) the vector field n^α fulfils the equations

$$\frac{D^2 n^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = \lambda \frac{Dn^\alpha}{d\tau} + m^\alpha, \tag{1.7}$$

and there exists a vector field ι^α for which

$$\text{a)}\ \frac{D^2 \iota^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta \iota^\gamma u^\delta = \lambda \frac{D\iota^\alpha}{d\tau} + k^\alpha, \qquad \text{b)}\ \frac{m_\alpha \iota^\alpha - n_\alpha k^\alpha}{\sqrt{|u_\lambda u^\lambda|}} = \frac{d\Phi_3}{d\tau}, \tag{1.8}$$

where Φ_3 is a scalar function of τ, then

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(\iota_\alpha \frac{Dn^\alpha}{d\tau} - n_\alpha \frac{D\iota^\alpha}{d\tau}\right) - \Phi_3(\tau) = \text{const} \tag{1.9}$$

along Γ.

Proof. Contracting (1.7) with ι_α and subtracting from the result Eqs (1.8a) multiplied by n_α yield

$$\frac{D}{d\tau}\left(\iota_\alpha \frac{Dn^\alpha}{d\tau} - n_\alpha \frac{D\iota^\alpha}{d\tau}\right) - \lambda\left(\iota_\alpha \frac{Dn^\alpha}{d\tau} - n_\alpha \frac{D\iota^\alpha}{d\tau}\right) = m_\alpha \iota^\alpha - n_\alpha k^\alpha.$$

Dividing then this by $\sqrt{|u_\lambda u^\lambda|}$ and taking into account Eq. (1.8b) as well as the definition of λ complete the proof. Obviously, Props 1.1–1.3 are trivially satisfied if Φ_1, Φ_2 and Φ_3 are constant.

The scheme used in the proofs of Props 1.1–1.3 forms a background of a general and simple method of finding first integrals for several equations of motion of the form (1.4) and for several laws of transport of the form (1.1) or (1.7) of some vector fields. In particular, if in an arbitrary V_n a world line Γ is defined by Eqs (1.4) with a constraint condition $\lambda = 0$, i.e. $\dfrac{Du^\alpha}{d\pi} = g^\alpha$, where π is a new parameter, then the vector field $\iota^\alpha \equiv u^\alpha$ satisfies the conditions (1.5) and generates the first integral of the form $\sqrt{|u_\lambda u^\lambda|} = \text{const.}$ It is not difficult to see however that in some pseudo-Riemannian manifolds there are vector fields ι^α, other than u^α, which have the properties mentioned in Props. 1.1–1.3.

PROPOSITION 1.4. If a pseudo-Riemannian manifold V_n admits a Killing vector field ξ^α, i.e., if

$$\pounds_\xi g_{\alpha\beta} = 0 = \xi_{\alpha;\beta} + \xi_{\beta;\alpha}, \tag{1.10}$$

then along the world line Γ defined by Eqs (1.4) the vector field $\iota^\alpha \equiv \xi^\alpha$ fulfils Eq. (1.5a) and Eqs (1.8a) with $k_\alpha := \xi_{\alpha;\beta} g^\beta$.

The proof of Eq.(1.5a) is straightforward. To prove the validity of Eqs (1.8a), one must use the integrability conditions

$$\xi_{\alpha;\beta\gamma} = R_{\alpha\beta\gamma\delta}\xi^\delta \tag{1.11}$$

of the Killing equations (1.10).

PROPOSITION 1.5. If a pseudo-Riemannian manifold V_n admits a symmetric Killing tensor field $K_{\alpha\beta}$ of the order two, determined by the equations

$$K_{(\alpha\beta;\gamma)} = 0, \tag{1.12}$$

then along a world line Γ defined by Eqs (1.4) the vector field

$$\zeta_\alpha := \frac{K_{\alpha\beta}u^\beta}{\sqrt{|u_\lambda u^\lambda|}} \tag{1.13}$$

fulfils both Eqs (1.8a), with $k_\alpha := \sqrt{|u_\lambda u^\lambda|}\,\dfrac{D}{d\tau}\left(\dfrac{\zeta_\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right) - \dfrac{K_{\beta\gamma;\alpha}u^\beta g^\gamma}{\sqrt{|u_\lambda u^\lambda|}}$, and Eq. (1.5a) provided

$$K_{\alpha\beta} g^\alpha u^\beta = 0. \tag{1.14}$$

The proof of Eq. (1.5a) follows from Eqs (1.12), (1.14). To prove the validity of (1.8a) one must use the equality

$$K_{\alpha\beta;\gamma\delta}u^{\beta}u^{\gamma}u^{\delta} = K_{\beta\varepsilon}R^{\varepsilon}{}_{\delta\gamma\alpha}u^{\beta}u^{\gamma}u^{\delta} \tag{1.15}$$

which follows from the integrability conditions

$$K_{\alpha\beta;\gamma\delta}-K_{\delta\gamma;\beta\alpha} = K_{\alpha\varepsilon}R^{\varepsilon}{}_{\delta\gamma\beta}+K_{\beta\varepsilon}R^{\varepsilon}{}_{\delta\gamma\alpha}+K_{\gamma\varepsilon}R^{\varepsilon}{}_{\alpha\delta\beta}+K_{\delta\varepsilon}R^{\varepsilon}{}_{\alpha\gamma\beta}$$

of Eqs (1.12).

