HAMILTONIAN FORMULATION FOR THE YANG-MILLS FIELDS ASSOCIATED WITH ELECTRIC AND MAGNETIC CHARGES IN THE COULOMB GAUGE*

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We consider particles carrying both the electric and magnetic charges in the Yang-Mills theory. The Hamiltonian formulation for the system has been carried out in the Coulomb gauge. The zero mode problem associated with the Coulomb gauge has been taken care of by introducing generalized Green's functions and imposing the restrictions of orthogonality of source terms and zero modes. It has been shown that the orthogonality imposes restrictions on the physical states of the system.

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1. *Introduction*

It is generally believed that quantum field theory is able to provide a simple, unified description of all microscopic phenomena. However, there are problems associated with the quantization of field theories containing a gauge symmetry. It is because of the fact that imposition of canonical commutation relations may lead to inconsistency with the gauge conditions [1]. According to usual interpretation of quantum theory, choosing the Coulomb gauge in the Yang-Mills theory eliminates the redundant degrees of freedom inherent in gauge theories [2]. Gribov [3], however, has discussed some difficulties associated with formulating a non-Abelian gauge theory in the Coulomb gauge, which are related to the existence of zero modes of a particular operator, when fields are sufficiently large. Tyburski [4], in considering a source-free first-order canonical quantization scheme for a non-Abelian gauge theory, has treated the zero modes by introducing a generalized Green's function, which has also been used [5] to remove the zero modes in quantizing the Yang-Mills theory in the Coulomb gauge. However, the sources in the latter paper are only 'electric' ones.

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In the present paper we consider the sources as being both electric and magnetic, and carry out the Hamiltonian formulation for the Yang-Mills theory in the Coulomb gauge by introducing generalized Green's functions. The magnetic sources are not of topological origin [6] but similar to the ones considered by Brandt and Neri [7] in formulating the non-Abelian classical Lagrangian for point electric and magnetic charges. For the inclusion of both the sources a non-Abelian field tensor [8] defined in terms of two potentials has been used. In Sect. 2 the field equations have been derived and time components of these equations have been written as the constraints to which the potentials and field tensor are subjected. The Gribov ambiguity and the necessity of introducing the generalized Green's functions to remove the zero modes has been taken up in Sect. 3. In Sect. 4 the Hamiltonian formulation has been carried out. Finally, the conclusions are given in Sect. 5.

2. *Field equations*

Since we have considered the sources to be both electric and magnetic, similar to point electric and magnetic charges assumed by Brandt et al. [7], we define a non-Abelian field tensor [8] as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e f^{abc} A^b_\mu A^c_\nu - \tfrac{1}{2}\,\delta_{\mu\nu\varrho\sigma}(\partial^\varrho B^{\sigma a} - \partial^\sigma B^{\varrho a} + g f^{abc} B^{\varrho b} B^{\sigma c}) \tag{1}$$

to describe the system of electric and magnetic charges in Yang-Mills theory. In these equations A^a_μ and B^a_μ are two potentials which have been used to avoid string variables, e and g are the corresponding gauge coupling constants, f^{abc} are the structure constants of the gauge group SU(2) and $\delta_{\mu\nu\varrho\sigma}$ is the antisymmetric tensor with μ, ν, ϱ, σ running from 0 to 3 and

$$\delta^{\mu\nu\varrho\sigma} = -\delta_{\mu\nu\varrho\sigma}. \tag{2}$$

In view of the use of two potentials in the field tensor (1) we choose two gauge functions U and U' for their gauge transformations

$$A_\mu \to U A_\mu U^{-1} - \frac{1}{e}\, U \partial_\mu U^{-1} \tag{3a}$$

and

$$B_\mu \to U' B_\mu U'^{-1} - \frac{1}{g}\, U' \partial_\mu U'^{-1}, \tag{3b}$$

where U and U' may be defined as

$$U = \exp\left[-ie\lambda^a(x) T^a\right] \tag{4a}$$

and

$$U' = \exp\left[-ig\lambda'^a(x) T^a\right] \tag{4b}$$

in which T^a represent the group generators of the gauge group SU(2) obeying

$$[T^a, T^b] = if^{abc}T^c, \tag{5}$$

$\lambda^a(x)$ and $\lambda'^a(x)$ are arbitrary real functions of x. It is easy to see from equations (4) that $e\lambda^a(x) \neq g\lambda'^a(x)$. The field tensor (1) is invariant under gauge functions (4).