Thus the vector ι^{α} in Props 1.1–1.3 may be replaced by a Killing vector ξ^{α} or by the vector ζ^{α} which generate the first integrals of Eqs (1.1), (1.4) and (1.7) provided conditions (1.2), (1.5b) and (1.8b) are satisfied correspondingly.

2. The Lorentzian world lines

Now, let us restrict ourselves to the case of a general relativistic Lorentz equations of motion in an arbitrary parametrization [6], which may be written in either one of the two equivalent forms:

$$\frac{D}{d\tau}\left(\frac{u^{\alpha}}{\sqrt{|u_{\lambda}u^{\lambda}|}}\right) = \sigma F^{\alpha}{}_{\beta}u^{\beta} \tag{2.1}$$

or

$$\frac{Du^{\alpha}}{d\tau} = \lambda u^{\alpha}+\sigma\sqrt{|u_{\lambda}u^{\lambda}|}\,F^{\alpha}{}_{\beta}u^{\beta}, \tag{2.2}$$

where $F_{\alpha\beta} = -F_{\beta\alpha}$ is the electromagnetic field tensor and $\sigma = \dfrac{q}{\mu c^2}$ is a constant. Then the following statements are true.

THEOREM 2.1. If in V_n from Prop. 1.4 the curve Γ is defined by the Lorentz equations (2.2) and if additionally the antisymmetric tensor field $F_{\alpha\beta}$ satisfies the conditions

$$\pounds_{\xi}F_{\alpha\beta} = 0, \tag{2.3}$$

then the Killing vector field ξ^{α} fulfils along the Lorentzian world line Γ the natural first e.m. deviation equations[1] in an arbitrary parametrization

$$\frac{D^2\xi^{\alpha}}{d\tau^2}+R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}\xi^{\gamma}u^{\delta} = \lambda\frac{D\xi^{\alpha}}{d\tau}+\sigma\sqrt{|u_{\lambda}u^{\lambda}|}\left(F^{\alpha}{}_{\beta;\gamma}u^{\beta}\xi^{\gamma}+F^{\alpha}{}_{\beta}\frac{D\xi^{\beta}}{d\tau}\right). \tag{2.4}$$

The proof follows from an immediate computation in which Eqs (2.2), (2.3) and Prop. 1.4 are taken into account.

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[1] That is, the general first e.m. deviation equations (cf. (3.2)) supplemented by the constraint condition $u_{\alpha}\dfrac{D\xi^{\alpha}}{d\tau} = 0$.

COROLLARY 2.1. A Killing vector field ξ^α is along an arbitrary Lorentzian world line a field of a natural first e.m. deviation provided the electromagnetic field tensor $F_{\alpha\beta}$ satisfies Eqs (2.3).

COROLLARY 2.2. In the particular case of a geodesic world line (i.e. for $\sigma = 0$) Eqs (2.4) assume the form

$$\frac{D^2\xi^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta\xi^\gamma u^\delta = \lambda\frac{D\xi^\alpha}{d\tau}, \tag{2.5}$$

or in other words, any Killing vector ξ^α is along a geodesic world line a field of a natural first geodesic deviation.

THEOREM 2.2. If in V_n from Prop. 1.5 the curve Γ is defined by the Lorentz equations (2.2) and if additionally the tensor fields $F_{\alpha\beta}$ and $K_{\alpha\beta}$ satisfy the conditions

$$F_{[\alpha\beta;\gamma]} = 0, \tag{2.6}$$

$$K^\gamma{}_{(\alpha}F_{\beta)\gamma} = 0, \tag{2.7}$$

then the vector ζ^α fulfils along the Lorentzian world line Γ the natural first e.m. deviation equations in an arbitrary parametrization

$$\frac{D^2\zeta^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta\zeta^\gamma u^\delta = \lambda\frac{D\zeta^\alpha}{d\tau} + \sigma\sqrt{|u_\lambda u^\lambda|}\left(F^\alpha{}_{\beta;\gamma}u^\beta\zeta^\gamma + F^\alpha{}_\beta\frac{D\zeta^\beta}{d\tau}\right). \tag{2.8}$$

The proof follows from an immediate computation which makes use of Eqs (2.2), (2.6), (2.7) and of Prop. 1.5.

COROLLARY 2.3. A symmetric Killing tensor $K_{\alpha\beta}$ generates along an arbitrary Lorentzian world line a field ζ^α of a natural first e.m. deviation provided the tensors $F_{\alpha\beta}$ and $K_{\alpha\beta}$ satisfy Eqs (2.7).

COROLLARY 2.4. In the particular case of a geodesic world line (i.e. for $\sigma = 0$) Eqs (2.8) read

$$\frac{D^2\zeta^\alpha}{d\tau^2} + R^\alpha{}_{\beta\gamma\delta}u^\beta\zeta^\gamma u^\delta = \lambda\frac{D\zeta^\alpha}{d\tau}. \tag{2.9}$$

In other words, any Killing tensor $K_{\alpha\beta}$ generates along the geodesic a field of a natural first geodesic deviation.

It must be noted that Corollaries 2.2 and 2.4 are generalizations to the case of an arbitrary parametrization of already known properties of ξ^α and $K_{\alpha\beta}$ (cf. e.g. [8–11]).

Now we are prepared to prove the following theorems.