$$F_{\mu\nu} \to UF_{\mu\nu}U^{-1} + U'F_{\mu\nu}U'^{-1} \tag{6}$$

provided that the conditions

$$U'(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef^{abc}A^b_\mu A^c_\nu)U'^{-1} = 0 \tag{7a}$$

and

$$U[\tfrac{1}{2}\delta_{\mu\nu\varrho\sigma}(\partial^\varrho B^{\sigma a} - \partial^\sigma B^{\varrho a} + gf^{abc}B^{\varrho b}B^{\sigma c})]U^{-1} = 0 \tag{7b}$$

are also obeyed.

Now the Lagrangian density for the system may be written as

$$\mathscr{L} = -\tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + j^a_\mu A^{\mu a} - k^a_\mu B^{\mu a}, \tag{8}$$

where j^a_μ and k^a_μ are the electric and magnetic source densities which transform as

$$j_\mu \to Uj_\mu U^{-1} \tag{9a}$$

and

$$k_\mu \to U'k_\mu U'^{-1}. \tag{9b}$$

Thus, under the transformations (3), (6) and (9) the Lagrangian density (8) is gauge covariant. Euler Lagrange variation of the Lagrangian density (8) with respect to the potential A^a_μ gives the field equations

$$D^{ac}_\nu \tilde{F}^{\mu\nu c} = j^{\mu a}, \tag{10}$$

where

$$D^{ac}_\nu = \partial_\nu \delta^{ac} + ef^{abc}A^b_\nu \tag{11}$$

is the covariant derivative. The variation of Eq. (8) with respect to the potential B^a_μ yields the field equations

$$D'^{ac}_\nu \tilde{F}^{\mu\nu c} = k^{\mu a}, \tag{12}$$

where

$$D'^{ac}_\nu = \partial_\nu \delta^{ac} + gf^{abc}B^b_\nu \tag{13}$$

is another covariant derivative and

$$\tilde{F}^{\mu\nu a} = \tfrac{1}{2}\delta^{\mu\nu\varrho\sigma}F_{\varrho\sigma} \tag{14}$$

is the dual of field tensor $F^{\mu\nu a}$ given by Eq. (1). The field equations derivable from Eq. (10) and (12) are

$$D_i^{ac}F^{0ic} = j^{0a}, \quad (15a)$$

$$D_i'^{ac}\tilde{F}^{0ic} = k^{0a}, \quad (15b)$$

and

$$D_0^{ac}F^{0ic} = -(j^{ia}-D_j^{ac}F^{ijc}), \quad (16a)$$

$$D_0'^{ac}\tilde{F}^{0ic} = -(k^{ia}-D_j'^{ac}\tilde{F}^{ijc}). \quad (16b)$$

From Eqs (15), the time components of field equations (10) and (12), the dependences between A_i^a and F^{0ia} as well as between B_i^a and $\tilde{F}^{0ia}$ may be observed. The time components imply the constraints for these dependences.

3. *The Coulomb gauge and zero modes*

The gauge transformations (3) shall maintain the Coulomb gauge condition

$$\partial_i A_i^a = 0 = \partial_i B_i^a \quad (17a)$$

in the form

$$\partial_i A_i'^a = 0 = \partial_i B_i'^a \quad (17b)$$

provided that

$$D_i\partial^i(UU^{-1}) = 0 = D_i'\partial^i(U'U'^{-1}). \quad (17c)$$

In the Coulomb gauge it is convenient to split the field strengths F^{0ia} and $\tilde{F}^{0ia}$ into transverse and longitudinal parts:

$$F^{0ia} = F^{0ia\mathrm{T}}-\partial^i\xi^a \quad (18a)$$

and

$$\tilde{F}^{0ia} = \tilde{F}^{0ia\mathrm{T}}-\partial^i\zeta^a, \quad (18b)$$

where

$$\partial_i F^{0ia\mathrm{T}} = 0 = \partial_i\tilde{F}^{0ia\mathrm{T}}. \quad (18c)$$

From the field tensor (1), we may write

$$F^{0ia} = \partial^0 A^{ia}-\partial^i A^{0a}+ef^{abc}A^{0b}A^{ic}+\varepsilon_{ijk}\left(\partial_j B_k^a+\frac{g}{2}f^{abc}B_j^b B_k^c\right) \quad (19)$$

and from Eq. (14)

$$\tilde{F}^{0ia} = \partial^0 B^{ia}-\partial^i B^{0a}+gf^{abc}B^{0b}B^{ic}-\varepsilon_{ijk}\left(\partial_j A_k^a+\frac{e}{2}f^{abc}A_j^b A_k^c\right). \quad (20)$$

In these equations we are interested in seeing whether the terms $(g/2)\varepsilon_{ijk}f^{abc}B_j^bB_k^c$ and $(e/2)\varepsilon_{ijk}f^{abc}A_j^bA_k^c$, which represent the crossproducts in the gauge group space, may be written in terms of the gradient of some scalar functions. This requires

$$\varepsilon_{ijk}\partial_j f^{abc}\varepsilon_{klm}A_l^bA_m^c = 0 \tag{21a}$$

as well as

$$\varepsilon_{ijk}\partial_j f^{abc}\varepsilon_{klm}B_1^bB_m^c = 0. \tag{21b}$$

It may then be observed that Eqs (21) are satisfied if the vector potentials $\vec{A}^a$ and $\vec{B}^a$ are such that

$$|\vec{A}^b| = |\vec{A}^c| \tag{22a}$$

and

$$|\vec{B}^b| = |\vec{B}^c|. \tag{22b}$$

Therefore, assuming the conditions (22) to exist we may write

$$\frac{g}{2}\varepsilon_{ijk}f^{abc}B_j^bB_k^c = -\partial^i\phi'^a \tag{23a}$$

and

$$\frac{e}{2}\varepsilon_{ijk}f^{abc}A_j^bA_k^c = -\partial^i\phi^a, \tag{23b}$$

where ϕ'^a and ϕ^a are the scalar functions.

From Eqs (19) and (20), we may write expressions for $\partial_i F^{0ia}$ and $\partial_i\tilde{F}^{0ia}$, respectively, which may then be used in Eqs (18) to obtain:

$$-\partial_i\partial^i\xi^a = -D_i^{ac}\partial^iA^{0c} - \partial_i\partial^i\phi'^a \tag{24a}$$

and

$$-\partial_i\partial^i\xi^a = -D_i'^{ac}\partial^iB^{0c} + \partial_i\partial^i\phi^a, \tag{24b}$$

where we have also used Eqs (23). Defining

$$f^a = \xi^a - \phi'^a \tag{25a}$$

and

$$f'^a = \xi^a + \phi^a, \tag{25b}$$

Eqs (24) may be written as

$$D_i^{ac}\partial^iA^{0c} = \partial_i\partial^if^a \tag{26a}$$

and

$$D_i'^{ac}\partial^iB^{0c} = \partial_i\partial^if'^a. \tag{26b}$$

Using Eqs (18), the field Eqs (15) may be written as

$$D_i^{ac}\partial^i\xi^c = -j^{0a}+D_i^{ac}F^{0icT} \tag{27a}$$

and

$$D_i'^{ac}\partial^i\xi^c = -k^{0a}+D_i'^{ac}\tilde{F}^{0icT}. \tag{27b}$$

Now solutions for Eqs (26a), (27a) and (26b), (27b) may be obtained by introducing Green's functions G^{ab} [5] and G'^{ab} respectively, which obey

$$D_i^{ab}\partial^i G^{bc}(\vec{x}, \vec{y}; A) = -\delta^{ac}\delta^3(\vec{x}-\vec{y}) \tag{28a}$$

and

$$D_i'^{ab}\partial^i G'^{bc}(\vec{x}, \vec{y}; B) = -\delta^{ac}\delta^3(\vec{x}-\vec{y}). \tag{28b}$$