THEOREM 2.3. If a pseudo-Riemannian manifold V_n admits a Killing vector field ξ^α and if additionally the electromagnetic field tensor $F_{\alpha\beta}$ admits a vector potential A_α which satisfies the conditions[2]

$$\pounds_\xi A_\alpha = 0, \tag{2.10}$$

[2] Let us note that conditions (2.10) are equivalent to conditions (2.3) (cf. [8]).

then there exists a first integral of the Lorentz equations of the form

$$\xi_\alpha\left(\frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}+\sigma A^\alpha\right)=C_1. \tag{2.11}$$

The proof follows either from (1.3) or from (1.6). Indeed, setting $\iota^\alpha \equiv \xi^\alpha$ and applying Prop. 1.1 to Eqs (2.1) or Prop. 1.2 to Eqs (2.3) yield

$$\Phi_1(\tau)=\Phi_2(\tau)=-\sigma A_\alpha\xi^\alpha,$$

and that ends the proof.

THEOREM 2.4. If V_n admits a symmetric Killing tensor field $K_{\alpha\beta}$ and the conditions (2.7) are satisfied, then there exists a first integral of the Lorentz equations of the form

$$\frac{K_{\alpha\beta}u^\alpha u^\beta}{u_\lambda u^\lambda}=C_2. \tag{2.12}$$

The proof follows either from (1.3) or from (1.6) after setting $\iota^\alpha \equiv \zeta^\alpha$ and taking into account that, due to Eqs (2.7), $\frac{d}{d\tau}\Phi_1=\frac{d}{d\tau}\Phi_2=0.$

Obviously, in the case of the natural parametrization of the Lorentzian world line the first integrals (2.11) and (2.12) reduce to customary forms quoted in [8, 11].

3. *The first e.m. deviation*

Now, let us supplement the Lorentz equations (2.1) by the first e.m. deviation equations in an arbitrary parametrization derived in [6] which may be represented in either one of the two equivalent forms

$$\frac{D}{d\tau}\left(\frac{h^\alpha{}_\beta}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dr^\beta}{d\tau}\right)=-\frac{1}{\sqrt{|u_\lambda u^\lambda|}}R^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta+\sigma(F^\alpha{}_{\beta;\gamma}u^\beta r^\gamma+F^\alpha{}_\beta\frac{Dr}{d\tau} \tag{3.1}$$

or

$$\frac{D^2r^\alpha}{d\tau^2}+R^\alpha{}_{\beta\gamma\delta}u^\beta r^\gamma u^\delta=\lambda\frac{Dr^\alpha}{d\tau}+u^\alpha\frac{d}{d\tau}\left(\frac{u_\beta}{u_\gamma u^\gamma}\frac{Dr^\beta}{d\tau}\right)$$
$$+\sigma\sqrt{|u_\lambda u^\lambda|}\left[F^\alpha{}_{\beta;\gamma}u^\beta r^\gamma+F^\alpha{}_\beta\frac{Dr^\beta}{d\tau}+F^\alpha{}_\beta u^\beta\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right], \tag{3.2}$$

where r^α is the first e.m. deviation vector. Then the following statements are true.

THEOREM 3.1. If V_n admits a Killing vector field ξ^α and if additionally the electromagnetic field tensor $F_{\alpha\beta}$ fulfils the conditions (2.3) and (2.6), then there exists a first integral of the system of the Lorentz and the first e.m. deviation equations of the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(h_{\alpha\beta}\xi^\alpha\frac{Dr^\beta}{d\tau}-r_\alpha\frac{D\xi^\alpha}{d\tau}\right)-\sigma F_{\alpha\beta}\xi^\alpha r^\beta=C_3. \tag{3.3}$$

Proof. The proof follows either from (1.3) or from (1.9). Indeed, Eqs (3.1) take the same form as Eqs (1.1). Therefore, after setting $\iota^\alpha \equiv \xi^\alpha$, applying Prop. 1.1 and making use of Eqs (1.10), (1.11), (2.3) and (2.6), we obtain

$$\Phi_1(\tau) = \frac{r_\alpha}{\sqrt{|u_\lambda u^\lambda|}} \frac{D\xi^\alpha}{d\tau} + \sigma F_{\alpha\beta}\xi^\alpha r^\beta,$$

and that finishes the first part of the proof. On the other hand after setting $n^\alpha \equiv r^\alpha$ Eqs (3.2) may be rewritten in the form (1.7). Applying, therefore, Prop. 1.3 and making use of Eqs (1.10), (2.4) and (2.6) yield

$$\Phi_3(\tau) = \frac{u_\alpha \xi^\alpha}{\sqrt{|u_\lambda u^\lambda|}} \left(\frac{u_\beta}{u_\gamma u^\gamma} \frac{Dr^\beta}{d\tau} \right) + \sigma F_{\alpha\beta}\xi^\alpha r^\beta,$$

which thus completes the proof.

COROLLARY 3.1. If r^α_{n} is a natural first e.m. deviation vector, then Eq. (3.3) takes the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(\xi_\alpha \frac{Dr^\alpha_{\mathrm{n}}}{d\tau} - r^\alpha_{\mathrm{n}} \frac{D\xi_\alpha}{d\tau} \right) - \sigma F_{\alpha\beta}\xi^\alpha r^\beta_{\mathrm{n}} = C_3. \tag{3.4}$$

For a first e.m. deviation vector r^α_{s}, which satisfies the constraint condition

$$\frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}} \frac{Dr^\alpha_{\mathrm{s}}}{d\tau} = \sigma F_{\alpha\beta} u^\alpha r^\beta_{\mathrm{s}}, \tag{3.5}$$

an therefore preserves its inclination $u_\alpha r^\alpha_{\mathrm{s}}$ with Γ, Eq. (3.3) reduces to

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left[\xi_\alpha \left(\frac{Dr^\alpha_{\mathrm{s}}}{d\tau} - u^\alpha \frac{k\sigma F_{\beta\gamma} u^\beta r^\gamma_{\mathrm{s}}}{\sqrt{|u_\lambda u^\lambda|}} \right) - r^\alpha_{\mathrm{s}} \frac{D\xi_\alpha}{d\tau} \right] - \sigma F_{\alpha\beta}\xi^\alpha r^\beta_{\mathrm{s}} = C_3, \tag{3.6}$$

where the positive sign of $k = \pm 1$ corresponds to timelike Lorentzian world lines.

COROLLARY 3.2. In the particular case of the first geodesic deviation equations following from Eqs (3.1) or (3.2) for $\sigma = 0$, Eqs (3.3) take the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(h_{\alpha\beta}\xi^\alpha \frac{Dr^\beta}{d\tau} - r_\alpha \frac{D\xi^\alpha}{d\tau} \right) = C_3, \tag{3.7}$$

and Eqs (3.5), (3.6) merge into a single expression

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}} \left(\xi_\alpha \frac{Dr^\alpha}{d\tau} - r_\alpha \frac{D\xi^\alpha}{d\tau} \right) = C_3,$$

which for timelike geodesic parametrized by the natural parameter s was derived in [1].

THEOREM 3.2. If V_n admits a symmetric Killing tensor field $K_{\alpha\beta}$ and the conditions (2.7) are fulfilled, then there exists a first integral of the system of the Lorentz and the first e.m. deviation equations of the form

$$\frac{u^\alpha}{u_\lambda u^\lambda}\left(K_{\alpha\beta}h^\beta{}_\gamma \frac{Dr^\gamma}{d\tau} - r^\beta \frac{DK_{\alpha\beta}}{d\tau}\right) = C_4. \tag{3.8}$$

Proof. The proof follows either from (1.3) or from (1.9). Setting $\iota^\alpha \equiv \zeta^\alpha$, applying Prop. 1.1 to Eqs (3.1), and making use of Eqs (1.12), (1.15), (2.1) and (2.7), we obtain

$$\Phi_1(\tau) = \frac{u^\alpha r^\beta}{|u_\lambda u^\lambda|}\frac{DK_{\alpha\beta}}{d\tau},$$

and that finishes the first part of the proof. On the other hand applying Prop. 1.3 to Eqs (3.2) and making use of Eqs (1.5a), (2.1), (2.6) and (2.7) result in

$$\Phi_3(\tau) = \frac{K_{\alpha\beta}u^\alpha u^\beta}{|u_\lambda u^\lambda|}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right) - \frac{K_{\alpha\beta}r^\beta}{\sqrt{|u_\lambda u^\lambda|}}\frac{D}{d\tau}\left(\frac{u^\alpha}{\sqrt{|u_\lambda u^\lambda|}}\right),$$

which completes the proof.

In the particular case of $\sigma = 0$, the system of the geodesic and the first geodesic deviation equations admits the first integral of the same form as (3.8).

COROLLARY 3.3. If r^α_{n} is a natural first e.m. deviation vector, then Eq. (3.8) takes the form

$$\frac{u^\alpha}{u_\lambda u^\lambda}\left(K_{\alpha\beta}\frac{Dr^\beta}{d\tau} - r^\beta \frac{DK_{\alpha\beta}}{d\tau}\right) = C_4. \tag{3.9}$$

For a first e.m. deviation vector r^α_{s}, which satisfies the condition (3.5), Eq. (3.8) reduces to

$$\frac{u^\alpha}{u_\lambda u^\lambda}\left[K_{\alpha\beta}\left(\frac{Dr^\beta_{\text{s}}}{d\tau} - u^\beta \frac{k\sigma F_{\gamma\delta}u^\gamma r^\delta_{\text{s}}}{\sqrt{|u_\lambda u^\lambda|}}\right) - r^\beta_{\text{s}}\frac{DK_{\alpha\beta}}{d\tau}\right] = C_4. \tag{3.10}$$

Let us observe that in the framework of the Σ-approach to families of Lorentzian world lines which was developed in [6] there exists a simple relation between first integrals of the Lorentz equations and those of the first e.m. deviation as it may be restated in the form of the following lemma.

LEMMA 3.1. If for each curve Γ_ε from a one-parametric family Σ of Lorentzian world lines the Lorentz equations admit the first integrals (2.11) and (2.12), i.e., if

$$\zeta_\alpha\left(\frac{u^\alpha(\tau,\varepsilon)}{\sqrt{|u_\lambda(\tau,\varepsilon)u^\lambda(\tau,\varepsilon)|}} + \sigma A^\alpha(\tau,\varepsilon)\right) = C_1(\varepsilon), \tag{3.11}$$

$$\frac{K_{\alpha\beta}u^\alpha(\tau,\varepsilon)u^\beta(\tau,\varepsilon)}{u_\lambda(\tau,\varepsilon)u^\lambda(\tau,\varepsilon)} = C_2(\varepsilon), \tag{3.12}$$

where $C_1(\varepsilon)$ and $C(\varepsilon)$ are regular functions of the parameter ε then the first integrals (3.3) and (3.8) follow from differentiation of Eqs (3.11) and (3.12) with respect to the parameter ε.