However, Green's functions are well defined if the operators $D_i\partial^i$ and $D_i'\partial^i$ have no zero modes. If the gauge functions U and U' (Eqs (4)) correspond to infinitesimal transformations

$$U = \exp(\varepsilon h) = 1+\varepsilon h \tag{29a}$$

and

$$U' = \exp(\varepsilon' h) = 1+\varepsilon' h, \tag{29b}$$

then gauge transformations (3) become

$$A_i' = A_i+\varepsilon D_i h \tag{30a}$$

and

$$B_i' = B_i+\varepsilon' D_i'^h. \tag{30b}$$

In that case the Coulomb gauge condition is maintained, provided that

$$D_i^{ab}\partial^i h^b = 0 = D_i'^{ab}\partial^i h^b. \tag{31}$$

Therefore, when gauge transformations are infinitesimal, the Eqs (31) become analogue to Eq. (17c). However, the Eqs (31) have non-trivial solutions [5] and there exist normalization zero modes of operators $D_i\partial^i$ and $D_i'\partial^i$ for sufficiently large A_i and B_i [3]. Due to the existence of the zero modes, the solutions of Eqs (26) and (27) cannot be obtained in terms of Green's functions given by Eqs (28a) and (28b), respectively. Instead, generalized Green's functions $\tilde{G}$ and $\tilde{G}'$ have to be introduced. These generalized Green's functions obey

$$D_i^{ab}\partial^i\tilde{G}^{bc}(\vec{x}-\vec{y}) = \delta^{ac}\delta^3(\vec{x}-\vec{y})-\sum_n h^{a(n)}(\vec{x}; A)h^{c(n)}(\vec{y}; A) \tag{32a}$$

and

$$D_i'^{ab}\partial^i G'^{bc}(x-y) = \delta^{ac}\delta^3(\vec{x}-\vec{y})-\sum_n h^{a(n)}(\vec{x}; B)h^{c(n)}(\vec{y}; B), \tag{32b}$$

where $h^{a(n)}$ are the zero modes of equations (30) and n is for number of zero modes. However, for simplicity we assume only one zero mode and also the source terms in equations (26a), (27a) and (26b), (27b) to be orthogonal to zero modes h^a. Therefore, equations (26a) and (27a) may be solved in terms of generalized Green's function $\tilde{G}$ as

$$A^0 = \tilde{G}\nabla^2 f \tag{33a}$$

and

$$\xi = \tilde{G}(D_i F^{0iT} - j^0). \tag{33b}$$

Eqs (25a) and (33b) may be solved to obtain

$$f^a = \tilde{G}^{ab} D_i^{bc} \left(F^{0icT} + \frac{g}{2} \varepsilon_{ijk} f^{abc} B_j^b B_k^c \right) - \tilde{G}^{ab} j^{0b} \tag{34}$$

and then A^0 may be obtained from Eq. (33a) using Eqs (34). Similarly, Eqs (26b) and (27b) may be solved by using generalized Green's function $\tilde{G}'$ Eq. (32b) as

$$B^0 = \tilde{G}'\nabla^2 f' \tag{35a}$$

and

$$\zeta = \tilde{G}'(D_i' \tilde{F}^{0iT} - k^0) \tag{35b}$$

again f' may be solved by using Eqs (25b) and (35b) to obtain the exact value of B^0 as

$$f'^a = \tilde{G}'^{ab} D_i'^{bc} \left(\tilde{F}^{0icT} - \frac{e}{2} \varepsilon_{ijk} f^{abc} A_j^b A_k^c \right) - \tilde{G}'^{ab} k^{0b}. \tag{36}$$