The proof follows from inspection, after the definitions

$$\frac{D}{\partial\varepsilon}(\cdot) := (\cdot)_{;\lambda}r^{\lambda}, \quad r^{\alpha}(\tau,\varepsilon) := \frac{\partial u^{\alpha}}{\partial\varepsilon}(\tau,\varepsilon), \quad C_3 = \frac{\partial C_1(\varepsilon)}{\partial\varepsilon},$$

$$C_4 = \tfrac{1}{2}\frac{\partial C_2(\varepsilon)}{\partial\varepsilon},$$

and the relations $\dfrac{Du^{\alpha}}{\partial\varepsilon} = \dfrac{Dr^{\alpha}}{\partial\tau}$ are taken into account.

4. The second e.m. deviation

Let us supplement the system of the Lorentz and the first e.m. deviation equations by the second e.m. deviation equations in an arbitrary parametrization [7] which may be also represented in either one of the two equivalent forms

$$\frac{D}{d\tau}\left(\frac{h^{\alpha}{}_{\beta}}{\sqrt{|u_{\lambda}u^{\lambda}|}}\frac{Dw^{\beta}}{d\tau}\right) = \frac{1}{\sqrt{|u_{\lambda}u^{\lambda}|}}\Bigg\{-R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}w^{\gamma}u^{\delta} + (R^{\alpha}{}_{\beta\gamma\delta;\varepsilon}$$
$$+R^{\alpha}{}_{\varepsilon\gamma\delta;\beta})u^{\beta}u^{\gamma}r^{\delta}r^{\varepsilon} + 4R^{\alpha}{}_{\beta\gamma\delta}\frac{Dr^{\beta}}{d\tau}u^{\gamma}r^{\delta} + 2\frac{Dr^{\alpha}}{d\tau}\frac{d}{d\tau}\left(\frac{u_{\beta}}{u_{\gamma}u^{\gamma}}\frac{Dr^{\beta}}{d\tau}\right)$$
$$+u^{\alpha}\frac{d}{d\tau}\left[\frac{1}{u_{\lambda}u^{\lambda}}\left(h_{\beta\gamma}\frac{Dr^{\beta}}{d\tau}\frac{Dr^{\gamma}}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^{\beta}r^{\gamma}r^{\delta}u^{\varepsilon}\right) - \left(\frac{u_{\beta}}{u_{\gamma}u^{\gamma}}\frac{Dr^{\beta}}{d\tau}\right)^2\right]\Bigg\}$$
$$+\sigma\Bigg\{F^{\alpha}{}_{\beta;\gamma\delta}u^{\beta}r^{\gamma}r^{\delta} + F^{\alpha}{}_{\beta;\gamma}\left[u^{\beta}w^{\gamma} + 2\frac{Dr^{\beta}}{d\tau}r^{\gamma} + 2u^{\beta}r^{\gamma}\left(\frac{u_{\delta}}{u_{\varepsilon}u^{\varepsilon}}\frac{Dr^{\delta}}{d\tau}\right)\right]$$
$$+(F^{\alpha}{}_{\beta}R^{\beta}{}_{\gamma\delta\varepsilon} - R^{\alpha}{}_{\gamma\delta\beta}F^{\beta}{}_{\varepsilon})r^{\gamma}r^{\delta}u^{\varepsilon} + F^{\alpha}{}_{\beta}\left[\frac{Dw^{\beta}}{d\tau} + 2\frac{Dr^{\beta}}{d\tau}\left(\frac{u_{\gamma}}{u_{\varepsilon}u^{\varepsilon}}\frac{Dr^{\gamma}}{d\tau}\right)\right.$$
$$\left.\left. + \frac{u^{\beta}}{u_{\lambda}u^{\lambda}}\left(h_{\gamma\delta}\frac{Dr^{\gamma}}{d\tau}\frac{Dr^{\delta}}{d\tau} + R_{\gamma\delta\varepsilon\tau}u^{\gamma}r^{\delta}r^{\varepsilon}u^{\tau}\right)\right]\right\} \tag{4.1}$$

or

$$\frac{D^2w^{\alpha}}{d\tau^2} + R^{\alpha}{}_{\beta\gamma\delta}u^{\beta}w^{\gamma}u^{\delta} = (R^{\alpha}{}_{\beta\gamma\delta;\varepsilon} + R^{\alpha}{}_{\varepsilon\gamma\delta;\beta})u^{\beta}u^{\gamma}r^{\delta}r^{\varepsilon}$$
$$+4R^{\alpha}{}_{\beta\gamma\delta}\frac{Dr^{\beta}}{d\tau}u^{\gamma}r^{\delta} + \frac{Dw^{\alpha}}{d\tau}\frac{d}{d\tau}\ln\sqrt{|u_{\lambda}u^{\lambda}|} + 2\frac{Dr^{\alpha}}{d\tau}\left(\frac{u_{\beta}}{u_{\gamma}u^{\gamma}}\frac{Dr^{\beta}}{d\tau}\right)$$
$$+u^{\alpha}\frac{d}{d\tau}\left\{\frac{1}{u_{\lambda}u^{\lambda}}\left[h_{\beta\gamma}\frac{Dr^{\beta}}{d\tau}\frac{Dr^{\gamma}}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^{\beta}r^{\gamma}r^{\delta}u^{\varepsilon} + u_{\beta}\frac{Dw^{\beta}}{d\tau}\right.\right.$$

$$- \frac{1}{u_\lambda u^\lambda}\left(u_\beta \frac{Dr^\beta}{d\tau}\right)^2\Bigg]\Bigg\} + \sigma\sqrt{|u_\lambda u^\lambda|}\Bigg\{F^\alpha{}_{\beta;\gamma\delta}u^\beta r^\gamma r^\delta + F^\alpha{}_{\beta;\gamma}\Bigg[u^\beta w^\gamma$$

$$+2\frac{Dr^\beta}{d\tau} r^\gamma + 2u^\beta r^\gamma\left(\frac{u_\delta}{u_\varepsilon u^\varepsilon}\frac{Dr^\delta}{d\tau}\right)\Bigg] + (F^\alpha{}_\beta R^\beta{}_{\gamma\delta\varepsilon} - R^\alpha{}_{\gamma\delta\beta}F^\beta{}_\varepsilon)r^\gamma r^\delta u^\varepsilon$$

$$+F^\alpha{}_\beta\Bigg[\frac{Dw^\beta}{d\tau} + 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\varepsilon u^\varepsilon}\frac{Dr^\gamma}{d\tau}\right) + \frac{u^\beta}{u_\lambda u^\lambda}\Bigg(h_{\gamma\delta}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta}{d\tau}$$

$$+R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau + u_\gamma\frac{Dw^\gamma}{d\tau}\Bigg)\Bigg]\Bigg\}, \tag{4.2}$$

where w^α is the second e.m. deviation vector. Analogously to Sect. 3 the following statements are valid.

THEOREM 4.1. If V_n admits a Killing vector field ξ^α and if additionally the electromagnetic field tensor $F_{\alpha\beta}$ fulfils the conditions (2.3) and (2.6), then there exists a first integral of the system of the Lorentz, the first, and the second e.m. deviation equations of the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\Bigg\{h_{\alpha\beta}\xi^\alpha\left[\frac{Dw^\beta}{d\tau} - 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right] + 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta r^\gamma u^\delta$$

$$- \frac{u_\alpha\xi^\alpha}{u_\lambda u^\lambda}\left(h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon\right) - \left[w_\alpha - 2r_\alpha\left(\frac{u_\beta}{u_\gamma u^\gamma}\frac{Dr^\beta}{d\tau}\right)\right]\frac{D\xi^\alpha}{d\tau}$$

$$-2\xi_{\alpha;\beta}r^\alpha\frac{Dr^\beta}{d\tau}\Bigg\} - \sigma[(F_{\alpha\beta}w^\beta + F_{\alpha\beta;\gamma}r^\beta r^\gamma)\xi^\alpha + \xi_{\alpha;\beta}F^\alpha{}_\gamma r^\beta r^\gamma] = C_5. \tag{4.3}$$

Proof. The proof follows again either from (1.3) or from (1.9). Setting $\iota^\alpha \equiv \xi^\alpha$, applying Prop. 1.1 to Eqs (4.1), making use of Eqs (1.10), (1.11), (2.1), (2.3), (2.6), (3.1) and of the relations $\underset{\xi}{£} F_{\alpha\beta;\gamma} = 0$ and $\underset{\xi}{£} R_{\alpha\beta\gamma\delta} = 0$, provided the Ricci and the Bianchi identities are taken into account, we obtain

$$\Phi_1(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\Bigg\{2h_{\alpha\beta}\xi^\alpha\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right) - 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta r^\gamma u^\delta$$

$$+ \frac{u_\alpha\xi^\alpha}{u_\lambda u^\lambda}\left(h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon\right) + \left[w_\alpha - 2r_\alpha\left(\frac{u_\beta}{u_\gamma u^\gamma}\frac{Dr^\beta}{d\tau}\right)\right]\frac{D\xi^\alpha}{d\tau}$$

$$+2\xi_{\alpha;\beta}r^\alpha\frac{Dr^\beta}{d\tau}\Bigg\} + \sigma[(F_{\alpha\beta}w^\beta + F_{\alpha\beta;\gamma}r^\beta r^\gamma)\xi^\alpha + \xi_{\alpha;\beta}F^\alpha{}_\gamma r^\beta r^\gamma],$$

and that finishes the first part of the proof. Analogously, setting $\iota^\alpha \equiv \zeta^\alpha$, applying Prop. 1.3 to Eqs (4.2) and making use of the equations mentioned above yield

$$\Phi_3(\tau) = \frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left[u_\alpha \xi^\alpha \left(\frac{u_\beta}{u_\gamma u^\gamma}\frac{Dw^\beta}{d\tau}\right) - w_\alpha \frac{D\xi^\alpha}{d\tau}\right] + \Phi_1(\tau),$$

which completes the proof.