The solutions of Eqs (26a) and (27a) obtained through Eqs (33) and (34) and those for Eqs (26b) and (27b) obtained through Eqs (35) and (36) are subjected to orthogonality requirements with respect to the zero modes as has been discussed above. The requirements impose the restrictions

$$Q = \mathrm{Tr} \int d^3\vec{x} h(\vec{x}) \left[D_i F^{0iT}(\vec{x}) - j^0 \right] = 0, \tag{37a}$$

$$R = \mathrm{Tr} \int d^3\vec{x} h(\vec{x}) \nabla^2 f(\vec{x}) = 0, \tag{37b}$$

and

$$Q' = \mathrm{Tr} \int d^3\vec{x} h(\vec{x}) \left[D_i' \tilde{F}^{0iT}(x) - k^0 \right] = 0, \tag{38a}$$

$$R' = \mathrm{Tr} \int d^3\vec{x} h(\vec{x}) \nabla^2 f'(\vec{x}) = 0, \tag{38b}$$

which are essential to obtain the solutions in terms of generalized Green's functions $\tilde{G}$ and $\tilde{G}'$ given by the Eqs (32a) and (32b), respectively.

4. *Hamiltonian*

Having solved it for A^0, ξ and f as well as B^0, ζ and f', we may now calculate the Hamiltonian for the system of electric and magnetic charges. From the Lagrangian density (8), the Hamiltonian density for the system may be calculated as

$$\mathscr{H} = \mathscr{H}_A + \mathscr{H}_B, \tag{39}$$

where

$$\mathscr{H}_A = \partial_0 A_i^a \frac{\partial \mathscr{L}}{\partial(\partial_0 A_i^a)} - \mathscr{L} \tag{40a}$$

and

$$\mathscr{H}_B = -\left[\partial_0 B_i^a \frac{\partial \mathscr{L}}{\partial(\partial_0 B_i^a)} - \mathscr{L}\right] \tag{40b}$$

to give the total energy of the system as

$$H = \int d^3\vec{x}\mathscr{H} = \int d^3\vec{x}[\mathscr{H}_A + \mathscr{H}_B]. \tag{41}$$

The negative sign in the Eq. (40b) is due to the fact that the variation of Lagrangian density with respect to B_μ^a corresponds to the dual of field tensor $F^{\mu\nu a}$ (Eq. (1)). From Eq. (40a) we may calculate the $\mathscr{H}_A$ by using Eqs (19) and (18a) as

$$\mathscr{H}_A = -\partial_0 A_i^a F^{0iaT} - \mathscr{L}. \tag{42}$$

Similarly, using Eqs (20) and (18b), $\mathscr{H}_B$ may be calculated as

$$\mathscr{H}_B = -(\partial_0 B_i^a \tilde{F}^{0iaT} - \mathscr{L}). \tag{43}$$

From Eqs (19), (18a), (23a), (25a) and (33a), we have

$$\partial_0 A_i = F_{0i}^T - (\partial_i - D_i \tilde{G} \nabla^2) f + \varepsilon_{ijk}(\partial_j B_k) \tag{44}$$

and similarly using Eqs (20), (18b), (25b) and (33b), we obtain

$$\partial_0 B_i = \tilde{F}_{0i}^T - (\partial_i - D_i' \tilde{G}' \nabla^2) f' - \varepsilon_{ijk}(\partial_j A_k). \tag{45}$$

Since $\tilde{G}$ and $\tilde{G}'$ obey Eqs (32a) and (32b), respectively, the right-hand sides of Eqs (44) and (45) are not explicitly transverse unless the Eqs (37b) and (38b) are satisfied. Then we obtain from the first terms of $\mathscr{H}_A$ (Eq. (42)), the following contribution to the Hamiltonian

$$-\int d^3\vec{x}\partial_0 A_i^a F^{0iaT} = -\int d^3\vec{x}[F_{0i}^{aT} F^{0iaT} + (D_i^{ab} \tilde{G}^{bc} \nabla^2 f^c) F^{0iaT} + \varepsilon_{ijk}(\partial_j B_k^a) F^{0iaT}]. \tag{46}$$