COROLLARY 4.1. If r^α_{n} and w^α_{n} are natural first and second deviation vectors, then Eq. (4.3) takes the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(\xi_\alpha \frac{Dw^\alpha_{\mathrm{n}}}{d\tau} + 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta_{\mathrm{n}} r^\gamma_{\mathrm{n}} u^\delta - w^\alpha_{\mathrm{n}} \frac{D\xi_\alpha}{d\tau} - 2\xi_{\alpha;\beta} r^\alpha_{\mathrm{n}} \frac{Dr^\beta_{\mathrm{n}}}{d\tau}\right)$$

$$-\sigma[(F_{\alpha\beta} w^\beta_{\mathrm{n}} + F_{\alpha\beta;\gamma} r^\beta_{\mathrm{n}} r^\gamma_{\mathrm{n}})\xi^\alpha + \xi_{\alpha;\beta} F^\alpha{}_\gamma r^\beta_{\mathrm{n}} r^\gamma_{\mathrm{n}}] + C_5. \tag{4.4}$$

For vectors r^α_{s} and w^α_{s} which satisfy the constraint conditions (3.5) and

$$\frac{u_\alpha}{\sqrt{|u_\lambda u^\lambda|}}\frac{Dw^\alpha_{\mathrm{s}}}{d\tau} = \sigma F_{\alpha\beta} u^\alpha w^\beta_{\mathrm{s}} \tag{4.5}$$

correspondingly, i.e. both conserve their inclinations with Γ, the subsequent form of first integral follows from Eq. (4.3), after inserting in it Eqs (3.5) and (4.5).

COROLLARY 4.2. In the particular case $\sigma = 0$, the system of the geodesic, the first, and the second geodesic deviation equations admits a first integral of the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left\{h_{\alpha\beta}\xi^\alpha\left[\frac{Dw^\beta}{d\tau} - 2\frac{Dr^\beta}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right] + 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta r^\gamma u^\delta\right.$$

$$- \frac{u_\alpha \xi^\alpha}{u_\lambda u^\lambda}\left(h_{\beta\gamma}\frac{Dr^\beta}{d\tau}\frac{Dr^\gamma}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma r^\delta u^\varepsilon\right) - \left[w_\alpha - 2r_\alpha\left(\frac{u_\beta}{u_\lambda u^\lambda}\frac{Dr^\beta}{d\tau}\right)\right]\frac{D\xi^\alpha}{d\tau}$$

$$\left. - 2\xi_{\alpha;\beta} r^\alpha \frac{Dr^\beta}{d\tau}\right\} = C_5. \tag{4.6}$$

If r^α_{n} and w^α_{n} are the natural first and second geodesic vectors, then (4.6) reduces to

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left(\xi_\alpha \frac{Dw^\alpha_{\mathrm{n}}}{d\tau} - w^\alpha_{\mathrm{n}}\frac{D\xi_\alpha}{d\tau} + 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta_{\mathrm{n}} r^\gamma_{\mathrm{n}} u^\delta - 2\xi_{\alpha;\beta} r^\alpha_{\mathrm{n}} \frac{Dr^\beta_{\mathrm{n}}}{d\tau}\right) = C_5, \tag{4.7}$$

and for the first and the second geodesic deviation vectors r^α_{s} and w^α_{s} fulfilling conditions (3.5) and (4.5) in the limiting case $\sigma = 0$ Eq. (4.6) takes the form

$$\frac{1}{\sqrt{|u_\lambda u^\lambda|}}\left[\xi_\alpha \frac{Dw^\alpha_{\mathrm{s}}}{d\tau} - w^\alpha_{\mathrm{s}}\frac{D\xi_\alpha}{d\tau} + 2R_{\alpha\beta\gamma\delta}\xi^\alpha r^\beta_{\mathrm{s}} r^\gamma_{\mathrm{s}} u^\delta - 2\xi_{\alpha;\beta} r^\alpha_{\mathrm{s}} \frac{Dr^\beta_{\mathrm{s}}}{d\tau}\right.$$

$$\left. - \frac{u_\alpha \xi^\alpha}{u_\lambda u^\lambda}\left(g_{\beta\gamma}\frac{Dr^\beta_{\mathrm{s}}}{d\tau}\frac{Dr^\gamma_{\mathrm{s}}}{d\tau} + R_{\beta\gamma\delta\varepsilon}u^\beta r^\gamma_{\mathrm{s}} r^\delta_{\mathrm{s}} u^\varepsilon\right)\right] = C_5. \tag{4.8}$$

THEOREM 4.2. If V_n admits a symmetric Killing tensor field $K_{\alpha\beta}$ and the conditions (2.7) are fulfilled, then there exists a first integral of the system of the Lorentz, the first, and the second e.m. deviation equations of the form

$$\frac{u^\alpha}{u_\lambda u^\lambda}\left\{K_{\alpha\beta}h^\beta{}_\gamma\frac{Dw^\gamma}{d\tau}-w^\beta\frac{DK_{\alpha\beta}}{d\tau}+2K_{\alpha\beta;\gamma}\frac{Dr^\beta}{d\tau}r^\gamma+r^\beta\frac{DK_{\alpha\beta}}{d\tau}\left(\frac{u_\gamma}{u_\lambda u^\lambda}\frac{Dr^\gamma}{d\tau}\right)\right.$$

$$-(K_{\alpha\beta;\gamma\delta}+K_{\alpha\varepsilon}R^\varepsilon{}_{\beta\gamma\delta})r^\beta u^\gamma r^\delta-\frac{K_{\alpha\beta}u^\beta}{u_\lambda u^\lambda}\left(h_{\gamma\delta}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta}{d\tau}+R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau\right.$$

$$+\frac{K_{\alpha\beta}h^\beta{}_\gamma}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau}\frac{Dr^\gamma}{d\tau}-3\frac{u^\alpha}{u_\lambda u^\lambda}\left(K_{\alpha\beta}h^\beta{}_\gamma\frac{Dr^\gamma}{d\tau}-r^\beta\frac{DK_{\alpha\beta}}{d\tau}\right)\left(\frac{u_\varepsilon}{u_\lambda u^\lambda}\frac{Dr^\varepsilon}{d\tau}\right)=C_6. \qquad (4.9)$$