Integrating this equation by parts and using Eqs (14), (15a), (16b), (23a), (27b) and imposing the conditions (37b), we obtain after performing simple calculations

$$-\int d^3x \partial_0 A_i^a F^{0ia\mathrm{T}} = \int d^3\vec{x}[F_{0i}^{a\mathrm{T}} F_{0i}^{a\mathrm{T}} - \xi^a \nabla^2 \xi^a + \tilde{G}^{ab} \nabla^2 f^b j^{0a} + B_i^a k_i^a - B_i^a D_0^{ab} \tilde{F}^{0ib}], \tag{47}$$

which may be used to obtain the Hamiltonian H_A from the density (40a) as

$$H_A = \int d^3\vec{x} \mathscr{H}_A = \int d^3\vec{x}[\tfrac{1}{2}\{(F_{0i}^{a\mathrm{T}})^2 + (\partial_i \xi^a)^2 + (F_{jk}^a)^2\} + j_i^a A_i^a. \tag{48}$$

Similarly, Hamiltonian H_B may be calculated from the density (40b) as

$$H_B = \int d^3\vec{x} \mathscr{H}_B = \int d^3\vec{x}[\tfrac{1}{2}\{(\tilde{F}_{0i}^{a\mathrm{T}})^2 + (\partial_i \zeta^a)^2 + (\tilde{F}_{jk}^a)^2\} + B_i^a k_i^a], \tag{49}$$

where use has been made of Eqs (14), (15b), (16a), (23b), (27a) and (38b). However, from Eq. (14),

$$F_{jk} = \tfrac{1}{2}\delta_{jk0i}\tilde{F}^{0ia} = -\tilde{F}_{0i}^a \tag{50a}$$

and

$$\tilde{F}_{jk}^a = \tfrac{1}{2}\delta_{jk0i}F^{0ia} = F_{0i}^a, \tag{50b}$$

we may calculate the total Hamiltonian from Eqs (48) and (49) as

$$H = \int d^3\vec{x}[(F_{0i}^{a\mathrm{T}})^2 + (\tilde{F}_{0i}^{a\mathrm{T}})^2 + (\partial_i \xi^a)^2 + (\partial_i \zeta^a)^2 + j_i^a A_i^a + k_i^a B_i^a] \tag{51}$$

which identifies the Coulombian interaction energy explicitly in the form

$$H_c = \int d^3\vec{x}[(\partial_i \xi^a)^2 + (\partial_i \zeta^a)^2]. \tag{52}$$

5. Conclusion

We have carried out the Hamiltonian formulation for the system of non-Abelian electric and magnetic charges in the Coulomb gauge. The quantization problem of non--Abelian gauge theories in the Coulomb gauge is not free from ambiguities [3] in view of the zero modes associated with the operators $D_i\partial^i$ and $D_i'\partial^i$. However, we have taken due care of the zero modes, by using generalized Green's functions and imposing the restrictions that the source terms must be orthogonal to the zero modes. The restrictions given by Eqs (37) and (38) may be interpreted as the restrictions on the physical states $Q|\psi\rangle = 0$, $R|\psi\rangle = 0$ and $Q'|\chi\rangle = 0$, $R'|\chi\rangle = 0$. Contrary to the magnetic charges of topological origin [6], we have introduced the point particles, which carry both electric and magnetic charges. The use of the two-potential approach has been preferred over the controversial string variables [9]. It may be observed from Eqs (24), (25) and (26) that contrary to electric case, we need to introduce another functions ϕ and ϕ' which are defined by Eqs (23). It may also be observed from Eq. (52) that contrary to electric case, the contribution to the Coulombian interaction energy is due to two different scalar functions ξ and ζ whose solutions are given by Eqs (33b) and (35b), respectively.

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