Proof. The proof follows again either from (1.3) or from (1.9). Applying Prop 1.1 to Eqs (4.1) and making use of Eqs (1.12), (1.15), (2.1), (2.7), (3.1), we obtain

$$\Phi_1(\tau)=\frac{u^\alpha}{|u_\lambda u^\lambda|}\left[w^\beta\frac{DK_{\alpha\beta}}{d\tau}-2K_{\alpha\beta;\gamma}\frac{Dr^\beta}{d\tau}r^\gamma-r^\beta\frac{DK_{\alpha\beta}}{d\tau}\left(\frac{u_\gamma}{u_\varepsilon u^\varepsilon}\frac{Dr^\gamma}{d\tau}\right)\right.$$

$$+(K_{\alpha\beta;\gamma\delta}+K_{\alpha\varepsilon}R^\varepsilon{}_{\beta\gamma\delta})r^\beta u^\gamma r^\delta+\frac{K_{\alpha\beta}u^\beta}{u_\lambda u^\lambda}\left(h_{\gamma\delta}\frac{Dr^\gamma}{d\tau}\frac{Dr^\delta}{d\tau}\right.$$

$$\left.\left.+R_{\gamma\delta\varepsilon\tau}u^\gamma r^\delta r^\varepsilon u^\tau\right)\right]-\frac{K_{\alpha\beta}h^\beta{}_\gamma}{u_\lambda u^\lambda}\frac{Dr^\alpha}{d\tau}\frac{Dr^\gamma}{d\tau}+3\frac{u_\alpha}{u_\lambda u^\lambda}\left(K_{\alpha\beta}h^\beta{}_\gamma\frac{Dr^\gamma}{d\tau}\right.$$

$$\left.-r^\beta\frac{DK_{\alpha\beta}}{d\tau}\right)\left(\frac{u_\delta}{u_\varepsilon u^\varepsilon}\frac{Dr^\delta}{d\tau}\right),$$

and that finishes the first part of the proof. Analogously, applying Prop. 1.3 to Eqs (4.2) and making use of the equations mentioned above yield

$$\Phi_3(\tau)=\frac{u^\alpha}{|u_\lambda u^\lambda|}\left[K_{\alpha\beta}u^\beta\left(\frac{u_\gamma}{u_\varepsilon u^\varepsilon}\frac{Dw^\gamma}{d\tau}\right)-w^\beta\frac{DK_{\alpha\beta}}{d\tau}\right]+\Phi_1(\tau),$$

which completes the proof.

COROLLARY 4.3. If r_n^α and w_n^α are the natural first and second e.m. deviation vectors, then Eq. (4.9) reads

$$\frac{u_\alpha}{u_\lambda u^\lambda}\left[K_{\alpha\beta}\frac{Dw_n^\beta}{d\tau}-w_n^\beta\frac{DK_{\alpha\beta}}{d\tau}+2K_{\alpha\beta;\gamma}\frac{Dr_n^\beta}{d\tau}r_n^\gamma-(K_{\alpha\beta;\gamma\delta}\right.$$

$$\left.+K_{\alpha\varepsilon}R^\varepsilon{}_{\beta\gamma\delta})r_n^\beta u^\gamma r_n^\delta\right]-\frac{K_{\alpha\beta}}{u_\lambda u^\lambda}\frac{Dr_n^\alpha}{d\tau}\frac{Dr_n^\beta}{d\tau}=C_6. \qquad (4.10)$$

The subsequent expression for the vectors r_s^α and w_s^α satisfying the constraint conditions (3.5) and (4.5), correspondingly, follows from (4.9), after inserting there these conditions.

In the particular case of $\sigma = 0$, the system of the geodesic the first, and the second geodesic deviation equations admits the first integral of the same form as (4.9).

Similarly like in Sect. 3 there is a simple relation between the integrals of the Lorentz equations and of the second e.m. deviation equations as it follows from the following lemma.

LEMMA 4.1. Under conditions of Lemma 3.1, the first integrals (4.3) and (4.9) follow from Eq. (3.11) and (3.12) after performing a twofold differentiation with respect to the parameter ε, in which the definitions

$$w^\alpha(\tau, \varepsilon) := \frac{\partial r^\alpha}{\partial \varepsilon}(\tau, \varepsilon), \quad C_5 := \frac{\partial^2 C_1(\varepsilon)}{\partial \varepsilon^2}, \quad C_6 := \tfrac{1}{2}\frac{\partial^2 C_2(\varepsilon)}{\partial \varepsilon^2}$$

and the Ricci identity

$$\frac{D^2 u^\alpha}{\partial\varepsilon\partial\tau} - \frac{D^2 u^\alpha}{\partial\tau\partial\varepsilon} = R^\alpha{}_{\beta\gamma\delta} u^\beta r^\gamma u^\delta$$

must be taken into account.

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